A

COLLECTION OF PROBLEMS

IN

ILLUSTRATION OF THE PRINCIPLES

OF

THEORETICAL MECHANICS

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"Examples give a quicker impression than arguments."—BACON.

THIRD EDITION.

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PREFACE TO THE FIRST EDITION.

THE design of this Work is to facilitate the study of Theoretical Mechanics, by presenting to the student a systematic collection of problems in illustration of the more important principles of the science. The want of any such treatise, it is believed, has been felt by many as a serious impediment to the acquisition of adequate ideas in this branch of mathematical philosophy. Much importance, it may be observed, was attached by the great discoverers of the mechanical theories to the full discussion of numerous problems, as will be evident from a reference to the works of the three Bernoullis, of Leibnitz, and of D'Alembert, and to the beautiful investigations scattered throughout so long a series of volumes of the *St. Petersburgh Transactions* by the liberal hand of Euler.

The author of this volume has endeavoured, as much as possible, to direct the attention of the student to the original memoirs of which he has so largely availed himself. This he has done, partly, to enable the beginner to obtain more detailed information than is compatible with the nature of this work, on particular questions which may excite an interest in his mind: his chief object, however, has been, to offer every facility to those, who have already overcome at least the elementary difficulties of the subject, for acquiring a practical familiarity with the historical development of the science. Although it be admitted that useful and exact knowledge may be obtained from even an exclusive perusal of the concise and methodical treatises which are generally adopted for the purpose of academic instruction; yet it may be asserted with confidence, that an excessive adherence to such a system

of study, must deprive the student of much delightful and most valuable information.

In regard to the mode in which the author of this treatise has completed the task which he has proposed to himself, he feels every degree of diffidence, and would willingly that it had been undertaken by an abler hand. In apology for the imperfections, of which either he is himself aware or which may have eluded his observation, he can plead only the fact of engrossing occupations, or of perhaps insufficient preparation for a work requiring greater research than was originally contemplated.

Many of the problems in this volume have been extracted, with appropriate modifications, from the Ancient Transactions of the various Academies and learned Societies of Europe; many have been selected from the Cambridge Senate-House Papers; and for not a few the author is under obligation to the contributions of his friends. In arriving at original sources of information, it is scarcely necessary to state that great assistance has been obtained from the historical matter of Lagrange's *Mécanique Analytique*, and from Montucla's *Histoire des Mathématiques.*

CAMBRIDGE, *October*, 1842.

PREFACE TO THE SECOND EDITION.

In preparing for the press a Second Edition of this Treatise, the author has adhered to the general design of the First Edition; he has, however, effected numerous alterations and corrections, many of which are due to the kind suggestions of readers of the work; he has also considerably augmented the matter of those chapters which in the former edition appeared to be inadequately supplied with problems. Certain entirely new chapters have also been written; one on the Attractions of Solid Bodies, two on Miscellaneous Problems, and one on Live Things.

Cambridge, *September* 8, 1855.

PREFACE TO THE THIRD EDITION.

In the Third Edition of the *Mechanical Problems* the whole Work has been carefully revised, and all errors, which have been pointed out to the author or detected by himself, have been corrected. Some of the matter of the last edition has been cancelled, and the Treatise has been enriched by the addition of a considerable number of new problems, taken principally, when not assigned to their authors, from the examination papers proposed in the various Colleges of the University of Cambridge, in the Senate-House for the Mathematical Tripos, and to the candidates for the Smith's Prizes. Several additional classes of Problems, in illustration of various Theorems, might have been introduced into the Work with advantage: the bulk of the Book would however have been thereby increased beyond the limits within which it would seem expedient to confine it.

Trinity Hall, Cambridge, *January* 15, 1876.

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STATICS.

CHAPTER I.

CENTRE OF GRAVITY.

LET dm represent an element, at any point x, y, z, of the mass of a body referred to any three co-ordinate axes, rectangular or oblique, and let $\bar{x}, \bar{y}, \bar{z}$, denote the co-ordinates of the centre of gravity of the body; then the formulæ for finding the values of $\bar{x}, \bar{y}, \bar{z}$, are

$$\bar{x} = \frac{\int x dm}{\int dm}, \quad \bar{y} = \frac{\int y dm}{\int dm}, \quad \bar{z} = \frac{\int z dm}{\int dm},$$

the limits of the integrations being determined by the form of the body.

If the body be bounded by a surface represented by a single algebraical equation in x, y, z, the evaluation of each of the expressions $\int x dm$, $\int y dm$, $\int z dm$, $\int dm$, will require the performance of the operation of integration on a single function of x, y, z, between appropriate limits; if, however, the body be bounded by discontinuous surfaces, the evaluation of each of these expressions will require the integration between proper limits of several functions of x, y, z, corresponding to the several discontinuous surfaces; the sum of the definite integrals of these functions being the required value of the expression.

The idea of the centre of gravity of material bodies is due to Archimedes, by whom the centres of gravity of various areas

were investigated in his treatise, entitled Ἐπιπέδων ἰσοῤῥοπικῶν ἢ κέντρα βαρῶν ἐπιπέδων. He likewise determined the centre of gravity of the parabolic conoid. Among the mathematical successors of Archimedes who have cultivated the science of the centre of gravity, may be mentioned Pappus[1], Guido Ubaldi[2], Lucas Valerius[3], La-Faille[4], Guldin[5], Wallis[6], Carré[7], Varignon[8], Clairaut[9].

Sect. 1. *Symmetrical Area.*

Let x be the abscissa and y the ordinate of any point in the circumference of a plane area, symmetrical with respect to the axis of x; the axes of co-ordinates being either rectangular or oblique. Then the centre of gravity of any portion of this area, intercepted between any assigned pair of double ordinates, will lie in the axis of x, and its distance $\bar{x}$ from the origin will be given by the formula

$$\bar{x} = \frac{\int xy\,dx}{\int y\,dx},$$

where the integrations are to be performed between limits depending upon the positions of the intercepting ordinates.

The value of $\bar{x}$ is sometimes more readily obtained by polar co-ordinates, when the formula will be

$$\bar{x} = \frac{\iint r^2 \cos\theta\,d\theta\,dr}{\iint r\,d\theta\,dr},$$

where r denotes the distance of any point within the area from

[1] *Mathemat. Collect.*, lib. 8, *published for the first time in* 1588.
[2] *In duos Archimedis Æquiponderantium libros Paraphrasis*, 1588.
[3] *De Centro Gravitatis Solidorum*, 1604.
[4] *De Centro Gravitatis partium Circuli et Ellipsis Theoremata*, 1632.
[5] *Centrobaryca*, 1635.
[6] *Opera*, tom. I. cap. 4 et 5, 1670.
[7] *Mésure des Surfaces*, 1700.
[8] *Mém. de l'Acad. des Sciences de Paris*, 1714.
[9] *Mém. de l'Acad. des Sciences de Paris*, 1731, p. 159.

the origin, and θ the inclination of r to the axis of x. The nature of the limits in the double integrations will depend upon the form of the area in each particular case.

Supposing the area to consist of several portions, the boundaries of which are defined by distinct equations, the above formulæ must be replaced by

$$\bar{x} = \frac{\Sigma \int xy\, dx}{\Sigma \int y\, dx},$$

$$\bar{x} = \frac{\Sigma \iint r^2 \cos\theta\, d\theta\, dr}{\Sigma \iint r\, d\theta\, dr},$$

where Σ represents the summation of the integrations performed in regard to the several portions of the area.

(1) To find the centre of gravity of the area of any portion BAC (fig. 1) of a parabola cut off by any chord BC.

Let Py be the tangent to the parabola, which is parallel to the chord CB, P being the point of contact; from P draw Px parallel to the axis of the parabola. Then, Px and Py being taken as the axes of x and y, the equation to the curve will be

$$y^2 = 4mx,$$

m being the distance of the point P from the focus.

Hence, if $PE = a$,

$$\bar{x} = \frac{\int_0^a xy\, dx}{\int_0^a y\, dx} = \frac{\int_0^a x^{\frac{3}{2}} dx}{\int_0^a x^{\frac{1}{2}} dx} = \frac{\frac{2}{5}a^{\frac{5}{2}}}{\frac{2}{3}a^{\frac{3}{2}}} = \frac{3}{5}a.$$

Archimedes, 'Επιπέδων ἰσοῤῥοπικῶν, Lib. II. Prop. 8; Guldin, *Centrobaryca*, Lib. I. cap. 9, p. 121.

(2) To find the centre of gravity of the area of the Cissoid of Diocles, EAE', (fig. 2).

The equation to the curve is

$$y^2 = \frac{x^3}{a-x};$$

hence
$$\bar{x} = \frac{\int xy\,dx}{\int y\,dx} = \frac{\int_0^a \frac{x^{\frac{5}{2}}}{(a-x)^{\frac{1}{2}}}\,dx}{\int_0^a \frac{x^{\frac{3}{2}}}{(a-x)^{\frac{1}{2}}}\,dx} \quad\ldots\ldots\ldots\ldots\ldots (1);$$

but
$$\int \frac{x^{\frac{5}{2}}}{(a-x)^{\frac{1}{2}}}\,dx = -2x^{\frac{5}{2}}(a-x)^{\frac{1}{2}} + 5\int x^{\frac{3}{2}}(a-x)^{\frac{1}{2}}\,dx,$$

and therefore
$$\int_0^a \frac{x^{\frac{5}{2}}}{(a-x)^{\frac{1}{2}}}\,dx = 5\int_0^a x^{\frac{3}{2}}(a-x)^{\frac{1}{2}}\,dx$$

$$= 5a\int_0^a \frac{x^{\frac{3}{2}}}{(a-x)^{\frac{1}{2}}}\,dx - 5\int_0^a \frac{x^{\frac{5}{2}}}{(a-x)^{\frac{1}{2}}}\,dx = \tfrac{5}{6}a\int_0^a \frac{x^{\frac{3}{2}}}{(a-x)^{\frac{1}{2}}}\,dx;$$

hence from (1) we have

$$\bar{x} = \tfrac{5}{6}a\,\frac{\int_0^a \frac{x^{\frac{3}{2}}}{(a-x)^{\frac{1}{2}}}\,dx}{\int_0^a \frac{x^{\frac{3}{2}}}{(a-x)^{\frac{1}{2}}}\,dx} = \tfrac{5}{6}a.$$

(3) To find the centre of gravity of the sector ABC (fig. 3) of a circle, of which C is the centre.

From C draw the straight line CEx bisecting the sectorial area; and draw Cy at right angles to Cx. Let $CE = a$, and $\angle ACx = \alpha$; then, Cx, Cy, being the axes of x and y,

$$\bar{x} = \frac{\Sigma\int xy\,dx}{\Sigma\int y\,dx} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now the equations to the straight line CA, and to the circle of which AEB is an arc, are respectively

$$y = x\tan\alpha, \quad y^2 = a^2 - x^2;$$

also CF is equal to $a \cos \alpha$; hence

$$\Sigma \int xy\, dx = \int_0^{a\cos\alpha} x^2 \tan\alpha\, dx + \int_{a\cos\alpha}^{a} x\,(a^2 - x^2)^{\frac{1}{2}}\, dx \ldots\ldots (2),$$

and $$\Sigma \int y\, dx = \int_0^{a\cos\alpha} x \tan\alpha\, dx + \int_{a\cos\alpha}^{a} (a^2 - x^2)^{\frac{1}{2}}\, dx \ldots\ldots (3).$$

Now, by the ordinary processes of the Integral Calculus,

$$\int_0^{a\cos\alpha} x^2 \tan\alpha\, dx = \tfrac{1}{3} a^3 \sin\alpha \cos^2\alpha,$$

and $$\int_{a\cos\alpha}^{a} x\,(a^2 - x^2)^{\frac{1}{2}}\, dx = \tfrac{1}{3} a^3 \sin^3\alpha;$$

hence from (2) we have

$$\Sigma \int xy\, dx = \tfrac{1}{3} a^3 \sin\alpha \ldots\ldots (4).$$

Again, $$\int_0^{a\cos\alpha} x \tan\alpha\, dx = \tfrac{1}{2} a^2 \sin\alpha \cos\alpha,$$

and $$\int_{a\cos\alpha}^{a} (a^2 - x^2)^{\frac{1}{2}}\, dx = \tfrac{1}{2} (a^2\alpha - a^2 \sin\alpha \cos\alpha);$$

hence from (3) we have

$$\Sigma \int y\, dx = \tfrac{1}{2} a^2 \alpha \ldots\ldots (5).$$

From the relations (1), (4), (5),

$$\bar{x} = \frac{\tfrac{1}{3} a^3 \sin\alpha}{\tfrac{1}{2} a^2 \alpha} = \tfrac{2}{3} a \frac{\sin\alpha}{\alpha}.$$

This result however may be obtained more readily by polar co-ordinates: let P be any point in the area of the sector; let $CP = r$, $\angle PCx = \theta$; then

$$\bar{x} = \frac{\int_{-\alpha}^{+\alpha}\int_0^a r^2 \cos\theta\, d\theta\, dr}{\int_{-\alpha}^{+\alpha}\int_0^a r\, d\theta\, dr} = \frac{\tfrac{1}{3} a^3 \int_{-\alpha}^{+\alpha} \cos\theta\, d\theta}{\tfrac{1}{2} a^2 \int_{-\alpha}^{+\alpha} d\theta} = \tfrac{2}{3} a \frac{\sin\alpha}{\alpha}.$$

We might have effected the double integration in a different order; thus

$$\bar{x} = \frac{\int_0^a \int_{-\alpha}^{+\alpha} r^2 \cos\theta \, dr \, d\theta}{\int_0^a \int_{-\alpha}^{+\alpha} r \, dr \, d\theta} = \frac{2\sin\alpha \int_0^a r^2 \, dr}{2\alpha \int_0^a r \, dr}$$

$$= \frac{2\sin\alpha \, . \, \frac{1}{3}a^3}{2\alpha \, . \, \frac{1}{2}a^2} = \tfrac{2}{3}a\frac{\sin\alpha}{\alpha} = \tfrac{2}{3}\frac{\text{radius} \times \text{chord}}{\text{arc}}.$$

According to the former order of integration, the sector ACB is conceived to be subdivided into an infinite series of infinitesimal triangles having a common vertex C, their bases being elements of the arc AEB; according to the latter order, we conceive the sector to be made up of a series of circular rings of indefinitely small breadth, having a common centre C.

Carré; *Mésure des Surfaces*, &c. p. 76.

(4) To find the centre of gravity of the segment $AEBF$ (fig. 3) of a circle.

The construction and notation remaining the same as in the preceding example, produce CP to cut the chord AB in Q and the arc AEB in R.

Then, if $CQ = r'$,

$$\bar{x} = \frac{\int_{-\alpha}^{+\alpha} \int_{r'}^{a} r^2 \cos\theta \, d\theta \, dr}{\int_{-\alpha}^{+\alpha} \int_{r'}^{a} r \, d\theta \, dr} \quad \text{.....................(1).}$$

Now, since $r' = a\dfrac{\cos\alpha}{\cos\theta}$, we have

$$\int_{r'}^{a} r^2 \cos\theta \, d\theta \, dr = \tfrac{1}{3}(a^3 - r'^3)\cos\theta \, d\theta = \tfrac{1}{3}a^3\left(1 - \frac{\cos^3\alpha}{\cos^3\theta}\right)\cos\theta \, d\theta;$$

hence $$\int_{-\alpha}^{+\alpha} \int_{r'}^{a} r^2 \cos\theta \, d\theta \, dr = \tfrac{1}{3}a^3 \int_{-\alpha}^{+\alpha}\left(\cos\theta - \frac{\cos^3\alpha}{\cos^2\theta}\right)d\theta$$

$$= \tfrac{1}{3}a^3(\sin\theta - \cos^3\alpha\tan\theta), \quad \text{from } \theta = -\alpha \text{ to } \theta = +\alpha,$$

$$= \tfrac{2}{3}a^3(\sin\alpha - \cos^2\alpha\sin\alpha) = \tfrac{2}{3}a^3\sin^3\alpha \quad \text{.....................(2).}$$

Again, $\int_r^a r\,d\theta\,dr = \frac{1}{2}(a^2 - r'^2)\,d\theta = \frac{1}{2}a^2\left(1 - \frac{\cos^2\alpha}{\cos^2\theta}\right)d\theta,$

and therefore

$$\int_{-\alpha}^{+\alpha}\int_{r'}^{a} r\,d\theta\,dr = \tfrac{1}{2}a^2(\theta - \cos^2\alpha\tan\theta),\quad \text{from } \theta = -\alpha \text{ to } \theta = +\alpha,$$

$$= a^2(\alpha - \sin\alpha\cos\alpha) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Hence, from (1), (2), (3), we get

$$\bar{x} = \tfrac{2}{3}a\,\frac{\sin^3\alpha}{\alpha - \sin\alpha\cos\alpha}.$$

This result may be obtained as easily by rectangular co-ordinates; thus, putting $a\cos\alpha = a'$,

$$\bar{x} = \frac{\int_{a'}^{a} xy\,dx}{\int_{a'}^{a} y\,dx} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4);$$

but $\quad y^2 = a^2 - x^2;$

hence $\quad \int_{a'}^{a} xy\,dx = \int_{a'}^{a} x(a^2 - x^2)^{\frac{1}{2}}\,dx$

$$= -\tfrac{1}{3}(a^2 - x^2)^{\frac{3}{2}}, \text{ between limits, } = \tfrac{1}{3}a^3\sin^3\alpha \ldots\ldots\ldots\ldots (5).$$

Again, $\quad \int_{a'}^{a} y\,dx = \int_{a'}^{a}(a^2 - x^2)^{\frac{1}{2}}\,dx$

$$= (a^2 - x^2)^{\frac{1}{2}}x + \int x^2(a^2 - x^2)^{-\frac{1}{2}}\,dx, \text{ between limits,}$$

$$= (a^2 - x^2)^{\frac{1}{2}}x + a^2\int_{a'}^{a}\frac{dx}{(a^2 - x^2)^{\frac{1}{2}}} - \int_{a'}^{a}(a^2 - x^2)^{\frac{1}{2}}\,dx$$

$$= \tfrac{1}{2}(a^2 - x^2)^{\frac{1}{2}}x + \tfrac{1}{2}a^2\sin^{-1}\frac{x}{a}, \text{ from } x = a' \text{ to } x = a,$$

$$= -\tfrac{1}{2}a^2\sin\alpha\cos\alpha + \tfrac{1}{2}a^2\alpha = \tfrac{1}{2}a^2(\alpha - \sin\alpha\cos\alpha) \ldots\ldots (6).$$

Hence from (4), (5), (6), there results

$$\bar{x} = \tfrac{2}{3}a\,\frac{\sin^3\alpha}{\alpha - \sin\alpha\cos\alpha}.$$

In the integrations by polar co-ordinates the segmental a
is conceived to be made up of frustums of an infinite numbe
infinitesimal triangles intercepted by the chord AB, C be
the common vertex of the triangles, and a series of element
the arc AEB being their bases; on the other hand, wh
rectangular co-ordinates are made use of, the segment is c
ceived to be made up of an infinite number of indefinitely th
parallelograms parallel to the chord AB.

Guldin; *Centrobaryca*, Lib. I. cap. 9, p. 107.

(5) To find the centre of gravity of any portion of a ser
cubical parabola comprised between the curve and a doul
ordinate.

The equation to the curve being $ay^2 = x^3$, we shall have

$$\bar{x} = \tfrac{5}{7} x.$$

(6) To find the centre of gravity of the whole area of t
curve of which the equation is

$$y^2 = b^2 \frac{a-x}{x}.$$

$$\bar{x} = \tfrac{1}{4} a.$$

(7) To find the centre of gravity of a semi-ellipse, the bisec
ing line being any diameter.

If the bisecting diameter be taken as the axis of y, an
the conjugate diameter as the axis of x, the equation to th
ellipse will be

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2),$$

and we shall have $\bar{x} = \dfrac{4a}{3\pi}.$

Guldin; *Centrobaryca*, Lib. I. cap. 9, p. 115.

(8) To find the centre of gravity of a loop of the Lemniscat
of James Bernoulli.

The equation to the curve being $r^2 = a^2 \cos 2\theta$, we shall have

$$\bar{x} = \frac{2^{\frac{1}{2}}\pi}{8} a.$$

(9) To find the centre of gravity of the whole area of a cycloid.

The equations to the cycloid being

$$x = a(1 - \cos\theta), \qquad y = a(\theta + \sin\theta),$$

we shall have $$\bar{x} = \tfrac{7}{6}a.$$

SECT. 2. *Area not Symmetrical.*

The formulæ for the determination of the co-ordinates of the centre of gravity of an area, not symmetrical with respect to either of the axes, are

$$\bar{x} = \frac{\iint x\,dx\,dy}{\iint dx\,dy}, \quad \bar{y} = \frac{\iint y\,dx\,dy}{\iint dx\,dy}:$$

x and y in these expressions are the co-ordinates of any point whatever within the area, and the limits of the double integration depend upon the form of the bounding curve.

It frequently happens that the method of polar co-ordinates is more convenient for the determination of $\bar{x}$ and $\bar{y}$ than that of rectangular co-ordinates: the formulæ are

$$\bar{x} = \frac{\iint r^2 \cos\theta\,d\theta\,dr}{\iint r\,d\theta\,dr}, \quad \bar{y} = \frac{\iint r^2 \sin\theta\,d\theta\,dr}{\iint r\,d\theta\,dr}.$$

(1) To find the centre of gravity of the area CPD (fig. 4) of an ellipse, where CP, CD, are two conjugate semi-diameters.

If $CP = a$, $CD = b$, and CP, CD, produced indefinitely, be taken as the axes of x, y, the equation to the ellipse will be

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

and for the position of the centre of gravity we have, indicating the limits of integration,

$$\bar{x} = \frac{\int_0^a \int_0^y x\,dx\,dy}{\int_0^a \int_0^y dx\,dy}, \quad \bar{y} = \frac{\int_0^a \int_0^y y\,dx\,dy}{\int_0^a \int_0^y dx\,dy},$$

the value of y in the limit being pm in the figure; in the integration indicated with respect to y, the figure $pqnm$ is considered as being made up of an infinite number of indefinitely small parallelograms $p'q'$; and, in the integration indicated with respect to x, the whole figure CPD is conceived to be composed of an infinite number of indefinitely thin figures such as $pqnm$.

$$\int_0^a \int_0^y x\,dx\,dy = \int_0^a xy\,dx = \frac{b}{a}\int_0^a x\,(a^2 - x^2)^{\frac{1}{2}}\,dx,$$

since the value of y in the limit coincides with the ordinate in the equation (1); hence

$$\int_0^a \int_0^y x\,dx\,dy = -\tfrac{1}{3}\,\frac{b}{a}\,(a^2 - x^2)^{\frac{3}{2}}, \text{ from } x = 0 \text{ to } x = a,$$

$$= \tfrac{1}{3}\,a^2 b.$$

Again, $$\int_0^a \int_0^y dx\,dy = \int_0^a y\,dx$$

$$= \frac{b}{a}\int_0^a (a^2 - x^2)^{\frac{1}{2}}\,dx = \tfrac{1}{4}\,\frac{b}{a}\,\pi a^2 = \tfrac{1}{4}\pi ab.$$

Hence, by the general formula for $\bar{x}$, we have

$$\bar{x} = \frac{\tfrac{1}{3}a^2 b}{\tfrac{1}{4}\pi ab} = \frac{4a}{3\pi}.$$

Again, $$\int_0^a \int_0^y y\,dx\,dy = \tfrac{1}{2}\int_0^a y^2\,dx$$

$$= \tfrac{1}{2}\,\frac{b^2}{a^2}\int_0^a (a^2 - x^2)\,dx = \tfrac{1}{2}\,\frac{b^2}{a^2}\,(a^3 - \tfrac{1}{3}\,a^3) = \tfrac{1}{3}\,ab^2,$$

and therefore $$\bar{y} = \frac{\tfrac{1}{3}ab^2}{\tfrac{1}{4}\pi ab} = \frac{4b}{3\pi},$$

a result which might have been foreseen from the value of $\bar{x}$.

Instead of the order of the limits which we have chosen, we might equally well have integrated, first with respect to x and

then with respect to y, when the formulæ for $\bar{x}$ and $\bar{y}$ would have been

$$\bar{x} = \frac{\int_0^b \int_0^x x\,dy\,dx}{\int_0^b \int_0^x dy\,dx}, \quad \bar{y} = \frac{\int_0^b \int_0^x y\,dy\,dx}{\int_0^b \int_0^x dy\,dx}.$$

(2) To find the centre of gravity of the segment $APBp$ (fig. 5) of an ellipse cut off by a quadrantal chord ApB.

Let $CA = a$, $CB = b$, $CM = x$, $PM = y$, $pM = y'$; then the equations to the ellipse and to the chord will be

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2), \quad y' = \frac{b}{a}(a - x) \ldots\ldots\ldots\ldots (1).$$

The formula for $\bar{x}$ will be, indicating the limits,

$$\bar{x} = \frac{\int_0^a \int_{y'}^y x\,dx\,dy}{\int_0^a \int_{y'}^y dx\,dy} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Now $$\int_0^a \int_{y'}^y x\,dx\,dy = \int_0^a (y - y')\,x\,dx$$

$$= \frac{b}{a}\int_0^a \{(a^2 - x^2)^{\frac{1}{2}} - (a - x)\}\,x\,dx, \text{ by the equations (1),}$$

$$= \frac{b}{a}\{-\tfrac{1}{3}(a^2 - x^2)^{\frac{3}{2}} - \tfrac{1}{2}ax^2 + \tfrac{1}{3}x^3\}, \text{ from } x = 0 \text{ to } x = a,$$

$$= \tfrac{1}{6}a^2b.$$

Also $$\int_0^a \int_{y'}^y dx\,dy = \int_0^a (y - y')\,dx = \frac{b}{a}\int_0^a \{(a^2 - x^2)^{\frac{1}{2}} - (a - x)\}\,dx$$

$$= \frac{b}{a}\int_0^a (a^2 - x^2)^{\frac{1}{2}}\,dx - \frac{b}{a}(a^2 - \tfrac{1}{2}a^2)$$

$$= \frac{b}{a}(\tfrac{1}{4}\pi a^2 - \tfrac{1}{2}a^2) = \tfrac{1}{4}(\pi - 2)\,ab.$$

Hence, from (2), $$\bar{x} = \tfrac{2}{3}\frac{a}{\pi - 2}.$$

Similarly we should evidently get

$$\bar{y} = \tfrac{2}{3}\frac{b}{\pi - 2}.$$

(3) To find the centre of gravity of the area KSL (fig. 6) of a parabola, of which S is the focus, and SK, SL, any two radii.

Take S as the origin of co-ordinates; also, A being the vertex of the parabola, let ASx be the axis of x, and Sy at right angles to Sx the axis of y. Let $SP=r$, $\angle ASP=\theta$ $AS=m$. Then for the position of the centre of gravity, if $\angle ASK=\alpha$, $\angle ASL=\beta$,

$$\bar{x}=\frac{\int_\alpha^\beta\int_0^r r^2\cos(\pi-\theta)\,d\theta\,dr}{\int_\alpha^\beta\int_0^r r\,d\theta\,dr},\quad \bar{y}=\frac{\int_\alpha^\beta\int_0^r r^2\sin(\pi-\theta)\,d\theta\,dr}{\int_\alpha^\beta\int_0^r r\,d\theta\,dr}.$$

Now $\int_0^r r^2\cos(\pi-\theta)\,d\theta\,dr=\frac{1}{3}r^3\cos(\pi-\theta)\,d\theta=-\frac{1}{3}r^3\cos\theta\,d\theta$;

but, by the nature of the parabola,

$$r=\frac{m}{\cos^2\frac{1}{2}\theta};$$

hence $$\int_0^r r^2\cos(\pi-\theta)\,d\theta\,dr=-\frac{1}{3}m^3\frac{\cos\theta}{\cos^6\frac{1}{2}\theta}d\theta,$$

and therefore $$\int_\alpha^\beta\int_0^r r^2\cos(\pi-\theta)\,d\theta\,dr=-\frac{1}{3}m^3\int_\alpha^\beta\frac{\cos\theta}{\cos^6\frac{1}{2}\theta}d\theta;$$

but $$\frac{\cos\theta}{\cos^6\frac{1}{2}\theta}=\frac{1-\tan^2\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta}\frac{1}{\cos^6\frac{1}{2}\theta}=\frac{1-\tan^2\frac{1}{2}\theta}{\cos^4\frac{1}{2}\theta}=(1-\tan^2\frac{1}{2}\theta)\sec^4\frac{1}{2}\theta;$$

hence

$$\int_\alpha^\beta\int_0^r r^2\cos(\pi-\theta)\,d\theta\,dr=-\frac{1}{3}m^3\int_\alpha^\beta\sec^2\tfrac{1}{2}\theta\,(1-\tan^2\tfrac{1}{2}\theta)\sec^2\tfrac{1}{2}\theta\,d\theta$$

$$=-\frac{1}{3}m^3\int_{\tan\frac{1}{2}\alpha}^{\tan\frac{1}{2}\beta}(1-\tan^4\tfrac{1}{2}\theta)\,2d\tan\tfrac{1}{2}\theta$$

$$=-\frac{2}{3}m^3\{\tan\tfrac{1}{2}\beta-\tan\tfrac{1}{2}\alpha-\tfrac{1}{5}(\tan^5\tfrac{1}{2}\beta-\tan^5\tfrac{1}{2}\alpha)\}.$$

Also $$\int_\alpha^\beta\int_0^r r\,d\theta\,dr=\frac{1}{2}\int_\alpha^\beta r^2\,d\theta=\frac{1}{2}m^2\int_\alpha^\beta\frac{d\theta}{\cos^4\frac{1}{2}\theta}$$

$$=\frac{1}{2}m^2\int_{\tan\frac{1}{2}\alpha}^{\tan\frac{1}{2}\beta}(1+\tan^2\tfrac{1}{2}\theta)\,2d\tan\tfrac{1}{2}\theta$$

$$=m^2\{\tan\tfrac{1}{2}\beta-\tan\tfrac{1}{2}\alpha+\tfrac{1}{3}(\tan^3\tfrac{1}{2}\beta-\tan^3\tfrac{1}{2}\alpha)\}.$$

Hence $\bar{x} = \frac{2m}{3} \cdot \frac{\frac{1}{5}(\tan^5 \frac{1}{2}\beta - \tan^5 \frac{1}{2}\alpha) - (\tan \frac{1}{2}\beta - \tan \frac{1}{2}\alpha)}{\frac{1}{3}(\tan^3 \frac{1}{2}\beta - \tan^3 \frac{1}{2}\alpha) + (\tan \frac{1}{2}\beta - \tan \frac{1}{2}\alpha)}$.

Again, $$\int_\alpha^\beta \int_0^r r^2 \sin(\pi - \theta)\, d\theta\, dr = \frac{1}{3}\int_\alpha^\beta r^3 \sin\theta\, d\theta$$

$$= \frac{1}{3} m^3 \int_\alpha^\beta \frac{\sin\theta}{\cos^6 \frac{1}{2}\theta}\, d\theta = \frac{2}{3} m^3 \int_\alpha^\beta \frac{\sin \frac{1}{2}\theta}{\cos^5 \frac{1}{2}\theta}\, d\theta$$

$$= -\frac{4}{3} m^3 \int_{\cos\frac{1}{2}\alpha}^{\cos\frac{1}{2}\beta} \frac{d\cos\frac{1}{2}\theta}{\cos^5\frac{1}{2}\theta} = \frac{1}{3} m^3 (\sec^4 \tfrac{1}{2}\beta - \sec^4 \tfrac{1}{2}\alpha),$$

and therefore

$$\bar{y} = \frac{1}{3} m \cdot \frac{\sec^4 \frac{1}{2}\beta - \sec^4 \frac{1}{2}\alpha}{\frac{1}{3}(\tan^3 \frac{1}{2}\beta - \tan^3 \frac{1}{2}\alpha) + (\tan \frac{1}{2}\beta - \tan \frac{1}{2}\alpha)}$$

$$= \frac{2m}{3} \cdot \frac{\frac{1}{2}(\tan^4 \frac{1}{2}\beta - \tan^4 \frac{1}{2}\alpha) + (\tan^2 \frac{1}{2}\beta - \tan^2 \frac{1}{2}\alpha)}{\frac{1}{3}(\tan^3 \frac{1}{2}\beta - \tan^3 \frac{1}{2}\alpha) + (\tan \frac{1}{2}\beta - \tan \frac{1}{2}\alpha)}.$$

Let SQ be a radius vector very near to SP; and let pq, $p'q'$, be two circular arcs described about S as a centre, with radii Sp, Sp', very nearly equal to each other. In the integrations which we have executed for the determination of the values of $\bar{x}$ and $\bar{y}$, we have first conceived the indefinitely thin triangle PSQ to be made up of an infinite series of infinitesimal parallelograms $pqp'q'$, and we have then conceived the whole area KSL to be composed of an infinite number of indefinitely thin triangles, such as PSQ: thus the expressions

$$r^2 \cos(\pi - \theta)\, d\theta\, dr, \quad r^2 \sin(\pi - \theta)\, d\theta\, dr,$$

represent the moments of the area $pqp'q'$ about the axes of y and x; the expressions

$$\int_0^r r^2 \cos(\pi - \theta)\, d\theta\, dr, \quad \int_0^r r^2 \sin(\pi - \theta)\, d\theta\, dr,$$

the moments of the area SPQ about the axes of y and x; and the expressions

$$\int_\alpha^\beta \int_0^r r^2 \cos(\pi - \theta)\, d\theta\, dr, \quad \int_\alpha^\beta \int_0^r r^2 \sin(\pi - \theta)\, d\theta\, dr,$$

the moments of the whole area KSL about the axes of y and x.

Also the expressions

$$r\,d\theta\,dr, \quad \int_0^r r\,d\theta\,dr, \quad \int_\alpha^\beta\int_0^r r\,d\theta\,dr,$$

denote respectively the areas $pqp'q'$, PSQ, KSL.

(4) To find the centre of gravity of the area of a quadrant of a circle.

The equation to the circle being

$$x^2 + y^2 = a^2,$$

we shall have $\bar{x} = \dfrac{4a}{3\pi}, \quad \bar{y} = \dfrac{4a}{3\pi}.$

(5) AB (fig. 7) is a parabola, of which the equation is $a^{m-1}y = x^m$; to find the centre of gravity of the area $PMNQ$, comprised between two ordinates.

If $AM = \alpha$, $AN = \alpha'$, we shall have

$$\bar{x} = \frac{m+1}{m+2} \cdot \frac{\alpha'^{m+2} - \alpha^{m+2}}{\alpha'^{m+1} - \alpha^{m+1}}, \quad \bar{y} = \frac{m+1}{2(2m+1)a^{m-1}} \cdot \frac{\alpha'^{2m+1} - \alpha^{2m+1}}{\alpha'^{m+1} - \alpha^{m+1}}.$$

Carré; *Mésure des Surfaces*, &c. p. 80.

(6) Cx, Cy, are asymptotes to an hyperbola EAF, (fig. 8); PM, QN, are parallel to yC; to find the centre of gravity of the area $PMNQ$.

If a, b, be the semiaxes of the hyperbola; Cx, Cy, be taken as the axes of x, y; and CM, CN, be denoted by α, α'; then

$$x = \frac{\alpha' - \alpha}{\log\alpha' - \log\alpha}, \quad \bar{y} = \tfrac{1}{8}\,\frac{a^2 + b^2}{\alpha\alpha'} \cdot \frac{\alpha' - \alpha}{\log\alpha' - \log\alpha}.$$

(7) To find the centre of gravity of the portion of the area of the curve $y = \sin x$, between $x = 0$ and $x = \pi$.

$$\bar{x} = \tfrac{1}{2}\pi, \quad \bar{y} = \tfrac{1}{8}\pi.$$

(8) To find the centre of gravity of the area included between the axes of co-ordinates and the parabola of which the equation is

$$\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1.$$

$$\bar{x} = \tfrac{1}{5}a, \quad \bar{y} = \tfrac{1}{5}b.$$

(9) To find the centre of gravity of the area intercepted between a straight line $y = \beta x$ and a parabola $y^2 = 4mx$.

$$\bar{x} = \frac{8m}{5\beta^2}, \quad \bar{y} = \frac{2m}{\beta}.$$

(10) To find the centre of gravity of a quadrant of the area of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, bounded by the axes of co-ordinates.

Each of the co-ordinates of the centre of gravity is equal to

$$\frac{256}{315} \cdot \frac{a}{\pi}.$$

SECT. 3. *Solid of Revolution.*

Let a solid of revolution be generated by the rotation of a plane curve about the axis of x; then the centre of gravity will be in the axis of x, its position being given by the formula

$$\bar{x} = \frac{\iint xy\,dx\,dy}{\iint y\,dx\,dy} = \frac{\int x\,(y^2 - y'^2)\,dx}{\int (y^2 - y'^2)\,dx},$$

y, y', being the limiting values of y for any assignable value of x; if $y' = 0$, we have

$$\bar{x} = \frac{\int xy^2\,dx}{\int y^2\,dx}.$$

If polar co-ordinates be adopted, which are frequently convenient, the formula will be

$$\bar{x} = \frac{\iint r^3 \sin\theta \cos\theta\,d\theta\,dr}{\iint r^2 \sin\theta\,d\theta\,dr},$$

the pole being taken at the origin of x, and θ being the angle of inclination of the radius vector r to the axis of x.

(1) To find the centre of gravity of the segment of a sphere.

The centre of the generating circle being taken as the origin, its equation will be

$$x^2 + y^2 = a^2 \text{..............................(1)};$$

and, c being the distance of the centre of the plane face of the segment from the origin,

$$\bar{x} = \frac{\int_c^a xy^2\, dx}{\int_c y^2\, dx} \text{ ....(2)};$$

but $$\int_c^a xy^2\, dx = \int_c^a (a^2 - x^2)\, x\, dx, \text{ from (1)},$$

$$= \tfrac{1}{2}\, a^2x^2 - \tfrac{1}{4}\, x^4, \text{ from } x = c \text{ to } x = a,$$

$$= \tfrac{1}{4}\, a^4 - \tfrac{1}{2}\, a^2c^2 + \tfrac{1}{4}\, c^4 = \tfrac{1}{4}\, (a^2 - c^2)^2;$$

also $$\int_c^a y^2\, dx = \int_c^a (a^2 - x^2)\, dx$$

$$= a^2x - \tfrac{1}{3}\, x^3, \text{ from } x = c \text{ to } x = a,$$

$$= \tfrac{2}{3}\, a^3 - a^2c + \tfrac{1}{3}\, c^3:$$

hence from (2),

$$\bar{x} = \tfrac{3}{4}\, \frac{(a^2 - c^2)^2}{2a^3 - 3a^2c + c^3} = \tfrac{3}{4}\, \frac{(a + c)^2}{2a + c}.$$

If the segment become a semi-circle, then $c = 0$, and therefore

$$\bar{x} = \tfrac{3}{8}\, a.$$

Lucas Valerius; *De Centro Gravitatis Solidorum*, Lib. II. Prop. 33, and Lib. III. Prop. 31. Guldin; *Centrobaryca*, Lib. I. cap. 11, p. 130. Wallis; *Opera*, tom. I. p. 728.

(2) To find the centre of gravity of the solid formed by the revolution of the sector of a circle about one of its extreme radii.

Let β denote the angle between the extreme radii of the sector; then, the centre of the circle being the origin of x, and a the radius,

$$\bar{x} = \frac{\int_0^\beta \int_0^a r^3 \sin\theta \cos\theta \, d\theta \, dr}{\int_0^\beta \int_0^a r^2 \sin\theta \, d\theta \, dr} \text{ ....(1);}$$

but $\int_0^\beta \int_0^a r^3 \sin\theta \cos\theta \, d\theta \, dr = \frac{1}{4} a^4 \int_0^\beta \sin\theta \cos\theta \, d\theta$

$$= \frac{1}{8} a^4 \int_0^\beta \sin 2\theta \, d\theta = \frac{1}{16} a^4 (1 - \cos 2\beta),$$

and $\int_0^\beta \int_0^a r^2 \sin\theta \, d\theta \, dr = \frac{1}{3} a^3 \int_0^\beta \sin\theta \, d\theta = \frac{1}{3} a^3 (1 - \cos\beta)$;

hence from (1) we have

$$\bar{x} = \frac{3}{16} a \frac{1 - \cos 2\beta}{1 - \cos\beta} = \frac{3}{8} a (1 + \cos\beta) = \frac{3}{4} a \cos^2 \tfrac{1}{2}\beta.$$

We might equally well have integrated the numerator and denominator of (1), first with respect to θ, and afterwards with respect to r. In the one order of integration, we conceive the sector to be made up of an infinite number of thin triangles, of which the centre of the circle is the common vertex; in the other order, the sector is conceived to be made up of an infinite number of infinitesimal rings, having the centre of the circle as their common centre.

Wallis; *Opera*, Tom. I. p. 728.

(3) To find the centre of gravity of the solid generated by the revolution of the parabolic area ABC (fig. 9), about the tangent Ax at the vertex A, BC being at right angles to the axis Ay of the parabola.

Taking Ax, Ay, as the axes of x, y, the equation to the curve will be

$$x^2 = 4my.$$

Let $AC = a$, $BC = b$; then

$$\bar{x} = \frac{\int_0^b \int_y^a xy \, dx \, dy}{\int_0^b \int_y^a y \, dx \, dy} = \frac{\int_0^b (a^2 - y^2) \, x \, dx}{\int_0^b (a^2 - y^2) \, dx}$$

$$= 2m^{\frac{1}{2}} \frac{\int_0^a (a^2 - y^2)\, dy}{\int_0^a (a^2 y^{-\frac{1}{2}} - y^{\frac{3}{2}})\, dy}, \text{ from (2)},$$

$$= 2m^{\frac{1}{2}} \frac{a^3 - \frac{1}{3}a^3}{2a^{\frac{5}{2}} - \frac{2}{5}a^{\frac{5}{2}}} = \tfrac{5}{6} m^{\frac{1}{2}} a^{\frac{1}{2}} = \tfrac{5}{12} b.$$

This is a case of a more general problem given by Carré, *Mésure des Surfaces*, &c. p. 93.

(4) To find the centre of gravity of the solid formed by the revolution, about the axis of x, of any parabola, of which the equation is

$$y^{m+n} = a^m x^n.$$

For any portion of the solid from $x = 0$ to $x = b$,

$$\bar{x} = \frac{m + 3n}{2m + 4n} b.$$

(5) To find the centre of gravity of the solid generated by the revolution about the axis of x of the curve corresponding to the equation

$$y = (a - x) \left(\frac{x}{a}\right)^2,$$

between the limits $x = 0$ and $x = a$.

$$\bar{x} = \tfrac{5}{8} a.$$

Carré; *Mésure des Surfaces*, &c. p. 99.

(6) To find the centre of gravity of the frustum of a paraboloid.

If a, b, be the radii of the less and of the greater ends, h the length of the frustum, and $\bar{x}$ the distance of the centre of gravity from the smaller end;

$$\bar{x} = \tfrac{1}{3} h \frac{a^2 + 2b^2}{a^2 + b^2}.$$

(7) To find the centre of gravity of an hyperboloid.

If the equation to the generating hyperbola be

$$y^2 = \frac{b^2}{a^2}(x^2 + 2ax),$$

we shall have for the volume between $x=0$ and $x=c$,

$$\bar{x}=\frac{8ac+3c^2}{4(3a+c)}.$$

Carré; *Ib.* p. 97.

(8) ABC (fig. 10) is a portion of the area of a common parabola, where BC is at right angles to the axis Ax of the parabola; to find the centre of gravity of the solid generated by the revolution of the area ABC about BC.

Let $BC=b$; then, G being the centre of gravity,

$$CG=\tfrac{5}{16}b.$$

Carré; *Ib.* p. 90.

(9) AC, BC, (fig. 11), are the semiaxes of an hyperbola, AD being a portion of the curve intercepted by BD drawn parallel to CA; to find the centre of gravity of the solid generated by the revolution of the area $ACBD$ about CB.

If $BC=b$, then, G being the position of the centre of gravity in BC,

$$CG=\tfrac{9}{16}b.$$

Carré; *Ib.* p. 97.

(10) To find the centre of gravity of the solid formed by scooping out a cone from a given paraboloid of revolution, the bases of the two volumes being coincident as well as their vertices.

The centre of gravity bisects the axis.

(11) To find the position of the centre of gravity of the volume included between the surfaces generated by the revolution of two parabolas, $y^2=lx$, $y^2=l'(a-x)$, round the axis of x.

$$\bar{x}=\tfrac{1}{3}a\,\frac{l+2l'}{l+l'}.$$

SECT. 4. *Any Solid.*

Let x, y, z, be the co-ordinates of any point whatever within any assigned solid; let $\bar{x}$, $\bar{y}$, $\bar{z}$, be the co-ordinates of the centre of gravity of this solid; then

$$\bar{x} = \frac{\iiint x\,dx\,dy\,dz}{\iiint dx\,dy\,dz}, \quad \bar{y} = \frac{\iiint y\,dx\,dy\,dz}{\iiint dx\,dy\,dz}, \quad \bar{z} = \frac{\iiint z\,dx\,dy\,dz}{\iiint dx\,dy\,dz},$$

where each of the triple integrations is to be performed in accordance with the nature of the bounding surface of the solid.

It is often more convenient to make use of polar co-ordinates and to divide the solid into polar elements instead of infinitesimal parallelepipeds.

(1) To find the centre of gravity of a portion of the cone, of which the equation is

$$y^2 + z^2 = \beta^2 x^2,$$

which is contained between the planes of zx, xy, and a given plane parallel to that of yz.

Let a be the length of the axis of the portion of the cone: then

$$\bar{x} = \frac{\int_0^a \int_0^{\beta x} \int_0^z x\,dx\,dy\,dz}{\int_0^a \int_0^{\beta x} \int_0^z dx\,dy\,dz}, \quad \bar{y} = \frac{\int_0^a \int_0^{\beta x} \int_0^z y\,dx\,dy\,dz}{\int_0^a \int_0^{\beta x} \int_0^z dx\,dy\,dz},$$

$$\bar{z} = \frac{\int_0^a \int_0^{\beta x} \int_0^z z\,dx\,dy\,dz}{\int_0^a \int_0^{\beta x} \int_0^z dx\,dy\,dz}.$$

Now
$$\int_0^a \int_0^{\beta x} \int_0^z dx\,dy\,dz = \int_0^a \int_0^{\beta x} z\,dx\,dy$$

$$= \int_0^a \int_0^{\beta x} (\beta^2 x^2 - y^2)^{\frac{1}{2}}\,dx\,dy$$

$$= \int_0^a \tfrac{1}{4}\pi\beta^2 x^2\,dx = \tfrac{1}{12}\pi\beta^2 a^3.$$

Also $$\int_0^a \int_0^{\beta x} \int_0^z x\,dx\,dy\,dz = \int_0^a \int_0^{\beta x} xz\,dx\,dy$$

$$= \int_0^a \int_0^{\beta x} x(\beta^2x^2 - y^2)^{\frac{1}{2}}\,dx\,dy = \int_0^a x \,.\, \tfrac{1}{4}\pi\beta^2x^2\,dx$$

$$= \tfrac{1}{16}\pi\beta^2a^4;$$

hence $$\bar{x} = \frac{\frac{1}{16}\pi\beta^2a^4}{\frac{1}{12}\pi\beta^2a^3} = \tfrac{3}{4}a.$$

Again, $$\int_0^a \int_0^{\beta x} \int_0^z y\,dx\,dy\,dz = \int_0^a \int_0^{\beta x} yz\,dx\,dy$$

$$= \int_0^a \int_0^{\beta x} (\beta^2x^2 - y^2)^{\frac{1}{2}}\,y\,dx\,dy$$

$$= \int_0^a \tfrac{1}{3}\beta^3x^3\,dx = \tfrac{1}{12}\beta^3a^4,$$

and therefore $$\bar{y} = \frac{\frac{1}{12}\beta^3a^4}{\frac{1}{12}\pi\beta^2a^3} = \frac{\beta}{\pi}a.$$

Similarly, $$\bar{z} = \frac{\beta}{\pi}a.$$

(2) To find the centre of gravity of half the solid intercepted between the surfaces of a hemisphere, and a paraboloid of revolution on the same base, the latus-rectum of the paraboloid coinciding with the diameter of the hemisphere, and the solid being bisected by a plane passing through its axis.

Take the centre of the sphere as the origin of co-ordinates, and the axis of the paraboloid as the axis of z; also let the axis of x be so taken that the plane of zx coincides with the bisecting plane; and take the axis of y at right angles to this plane. Then, if a be the radius of the sphere, the equation to the sphere will be

$$x^2 + y^2 + z^2 = a^2,$$

and to the paraboloid,

$$x^2 + y^2 = a(a - 2z).$$

The centre of gravity will be somewhere in the plane of yz, and is to be determined by the formulæ

$$\bar{y} = \frac{\int_{-a}^{+a}\int_0^{x'}\int_{z'}^{z} y\,dx\,dy\,dz}{\int_{-a}^{+a}\int_0^{x'}\int_{z'}^{z} dx\,dy\,dz}, \quad \bar{z} = \frac{\int_{-a}^{+a}\int_0^{x'}\int_{z'}^{z} z\,dx\,dy\,dz}{\int_{-a}^{+a}\int_0^{x'}\int_{z'}^{z} dx\,dy\,dz};$$

where x' is taken to represent $(a^2 - x^2)^{\frac{1}{2}}$, and where the limits z', z, of the general value of z, are its values for any assigned values of x and y in the paraboloid and sphere respectively.

Now

$$\int_{z'}^{z} dx\,dy\,dz = dx\,dy\,(z - z') = dx\,dy\,\{(a^2 - x^2 - y^2)^{\frac{1}{2}} - \frac{1}{2a}(a^2 - x^2 - y^2)\};$$

hence $\int_0^{x'}\int_{z'}^{z} dx\,dy\,dz = \int_0^{x'} dx\,dy\,\{(a^2 - x^2 - y^2)^{\frac{1}{2}} - \frac{1}{2a}(a^2 - x^2 - y^2)\}$

$$= dx\,\{\tfrac{1}{4}\pi(a^2 - x^2) - \frac{1}{3a}(a^2 - x^2)^{\frac{3}{2}}\},$$

and therefore

$$\int_{-a}^{+a}\int_0^{x'}\int_{z'}^{z} dx\,dy\,dz = \int_{-a}^{+a} dx\,\{\tfrac{1}{4}\pi(a^2 - x^2) - \frac{1}{3a}(a^2 - x^2)^{\frac{3}{2}}\}$$

$$= \tfrac{1}{3}\pi a^3 - \frac{1}{3a}\int_{-a}^{+a}(a^2 - x^2)^{\frac{3}{2}}\,dx = \tfrac{1}{3}\pi a^3 - \tfrac{1}{8}\pi a^3 = \tfrac{5}{24}\pi a^3.$$

Again, $\int_{z'}^{z} y\,dx\,dy\,dz = y\,dx\,dy\,\{(a^2 - x^2 - y^2)^{\frac{1}{2}} - \frac{1}{2a}(a^2 - x^2 - y^2)\}$,

$$\int_0^{x'}\int_{z'}^{z} y\,dx\,dy\,dz = \int_0^{x'} y\,dx\,dy\,\{(a^2 - x^2 - y^2)^{\frac{1}{2}} - \frac{1}{2a}(a^2 - x^2 - y^2)\}$$

$= dx\,\{-\tfrac{1}{3}(a^2 - x^2 - y^2)^{\frac{3}{2}} - \frac{1}{4a}(a^2 - x^2)\,y^2 + \frac{1}{8a}y^4\}$, between limits,

$$= dx\,\{\tfrac{1}{3}(a^2 - x^2)^{\frac{3}{2}} - \frac{1}{8a}(a^2 - x^2)^2\};$$

hence

$$\int_{-a}^{+a}\int_0^{x'}\int_{z'}^{z} y\,dx\,dy\,dz = \tfrac{1}{3}\int_{-a}^{+a}(a^2 - x^2)^{\frac{3}{2}}\,dx - \frac{1}{8a}\int_{-a}^{+a}(a^4 - 2a^2x^2 + x^4)\,dx$$

$$= \tfrac{1}{8}\pi a^4 - \tfrac{2}{15}a^4 = \frac{15\pi - 16}{120}a^4.$$

Again

$$\int_{z'}^{z} z\,dx\,dy\,dz=\tfrac{1}{2}\,dx\,dy\,(z^2-z'^2)=\tfrac{1}{2}\,dx\,dy\,\{(a^2-x^2-y^2)-\frac{1}{4a^2}(a^2-x^2-y^2)^2\},$$

$$\int_0^{x'}\int_{z'}^{z} z\,dx\,dy\,dz$$

$$=\tfrac{1}{2}\int_0^{x'} dx\,dy\,\{a^2-x^2-y^2-\frac{1}{4a^2}(a^2-x^2)^2+\frac{1}{2a^2}(a^2-x^2)\,y^2-\frac{1}{4a^2}y^4\}$$

$$=\tfrac{1}{2}\,dx\{(a^2-x^2)^{\frac{3}{2}}-\tfrac{1}{3}(a^2-x^2)^{\frac{3}{2}}-\frac{1}{4a^2}(a^2-x^2)^{\frac{5}{2}}+\frac{1}{6a^2}(a^2-x^2)^{\frac{5}{2}}-\frac{1}{20a^2}(a^2-x^2)^{\frac{5}{2}}\}$$

between limits,

$$=\tfrac{1}{3}\,dx\,\{(a^2-x^2)^{\frac{3}{2}}-\frac{1}{5a^2}(a^2-x^2)^{\frac{5}{2}}\},$$

$$\int_{-a}^{+a}\int_0^{x'}\int_{z'}^{z} z\,dx\,dy\,dz=\tfrac{1}{3}\int_{-a}^{+a}dx\,\{(a^2-x^2)^{\frac{3}{2}}-\frac{1}{5a^2}(a^2-x^2)^{\frac{5}{2}}\}$$

$$=\tfrac{1}{3}\int_{-a}^{+a}dx\,(a^2-x^2)^{\frac{3}{2}}-\frac{1}{15a^2}\int_{-a}^{+a}(a^2-x^2)^{\frac{5}{2}}\,dx$$

$$=\tfrac{1}{8}\pi a^4-\tfrac{1}{48}\pi a^4=\tfrac{5}{48}\pi a^4.$$

From the formulæ for $\bar{y}$ and $\bar{z}$ then we have

$$\bar{y}=\frac{\frac{1}{120}(15\pi-16)\,a^4}{\frac{5}{24}\pi a^3}=\frac{15\pi-16}{25\pi}\,a,$$

$$\bar{z}=\frac{\frac{5}{48}\pi a^4}{\frac{5}{24}\pi a^3}=\tfrac{1}{2}a.$$

(3) AOC (fig. 12) is a right-angled triangle, O being the right angle; $AOBD$ is a rectangle, of which the plane is perpendicular to that of the triangle; from every point R in the line AC a straight line RQ is drawn to meet BD at Q, in a plane at right angles to the areas of the rectangle and triangle; to find the centre of gravity of the volume so generated.

Let OAx, OBy, OCz, be taken as axes of x, y, z; from R draw RM at right angles to OA, join QM, and draw PN at right angles to QM; let $OA=a$, $OB=b$, $OC=c$; $OM=x$, $MN=y$, $NP=z'$, z being the distance of any point in the line PN from the point N: then for the determination of the centre of gravity we have

$$\bar{x} = \frac{\int_0^a \int_0^b \int_0^{z'} x\,dx\,dy\,dz}{\int_0^a \int_0^b \int_0^{z'} dx\,dy\,dz}, \quad \bar{y} = \frac{\int_0^a \int_0^b \int_0^{z'} y\,dx\,dy\,dz}{\int_0^a \int_0^b \int_0^{z'} dx\,dy\,dz},$$

$$\bar{z} = \frac{\int_0^a \int_0^b \int_0^{z'} z\,dx\,dy\,dz}{\int_0^a \int_0^b \int_0^{z'} dx\,dy\,dz}.$$

From the geometry it is evident that

$$z' = c\,\frac{a-x}{a}\,\frac{b-y}{b};$$

hence we have

$$\bar{x} = \frac{\int_0^a \int_0^b x\,(a-x)\,(b-y)\,dx\,dy}{\int_0^a \int_0^b (a-x)\,(b-y)\,dx\,dy}$$

$$= \frac{\int_0^a x\,(a-x)\,dx}{\int_0^a (a-x)\,dx} = \frac{\frac{1}{6}a^3}{\frac{1}{2}a^2} = \tfrac{1}{3}a,$$

$$\bar{y} = \frac{\int_0^a \int_0^b (a-x)\,(b-y)\,y\,dx\,dy}{\int_0^a \int_0^b (a-x)\,(b-y)\,dx\,dy} = \frac{\int_0^b (b-y)\,y\,dy}{\int_0^b (b-y)\,dy} = \frac{\frac{1}{6}b^3}{\frac{1}{2}b^2} = \tfrac{1}{3}b,$$

$$\bar{z} = \frac{\int_0^a \int_0^b \frac{1}{2}c^2\,\frac{(a-x)^2}{a^2}\,\frac{(b-y)^2}{b^2}\,dx\,dy}{\int_0^a \int_0^b c\,\frac{a-x}{a}\,\frac{b-y}{b}\,dx\,dy} = \frac{c}{2ab}\,\frac{\int_0^a \int_0^b (a-x)^2\,(b-y)^2\,dx\,dy}{\int_0^a \int_0^b (a-x)\,(b-y)\,dx\,dy}$$

$$= \frac{c}{2ab}\,\frac{\frac{1}{3}a^3\,.\,\frac{1}{3}b^3}{\frac{1}{2}a^2\,.\,\frac{1}{2}b^2} = \frac{2c}{9}.$$

(4) To find the centre of gravity of the portion of the sphere

$$x^2 + y^2 + z^2 = a^2,$$

which is cut off by three planes, $x = 0$, $dy = 0$, $z = 0$.

$$\bar{x} = \bar{y} = \bar{z} = \tfrac{3}{8}a.$$

(5) To find the centre of gravity of a portion of the paraboloid

$$y^2 + z^2 = 4mx,$$

which is cut off by the three planes $x = a$, $y = 0$, $z = 0$.

If b be the radius of the section of the paraboloid made by the plane $x = a$, then

$$\bar{x} = \tfrac{2}{3}a, \quad \bar{y} = \bar{z} = \frac{16b}{15\pi}.$$

(6) To find the centre of gravity of a portion of the solid $z^2 = xy$, which is cut off by the five planes $x = 0$, $y = 0$, $z = 0$, $x = a$, $y = b$.

$$\bar{x} = \tfrac{3}{5}a, \quad \bar{y} = \tfrac{3}{5}b, \quad \bar{z} = \tfrac{9}{32}a^{\frac{1}{2}}b^{\frac{1}{2}}.$$

(7) To find the centre of gravity of the volume of the cylinder $y^2 = 2ax - x^2$, which is cut off between the two planes $z = \beta x$, $z = \beta' x$.

$$\bar{x} = \tfrac{5}{4}a, \quad \bar{y} = 0, \quad \bar{z} = \tfrac{5}{8}(\beta + \beta')\, a.$$

(8) A solid is generated by a variable rectangle moving parallel to itself along an axis perpendicular to its plane through its centre; one side of the rectangle varies as the distance from a fixed point in the axis, while half the other is the sine of a circular arc, of which this distance is the versed sine; to determine the distance of the centre of gravity of the whole solid from the fixed point.

The required distance is equal to five-eighths of the length of the axis.

(9) Through a given point in the circumference of a circle, and at right angles to its plane, is drawn a straight line, equal in length to the diameter of the circle, the given point bisecting the line: to find the centre of gravity of a solid bounded by a surface, which is the locus of a semi-ellipse terminating at the ends of the straight line, which is the major axis of the ellipse, the minor semi-axis being a chord drawn in the circle from the given point.

If c be the circumference of the circle, the distance of the centre of gravity of the solid from the given point is equal to

$$\frac{9c}{128}.$$

SECT. 5. *A Plane Curve.*

Let x, y, be the co-ordinates of any point of a plane curve, and let ds denote an element of the length of the curve at that point; then, $\bar{x}$, $\bar{y}$, denoting the co-ordinates of the centre of gravity of any assigned portion of the curve,

$$\bar{x} = \frac{\int x\,ds}{\int ds}, \quad \bar{y} = \frac{\int y\,ds}{\int ds},$$

the integrations being performed in accordance with the limits of the portion.

The idea of the determination of the centres of gravity of curve lines is due to La-Faille, a Flemish mathematician, by whom it was applied in the instances of portions of the circle and the ellipse, in a work entitled "*De centro gravitatis partium circuli et ellipsis theoremata,*" published in the year 1632. The theorems of La-Faille were afterwards published in a somewhat more elegant form, and with amplifications, by Guldin; *Centrobaryca*, Lib. I. caps. 4, 5, 6, 7.

(1) To find the centre of gravity of the arc of the curve $y = \sin x$ from $x = 0$ to $x = \pi$.

From the equation to the curve we have

$$ds^2 = dx^2 + dy^2 = (1 + \cos^2 x)\,dx^2;$$

hence
$$\bar{y} = \frac{\int y\,ds}{\int ds} = \frac{\int_0^\pi \sin x\,(1 + \cos^2 x)^{\frac{1}{2}}\,dx}{\int_0^\pi (1 + \cos^2 x)^{\frac{1}{2}}\,dx}.$$

Now, by the ordinary processes of the integral calculus,

$$\int_0^\pi \sin x\,(1 + \cos^2 x)^{\frac{1}{2}}\,dx = 2^{\frac{1}{2}} + \log(2^{\frac{1}{2}} + 1);$$

also, c denoting the length of the curve from $x=0$ to $x=\pi$,

$$c=\int_0^{\pi}(1+\cos^2 x)^{\frac{1}{2}}\,dx=2^{\frac{1}{2}}\int_0^{\pi}(1-\tfrac{1}{2}\sin^2 x)^{\frac{1}{2}}\,dx,$$

an elliptic function of the second order: hence $\bar{y}$ is given by the equation

$$c\bar{y}=2^{\frac{1}{2}}+\log(2^{\frac{1}{2}}+1).$$

(2) To find the centre of gravity of any arc of a circle.

Let the centre of the circle be taken as the origin of co-ordinates, and let the axis of x bisect the arc; then, if a be the radius of the circle, c the chord of the arc, and s the length of the arc,

$$\bar{x}=\frac{ac}{s},\quad \bar{y}=0.$$

Guldin; *Centrobaryca*, Lib. I. cap. 5, p. 59.
Wallis; *Opera*, Tom. I. p. 712.

(3) To find the centre of gravity of the arc, between two successive cusps, of the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$.

Each of the co-ordinates of the centre of gravity is equal to $\frac{2}{5}a$.

(4) To find the centre of gravity of the arc of a semicycloid.

The equations to the curve being

$$x=a(1-\cos\theta),\quad y=a(\theta+\sin\theta),$$

we shall have

$$\bar{x}=\tfrac{4}{3}a,\quad \bar{y}=(\pi-\tfrac{4}{3})\,a.$$

Wallis; *Opera*, Tom. I. p. 520.

(5) To find the centre of gravity of the arc of a catenary

$$y=\tfrac{1}{2}a\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right),$$

cut off by any assigned double ordinate.

If $2s$ be the whole length of the intercepted arc,

$$\bar{x}=0,\quad \bar{y}=\frac{ax+sy}{2s}.$$

(6) To find the centre of gravity of the arc of a parabola $y^2 = 4mx$, cut off by the latus rectum.

$$\bar{x} = \tfrac{1}{4} m \frac{3 \cdot 2^{\frac{1}{2}} - \log (1 + 2^{\frac{1}{2}})}{2^{\frac{1}{2}} + \log (1 + 2^{\frac{1}{2}})}, \quad \bar{y} = 0.$$

(7) To find the centre of gravity of the semi-arc of a loop of the Lemniscata of James Bernoulli.

If the axis of the loop be taken as the axis of x, the node being the origin; then, a being the length of the axis and l of the semi-arc,

$$\bar{x} = \frac{a^2}{2^{\frac{1}{2}} l}, \quad \bar{y} = \frac{(2^{\frac{1}{2}} - 1) a^2}{2^{\frac{1}{2}} l}.$$

(8) AP is any portion of a plane curvilinear wire, reckoned from a fixed point A in the wire: O is a fixed point in the plane of the wire: the centre of gravity of AP always lies in the straight line which bisects the angle AOP: to find the form of the wire.

The arc AP is a portion of a lemniscate of which O is the pole.

SECT. 6. *Curve of Double Curvature.*

The formulæ for the determination of the centre of gravity of a curve of double curvature, are

$$\bar{x} = \frac{\int x\, ds}{\int ds}, \quad \bar{y} = \frac{\int y\, ds}{\int ds}, \quad \bar{z} = \frac{\int z\, ds}{\int ds};$$

where x, y, z, are the co-ordinates of any point in the curve, and ds an element of the arc at that point: the limits of the integrations will depend upon the positions of the ends of that portion of the curve of which the centre of gravity is required.

Ex. To find the centre of gravity of the Helix.

The equations to the curve are

$$x^2 + y^2 = a^2, \quad z = b \cos^{-1} \frac{x}{a};$$

and for the centre of gravity of any length of the curve, beginning at the plane of xy,

$$\bar{x} = \frac{by}{z}, \quad \bar{y} = \frac{b(a-x)}{z}, \quad \bar{z} = \tfrac{1}{2}z.$$

SECT. 7. *Surface of Revolution.*

Let x, y, be the co-ordinates of any point of a curve, by the revolution of which about the axis of x a surface is supposed to be generated; then, if ds denote an element of the generating curve at the point, and $\bar{x}$ the abscissa of the centre of gravity of the surface of revolution,

$$\bar{x} = \frac{\int xy\,ds}{\int y\,ds};$$

the integrations being performed between appropriate limits.

(1) To find the centre of gravity of the surface of a segment of a sphere.

If the equation to the generating circle be

$$y = (2ax - x^2)^{\frac{1}{2}},$$

we shall have $$dy = \frac{a-x}{(2ax-x^2)^{\frac{1}{2}}}\,dx,$$

and therefore

$$ds^2 = dx^2 + dy^2 = \frac{a^2}{2ax - x^2}\,dx^2 = \frac{a^2 dx^2}{y^2}, \text{ or } y\,ds = a\,dx;$$

hence for any segment, of which the limiting abscissa is c,

$$\bar{x} = \frac{\int_0^c ax\,dx}{\int_0^c a\,dx} = \frac{\frac{1}{2}c^2}{c} = \frac{1}{2}c.$$

(2) To find the centre of gravity of the surface generated by the revolution of one of the loops of the curve $r^2 = a^2 \cos 2\theta$ about its axis.

$$\bar{x} = \frac{\int xy\,ds}{\int y\,ds} = \frac{\int_0^{\frac{1}{4}\pi} r^2 \sin\theta\cos\theta\,(dr^2 + r^2\,d\theta^2)^{\frac{1}{2}}}{\int_0^{\frac{1}{4}\pi} r\sin\theta\,(dr^2 + r^2\,d\theta^2)^{\frac{1}{2}}}.$$

But $\qquad r = a(\cos 2\theta)^{\frac{1}{2}}, \qquad dr = -a\dfrac{\sin 2\theta}{(\cos 2\theta)^{\frac{1}{2}}}\,d\theta,$

and therefore $\qquad dr^2 + r^2\,d\theta^2 = \dfrac{a^2\,d\theta^2}{\cos 2\theta}.$

Hence
$$\bar{x} = a\cdot\frac{\int_0^{\frac{1}{4}\pi} \sin\theta\cos\theta\,(\cos 2\theta)^{\frac{1}{2}}\,d\theta}{\int_0^{\frac{1}{4}\pi}\sin\theta\,d\theta}$$

$$= \frac{1}{4}a\frac{\int (\cos 2\theta)^{\frac{1}{2}}\,d\cos 2\theta}{\int d\cos\theta}$$

$$= \frac{1}{4}a\frac{\left\{\frac{2}{3}(\cos 2\theta)^{\frac{3}{2}}\right\}_0^{\frac{1}{4}\pi}}{\left\{\cos\theta\right\}_0^{\frac{1}{4}\pi}} = \frac{1}{4}a\frac{-\frac{2}{3}}{\frac{1}{\sqrt{2}} - 1} = \frac{a}{6}\cdot\frac{\sqrt{2}}{\sqrt{(2)} - 1}.$$

(3) To find the centre of gravity of the surface of a cone.

Let the equation to the generating straight line be $y = ax$; then, c being the length of the axis of the cone,

$$\bar{x} = \frac{2}{3}c.$$

Guldin; *Centrobaryca*, Lib. I. cap. 10, prop. 3.

(4) To find the centre of gravity of the surface generated by the revolution of a semicycloid about its axis.

The equations to the curve being

$$x = a(1 - \cos\theta), \qquad y = a(\theta + \sin\theta),$$

we shall have

$$\bar{x} = \tfrac{2}{15} a \, \frac{15\pi - 8}{3\pi - 4}.$$

(5) To find the centre of gravity of the surface generated by the revolution of the parabola $y^2 = 4mx$ about the axis of x.

$$\bar{x} = \tfrac{1}{5} \, \frac{(3x - 2m)(x + m)^{\frac{3}{2}} + 2m^{\frac{5}{2}}}{(x + m)^{\frac{3}{2}} - m^{\frac{3}{2}}}.$$

(6) A cardioid revolves about its axis: to find the centre of gravity of the surface generated.

The equation to the cardioid being $r = a(1 - \cos\theta)$, the distance of the required centre of gravity from the pole is equal to $\frac{50}{63}a$.

SECT. 8. *Any Surface.*

Let x, y, z, be the co-ordinates of any point of a surface referred to three rectangular axes; and let $\frac{dz}{dx} = p$, $\frac{dz}{dy} = q$; then, for the centre of gravity of any portion of the surface,

$$\bar{x} = \frac{\iint x (1 + p^2 + q^2)^{\frac{1}{2}} \, dx \, dy}{\iint (1 + p^2 + q^2)^{\frac{1}{2}} \, dx \, dy},$$

$$\bar{y} = \frac{\iint y (1 + p^2 + q^2)^{\frac{1}{2}} \, dx \, dy}{\iint (1 + p^2 + q^2)^{\frac{1}{2}} \, dx \, dy},$$

$$\bar{z} = \frac{\iint z (1 + p^2 + q^2)^{\frac{1}{2}} \, dx \, dy}{\iint (1 + p^2 + q^2)^{\frac{1}{2}} \, dx \, dy},$$

the integrations being performed between limits corresponding to the boundary of the surface.

(1) Suppose the surface to be any portion of the superficies of a sphere, of which the equation is

$$x^2 + y^2 + z^2 = r^2.$$

Then we have

$$p = -\frac{x}{z}, \quad q = -\frac{y}{z};$$

and therefore $$\bar{z} = r\,\frac{\iint dx\,dy}{\iint dx\,dy\,(1+p^2+q^2)^{\frac{1}{2}}}.$$

Now it is evident that, the integrations being performed within the given limits, the denominator of this expression for $\bar{z}$ represents the area of the given portion of the surface, while the numerator represents the area of the projection of this same portion upon the plane of x and y. Hence, in general language, the distance of the centre of gravity of any portion whatever of the surface of a sphere from the plane of any one of its great circles, is a fourth proportional to the area of the portion itself, the area of its projection on this plane, and the radius of the sphere.

The truth of this proposition depends solely upon the property expressed by the equation

$$z\,(1+p^2+q^2)^{\frac{1}{2}} = r;$$

but this equation holds good for the whole class of surfaces generated by the motion of a sphere of invariable radius, of which the centre describes a plane curve traced arbitrarily in the plane of x and y; hence we may evidently extend the preceding proposition to all these surfaces under the following enunciation:—

"Upon any surface whatever, generated by the motion of a sphere of which the centre never departs from a given plane, let any portion S be taken, and let S' be the projection of S upon the given plane; then the distance of the centre of gravity

of S from this plane will be a fourth proportional to S, S', and the radius of the generating sphere."

(2) To find the centre of gravity of any spherical triangle formed by three great circles.

Let ABC (fig. 13) be any spherical triangle, O the centre of the sphere; and OA, OB, OC, the three radii at the angles A, B, C, of the triangle. Let Z_a, Z_b, Z_c, denote the distances of the centre of gravity of this triangle from the three planes BOC, COA, AOB; then, by the proposition of the preceding Article, if r be the radius of the sphere, S the area of the spherical triangle ABC, and S' of its projection upon the plane BOC,

$$Z_a = \frac{S'}{S} r.$$

But, by the principles of spherical trigonometry,

$$S = \frac{\pi r^2}{180}(A + B + C - 180);$$

also it is clear that, a, b, c, being the number of degrees subtended at the centre of the sphere by the sides of the spherical triangle opposite to the angles A, B, C,

$$S' = \text{area } BOC - \text{area } AOB \times \cos B - \text{area } AOC \times \cos C$$

$$= \frac{\pi r^2}{360}(a - c\cos B - b\cos C),$$

and therefore $$Z_a = \tfrac{1}{2} r \frac{a - b\cos C - c\cos B}{A + B + C - 180}.$$

Similarly, $$Z_b = \tfrac{1}{2} r \frac{b - c\cos A - a\cos C}{A + B + C - 180}.$$

$$Z_c = \tfrac{1}{2} r \frac{c - a\cos B - b\cos A}{A + B + C - 180}.$$

The position of the centre of gravity of the spherical triangle may be elegantly expressed likewise in terms of its distances from three great circles of the sphere, at right angles to the

three edges OA, OB, OC, of the spherical pyramid $ABCO$. Let D_a, D_b, D_c, denote these distances; then by Art. (1) we have

$$D_a = r\frac{S_a}{S}, \quad D_b = r\frac{S_b}{S}, \quad D_c = r\frac{S_c}{S},$$

where S denotes the spherical area ABC, and S_a, S_b, S_c, its projections upon the three great circles at right angles to OA, OB, OC.

Now it is evident that the projections of the spherical area ABC, and of the sector BOC, upon the great circle which is at right angles to OA, are identically the same, and therefore, if the arc Aa be drawn at right angles to BC, we have

$$S_a = \text{area of sector } BOC \times \cos\left(\frac{\pi}{2} - \frac{Aa}{r}\right)$$

$$= \frac{\pi}{360} ar^2 \sin\frac{Aa}{r} = \frac{\pi}{360} ar^2 \sin B . \sin c:$$

but $$S = \frac{\pi r^2}{180}(A + B + C - 180);$$

hence $$D_a = \tfrac{1}{2} r \frac{a \sin B \sin c}{A + B + C - 180}.$$

Similarly, $$D_b = \tfrac{1}{2} r \frac{b \sin C \sin a}{A + B + C - 180},$$

$$D_c = \tfrac{1}{2} r \frac{c \sin A \sin b}{A + B + C - 180}.$$

If we desire to determine the position of the centre of gravity of the triangle by means of three rectangular co-ordinates x, y, z, let the plane of the side c be taken as the plane of x and y, and let the radius OA be taken to coincide with the axis of x. Then from the preceding results we have at once

$$x = \tfrac{1}{2} r \frac{a \sin B \sin c}{A + B + C - 180}, \quad z = \tfrac{1}{2} r \frac{c - b \cos A - a \cos B}{A + B + C - 180}.$$

Again, let the great circle, of which BC is an arc, meet the plane of x, z, in the point D, as in fig. 14; join A and D by an arc of a great circle. Then clearly the projection of the spherical triangle ABC upon the plane of x and z is equal to the

difference of the projections of the sectors AOC, BOC, upon this plane, and therefore to the expression

$$\frac{\pi}{360} r^2 b \cos CAD - \frac{\pi}{360} r^2 a \cos D$$
$$= \frac{\pi r^2}{360} (b \sin A - a \sin B \cos c);$$

hence, by the principles of Art. (1), we have

$$y = \tfrac{1}{2} \frac{b \sin A - a \sin B \cos c}{A + B + C - 180}.$$

(3) The general formula

$$\bar{z} = \frac{\iint z (1 + p^2 + q^2)^{\frac{1}{2}} \, dx \, dy}{\iint (1 + p^2 + q^2)^{\frac{1}{2}} \, dx \, dy},$$

furnishes us with the following general proposition :—

"Upon the surface (A), generated by the revolution of the curve of equilibrium of a homogeneous catenary about the vertical line which passes through its lowest point, trace arbitrarily a perimeter enclosing a portion S of the surface; project this perimeter upon a horizontal plane which intersects the axis of revolution at a distance c below the lowest point of the surface, where c is equal to the horizontal tension of the catenary divided by the mass of a unit of its length; let V be the volume contained between the surface S, its projection, and the cylindrical surface formed by the perpendiculars from the perimeter of S upon the plane of projection. Then the altitude of the centre of gravity of S above this plane will be double of that of the centre of gravity of V."

In fact, the plane touching the surface (A) in a point situated at an altitude z above the plane of projection, which we shall take for the plane of x and y, makes with this plane an angle, of which the cosine is $\frac{c}{z}$; and therefore we have the equation

$$(1 + p^2 + q^2)^{\frac{1}{2}} = \frac{z}{c};$$

hence, by the formula for $\bar{z}$, we obtain

$$\bar{z} = \frac{\iint z^2\, dx\, dy}{\iint z\, dx\, dy}.$$

But calling $\underline{z}$ the altitude of the centre of gravity of V above the same plane, we have

$$\underline{z} = \frac{\iint \frac{1}{2} zz\, dx\, dy}{\iint z\, dx\, dy},$$

and, the limits of the integrations being the same in the expressions for $\bar{z}$ and $\underline{z}$, we see that $\bar{z} = 2\underline{z}$.

The property expressed by the partial differential equation

$$(1 + p^2 + q^2)^{\frac{1}{2}} = \frac{z}{c},$$

being common to all the surfaces which can be generated by the surface (A) when it moves in such a manner that its axis always remains vertical, and that one of its points describes a plane curve traced arbitrarily upon a horizontal plane, the proposition which we have demonstrated is susceptible of the same extension as that of (1).

The illustrations of the general formulæ for the determination of the centre of gravity of any surface, which we have given in (1), (2), (3), are extracted from a memoir by Professor Giulio, of Turin, which may be seen in Liouville's *Journal de Mathématiques*, Tom. IV. p. 386.

(4) To find the centre of gravity of the surface of a cone

$$y^2 + z^2 = \beta^2 x^2,$$

intercepted by the planes $x = 0$, $y = 0$, $z = 0$, $x = a$.

$$\bar{x} = \tfrac{2}{3}a, \quad \bar{y} = \bar{z} = \tfrac{4}{3}\,\frac{\beta a}{\pi}.$$

SECT. 9. *Heterogeneous Bodies.*

In the preceding sections we have determined the centres of gravity of various classes of homogeneous bodies; we will now give a few examples of the determination of the centre of gravity when the density is variable.

(1) To find the centre of gravity of the surface of a hemisphere, when the density of each point in the surface varies as its perpendicular distance from the circular base of the hemisphere.

Let the equation to the quadrantal arc, by the revolution of which the hemispherical surface may be generated, be

$$x^2 + y^2 = a^2 \quad \text{.......................} \quad (1),$$

the axis of x being the axis of revolution.

The area of the strip of the surface, which is generated by the element ds of the arc, will be equal to $2\pi y ds$; and, if ρ be its density, its mass will be equal to $2\pi\rho\, y\, ds$: hence

$$\bar{x} = \frac{\int 2\pi\rho\, y\, ds \,.\, x}{\int 2\pi\rho\, y\, ds} = \frac{\int \rho\, xy\, ds}{\int \rho\, y\, ds};$$

but $\rho \propto x$; hence $$\bar{x} = \frac{\int x^2 y\, ds}{\int xy\, ds};$$

and therefore, since from (1) it is easily seen that

$$y\, ds = a\, dx,$$

we have $$\bar{x} = \frac{\int_0^a x^2\, dx}{\int_0^a x\, dx} = \frac{\frac{1}{3}a^3}{\frac{1}{2}a^2} = \tfrac{2}{3}a.$$

(2) To find the centre of gravity of a physical line, the density of which at any point varies as the n^{th} power of the distance of the point from a given point in the line produced.

Let a, b, be the distances of the given point from the two extremities, and $\bar{x}$ its distance from the centre of gravity of the physical line; then

$$\bar{x} = \frac{n+1}{n+2} \cdot \frac{b^{n+2} - a^{n+2}}{b^{n+1} - a^{n+1}}.$$

(3) To find the centre of gravity of a chain the form of which is that of a common catenary, the density of the chain varying as the curvature, and its extremities being equidistant from its directrix.

If a be the distance of either end of the chain from its axis, and α the inclination of the tangent at either end to the directrix, the centre of gravity lies in the axis of the catenary at a distance from the directrix equal to $\frac{a}{\alpha}$.

Haton de la Goupillière; *Nouvelles Annales de Mathématiques*, 2me Série, Tome VII. p. 39.

(4) To find the centre of gravity of an arc of a logarithmic spiral the density of which varies as the curvature.

Let c be the length of the chord of the arc, ω the angle between its extreme radii vectores; and let α be the constant inclination of the curve at any point to the radius vector of the point. Draw a tangent to the curve, parallel to the chord of the said arc. The centre of gravity will lie in the radius vector of the point of contact at a distance from the pole equal to $\frac{c}{\omega} \sin \alpha$.

Haton de la Goupillière; *Nouvelles Annales de Mathématiques*, 2me Série, Tome VII. p. 128.

(5) To find the centre of gravity of an arc of a cycloidal wire, symmetrical in relation to the vertex, the density of the wire being supposed to vary, from point to point, as the curvature.

If $x = a(1 - \cos\theta)$ be the distance of the chord of the arc from the vertex, the distance of the centre of gravity above the middle point of the axis of the cycloid is equal to $\dfrac{a \sin\theta}{\theta}$.

Haton de la Goupillière; *Nouvelles Annales de Mathématiques*, 2me Série, Tome VII. p. 131.

(6) To find the centre of gravity of an arc of a wire in the form of a catenary, starting from the vertex, the density being supposed to vary inversely as the distance from the directrix.

The equation to the catenary being

$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right),$$

the co-ordinates of the required centre of gravity are equal to $\frac{1}{2}x$ and $\dfrac{cs}{x}$, s being the length of the arc.

Haton de la Goupillière; *Nouvelles Annales de Mathématiques*, 2me Série, Tome VII. p. 131.

(7) To find the centre of gravity of a quadrant of a circle, the density at any point of which varies as the n^{th} power of its distance from the centre.

Let a denote the radius of the circle, and $\bar{x}$, $\bar{y}$, the co-ordinates of the centre of gravity of the quadrant referred to its two extreme radii as axes; then

$$\bar{x} = \frac{n+2}{n+3} \cdot \frac{2a}{\pi} = \bar{y}.$$

(8) To find the centre of gravity of a square $ABCD$, the density of which at any point varies as the distance of the point from a line through A parallel to BD.

The distance of the centre of gravity from A is equal to $\frac{7}{12}a$, a denoting the length of a diagonal.

(9) To find the centre of gravity of a hemisphere, the density of which varies as the distance from the centre of the sphere.

The distance of the centre of gravity from the centre of the sphere is equal to two-fifths of the radius.

(10) To find the centre of gravity of a sphere, the density of which varies inversely as the distance from a point in its surface.

The centre of gravity divides the diameter through the said point into two parts which bear to each other the ratio of 2 to 3.

(11) To find the centre of gravity of a solid sphere, the density of which varies inversely as the fifth power of the distance from an external point.

If a be the radius of the sphere, and c the distance of the external point from the centre of the sphere, the centre of gravity is in the line joining the centre of the sphere and the external point, at a distance $\frac{a^2}{c}$ from the centre.

SECT. 10. *Centre of Parallel Forces.*

When any number of parallel forces act on a system of rigidly connected points, they generally have a single resultant acting on a point of which the position is invariable while their common direction is changed in every possible way. This point is called the Centre of the Parallel Forces: the Centre of Gravity of a body is a particular case of this. Let x, y, z, denote the co-ordinates of the point of application of any force P of the system referred to any axes, rectangular or oblique; and let $\bar{x}$, $\bar{y}$, $\bar{z}$, be the co-ordinates of the Centre of Parallel Forces. Then, R representing the resultant,

$$R = \Sigma(P), \quad \bar{x} = \frac{\Sigma(Px)}{\Sigma(P)}, \quad \bar{y} = \frac{\Sigma(Py)}{\Sigma(P)}, \quad \bar{z} = \frac{\Sigma(Pz)}{\Sigma(P)}.$$

Whenever $\Sigma(P)$ is equal to zero, these formulæ cease to be applicable, there being in this case no single resultant; the forces will be reducible to a resultant couple. For the complete development of the theory of Statical Couples, the reader is referred to Poinsot's beautiful work entitled *Elémens de Statique.*

The formulæ for $\bar{x}$, $\bar{y}$, $\bar{z}$, were first given by Varignon, in the *Mémoires de l'Académie des Sciences de Paris* for the year 1714.

(1) Three parallel forces, acting at the angular points $A, B, C,$ of a plane triangle, are respectively proportional to the opposite sides a, b, c; to find the distance of the centre of parallel forces from A.

Produce $AB, AC,$ indefinitely to points $x, y,$ and let $Ax, Ay,$ be taken as co-ordinate axes.

Let $\mu a, \mu b, \mu c,$ be the forces applied at $A, B, C,$ where $a, b, c,$ denote the opposite sides of the triangle. The co-ordinates of the points of application of these three forces are $0, 0$; $c, 0$; $0, b$; hence

$$\bar{x} = \frac{c \cdot \mu b}{\mu a + \mu b + \mu c} = \frac{bc}{a+b+c},$$

$$\bar{y} = \frac{b \cdot \mu c}{\mu a + \mu b + \mu c} = \frac{bc}{a+b+c} = \bar{x}.$$

Let r be the distance of the centre of parallel forces from A; then

$$r^2 = \bar{x}^2 + \bar{y}^2 + 2\bar{x}\bar{y}\cos A = 2\bar{x}^2(1+\cos A) = 4\bar{x}^2\cos^2\tfrac{1}{2}A,$$

and therefore

$$r = 2\bar{x}\cos\tfrac{1}{2}A = \frac{2bc\cos\frac{1}{2}A}{a+b+c}.$$

(2) Three parallel forces $P, Q, R,$ act at the angles $A, B, C,$ of a given triangle, and are to each other as the reciprocals of the opposite sides a, b, c; to determine the distance of their centre from A.

$$\text{Required distance} = a\,\frac{(b^4 + 2b^2c^2\cos A + c^4)^{\frac{1}{2}}}{bc + ca + ab}.$$

(3) Upon a horizontal triangular lamina, supported at its three angular points, is placed a weight equal to that of the lamina: to find the position of the weight if the pressures on the points of support are proportional to $4a+b+c$, $4b+c+a$, $4c+a+b$, where $a, b, c,$ are the lengths of the sides of the triangle.

The required position is the centre of the inscribed circle.

(4) At the corners $B, C, D,$ of a quadrilateral pyramid $ABCD$, three parallel forces $P, Q, R,$ are applied; to find the distance of their centre from the corner A.

Let $AB=b$, $AC=c$, $AD=d$; $\angle CAD=(c,d)$, $\angle DAB=(d,b)$, $\angle BAC=(b,c)$; $r=$ the required distance; then

$$r^2(P+Q+R)^2=P^2b^2+Q^2c^2+R^2d^2+2QRcd\cos(c,d)$$
$$+2RPdb\cos(d,b)+2PQbc\cos(b,c).$$

(5) At three fixed points (a, b), (a', b'), (a'', b''), in the plane of x, y, are applied three parallel forces p, p', p''; supposing the magnitude of p'' to vary in every possible way, to find the locus of the centre of parallel forces.

The locus will be a straight line of which the equation is

$$(ap+a'p')\,b''+\{(a''-a)\,p+(a''-a')\,p'\}\,y$$
$$=(bp+b'p')\,a''+\{(b''-b)\,p+(b''-b')\,p'\}\,x.$$

SECT. 11. *The Properties of Pappus.*

I. If any plane area revolve about any external axis in its own plane through any assigned angle, the volume of the surface generated by the motion of the area will be equal to a prism, of which the base is equal to the revolving area, and the altitude to the length of the path described by the centre of gravity of the area during its revolution.

II. If any plane area revolve through any angle about any external axis in its own plane, the area of the surface generated by its perimeter will be equal to a rectangle, of which one side is the length of the perimeter, and the other the length of the path described by the centre of gravity of the perimeter.

The enunciation of these properties, which are generally called Guldin's properties, is due to Pappus[1], and may be seen at the

[1] The words of Pappus in the Latin translation are: "Perfectorum utrorumque ordinum proportio composita est ex proportione amphismatum, et rectarum linearum similiter ad axes ductarum a punctis, quæ in ipsis gravitatis centra sunt. Imperfectorum autem proportio composita est ex proportione amphismatum, et circumferentiarum a punctis quæ in ipsis sunt centra gravitatis, factarum, &c." In the former case he is alluding to those solids which are formed by the entire revolution of the generating figures through 360°; in the latter, to those which are formed by revolution through any smaller angle.

end of the Preface to the seventh book of his *Mathematical Collections*, of which the first edition appeared in the form of a Latin translation in the year 1588. They were afterwards published, with various applications, by Guldin, in his treatise *De Centro Gravitatis*, Lib. 2 and 3, which appeared for the first time in the year 1635. Cavalieri[1], in reply to objections advanced by Guldin against his method of indivisibles, gave a demonstration of these properties by this method; stating likewise, in allusion to Guldin's claims as a discoverer, that they had been communicated to him, long before the publication of Guldin's work, by a pupil of his, Antonio Roocha. Elegant demonstrations of these properties were given also by Varignon in the *Mémoires de l'Académie des Sciences de Paris* for the year 1714, p. 77.

(1) From any point P (fig. 15) in a parabola, is drawn a straight line PM at right angles to the axis, and meeting it in the point M; to find the content of the solid generated by the complete revolution of the area APM about PM.

Let $AM = x$, $PM = y$; $V =$ the required volume; and $\bar{x} =$ the distance of the centre of gravity of the area APM from PM. Then the whole path described by the centre of gravity will be equal to $2\pi\bar{x}$; hence, by (I.),

$$V = 2\pi\bar{x} \times \text{area of } PAM:$$

but $\bar{x} = \frac{2}{5}x$, and area of $PAM = \frac{2}{3}xy$; and therefore

$$V = 2\pi \,.\, \tfrac{2}{5}x \,.\, \tfrac{2}{3}xy = \tfrac{8}{15}\pi x^2 y.$$

Complete the parallelogram $MPmA$; then the area of this parallelogram will be equal to xy, and the distance of its centre of gravity from PM will be equal to $\frac{1}{2}x$. Conceive this parallelogram to make an entire revolution about PM; then the path of its centre of gravity will be equal to

$$2\pi \,.\, \tfrac{1}{2}x = \pi x;$$

and therefore, if U denote the volume of the cylinder which is generated by the revolution,

$$U = \pi x \,.\, xy = \pi x^2 y.$$

[1] *Exercitationes Geometricæ Sex*, Exercit. 1 & 2; Bononiæ, 1647.

Hence $U : V :: 15 : 8.$

This is one of the problems proposed in Kepler's *Stereometria.*

Guldinus; *De Centro Gravitatis,* Lib. II. cap. 12, prop. 6.

(2) To find the surface of a sphere.

Let BAb (fig. 16) be a semicircle, by the revolution of which about the diameter BCb the sphere is generated.

Let CA be at right angles to Bb, C being the centre of the circle, and let G be the centre of gravity of the semicircular arc BAb. Let $CA = a$, surface required $= S$; then, by (II.),

$$2\pi CG \,.\, \text{arc}\, BAb = S;$$

but $CG = \frac{2a}{\pi}$, and arc $BAb = \pi a$; hence

$$S = 2\pi \,.\, \frac{2a}{\pi} \,.\, \pi a = 4\pi a^2.$$

Now πa^2 is the area of a great circle of the sphere; and thus we find that the whole surface of a sphere is four times as great as that of one of its great circles. This proposition was first proved by Archimedes, *Περὶ Σφαίρας καὶ Κυλίνδρου*, Βιβλ. Α, *πρότα*. Λ; and afterwards, according to the method which we have given, by Guldin, *De Centro Gravitatis,* Lib. IV. cap. 1, prop. 7.

(3) To find the distance of the centre of gravity of the area of a semi-parabola from the axis of the parabola.

Let APM (fig. 15) be the semi-parabola.

Let $AM = x$, $PM = y$, $\bar{y} =$ the distance of the centre of gravity of the area APM from AM. Then, since the area of APM is equal to $\frac{2}{3}xy$, and since the volume generated by the revolution of APM about AM is equal to

$$\int \pi y^2 dx = 4\pi m \int x dx = 2\pi m x^2;$$

we have, by (I.),

$$\frac{2}{3}xy \,.\, 2\pi\bar{y} = 2\pi m x^2,$$

$$\frac{2}{3}xy \,.\, \bar{y} = m x^2,$$

$$\frac{8}{3} m x^2 \bar{y} = m x^2 y,$$

$$\bar{y} = \frac{3}{8} y.$$

(4) To find the volume and the surface of the solid ring generated by the complete revolution of a circle about any external line in its own plane.

Let b be the distance of the centre of the circle from the axis of revolution, and a the radius of the circle; then

$$\text{the volume} = 2\pi^2 a^2 b, \text{ and the surface} = 4\pi^2 ab.$$

(5) To find the volume of the solid ring generated by the revolution of an ellipse about an external axis in its own plane through an angle of 180°.

If a, b, be the semiaxes of the ellipse, and c the distance of its centre from the axis of rotation, then

$$\text{the volume} = \pi^2 abc.$$

(6) To find the volume generated by the revolution through a given angle of a portion APM (fig. 15) of a parabola about a tangent at its vertex A, PM being parallel to the tangent, and AM at right angles to it.

If $AM = x$, $PM = y$, and β be the angle through which the revolution takes place; then

$$\text{the volume} = \tfrac{2}{5}\beta x^2 y.$$

(7) To find the volume and the surface of the solid generated by the complete revolution of a cycloid about its axis.

If a be the radius of the generating circle,

$$\text{the volume} = \pi a^3 \left(\tfrac{3}{2}\pi^2 - \tfrac{8}{3}\right), \quad \text{the surface} = 8\pi a^2 \left(\pi - \tfrac{4}{3}\right).$$

(8) To find the volume and the surface of the solid generated by the complete revolution of a cycloid about its base.

$$\text{The volume} = 5\pi^2 a^3, \quad \text{the surface} = \tfrac{64}{3}\pi a^2.$$

(9) To find the content of the solid generated by the complete revolution of a right-angled triangle about its hypotenuse.

If a, b, denote the two sides of the triangle, the content will be equal to

$$\frac{\pi a^2 b^2}{3\,(a^2 + b^2)^{\frac{1}{2}}}.$$

CHAPTER II.

EQUILIBRIUM OF A PARTICLE.

LET P denote any one of a system of forces acting on a particle; and let α, β, γ, be the angles which the direction of this force makes with any three proposed straight lines, no two of which are parallel; then the sufficient and necessary conditions for the equilibrium of the particle are expressed by the three following equations,

$$\Sigma (P\cos\alpha) = 0, \quad \Sigma (P\cos\beta) = 0, \quad \Sigma (P\cos\gamma) = 0,$$

where the Σ represents the summation of all such quantities as $P\cos\alpha$, $P\cos\beta$, $P\cos\gamma$, for all the different forces of the system; or the algebraical sum of the resolved parts of all the forces of the system estimated parallel to each of the three straight lines must be equal to zero. If all the forces acting on the particle lie within a single plane, then two of the three straight lines being taken in this plane, the three equations of equilibrium will evidently be reduced to two.

The conditions for the equilibrium of a particle acted on by oblique forces, appear to have been first distinctly conceived by Stevin of Bruges[1]. He establishes by reasoning which, although indirect, is satisfactory and ingenious, the ratio which the weight of a particle supported on an inclined plane bears to the force by which it is sustained, the force being supposed to act along the plane. He then announces generally, without however supplying a corresponding extension of demonstration, that the condition of equilibrium of any three forces acting on a particle, consists in the proportionality of the forces to the sides of a triangle to which they are parallel. The first rigorous

[1] Beghinselen der Waaghconst, 1586. I. *Livre de la Statique*, prop. 19.

demonstration of Stevin's theorem in its general form, was obtained by Roberval[1] from the nature of the lever. The idea of a triangle of equilibrium had occurred indeed somewhat earlier to Michael Varro[2], of Geneva, in application to the equilibrium of forces acting on the sides of a right-angled triangular wedge: it does not appear, however, that Varro's notion was based upon any very distinct conception of the nature of statical pressure. The Principle of the Parallelogram of Forces, which is in fact a mere modification of Stevin's theorem, was announced almost simultaneously by Newton[3] and Varignon[4]; by whom it was inferred from the consideration of the composition of motions. In the same year was published by Lami, in a little treatise entitled *Nouvelle manière de démontrer les principaux Théorèmes des élémens des Mécaniques*, a theorem in which it is asserted, that if three forces P, Q, R, keep a particle at rest, then

$$P : Q : R :: \sin(Q, R) : \sin(R, P) : \sin(P, Q),$$

where (Q, R), (R, P), (P, Q), denote the angles between the directions of Q and R, R and P, P and Q, respectively. The virtual coincidence of this theorem with the Principle of the Parallelogram of Forces, subjected Lami to the imputation of plagiarism, an aspersion cast upon him by the author of the *Histoire des Ouvrages des Savans*, (April 1688). Lami combated this insinuation in a letter published in the *Journal des Savans*, (Sept. 13, 1688), to which the Journalist replied in the following December, when the controversy appears to have terminated. The first unexceptionable demonstration of the Parallelogram of Forces on pure statical principles, without the introduction of the idea of motion, was given by Daniel Bernoulli[5]. Many other proofs of the proposition have been since given. Eighteen demonstrations have been collected and

[1] *Traité de Mécanique*, printed in 1636, in the *Harmonie Universelle de Mersenne*, and in a work also by Mersenne, entitled *Cogitata Physico-Mathematica*, published in 1644.

[2] *Tractatus de Motu*, 1584.

[3] *Principia*, lex iii. cor. 2, 1687.

[4] *Projet de la Nouvelle Mécanique*, 1687.

[5] *Comment. Petrop.*, Tom. I. p. 126, 1726.

examined by Jacobi[1], by the following authors: D. Bernoulli, 1726; Lambert, 1771; Scarella, 1756; Venini, 1764; Araldi, 1806; Wachter, 1815; Kœstner, Marini, Eytelwein, Salimbeni, Duchayla[2]; two different proofs by Foncenex, 1760; three by D'Alembert; and those of Laplace and Poisson.

Sect. 1. *No Friction.*

(1) P and W (fig. 17) are two heavy particles; W is attached to the end of a fine thread, and P is suspended from a fixed point C of the thread; the thread has one extremity attached to a fixed point A, and passes through a smooth small ring at B in the same horizontal line with A; to find the ratio between P and W, that the vertical line through C may bisect AB in D.

From the supposition it is evident that $\angle ACD = \angle BCD$; let each of these angles be denoted by θ: let T = the tension of the string CA; $CA = b$, $AB = a$; the ring B being perfectly smooth, W will be the tension of the string BC.

Hence for the equilibrium of the point C we have, resolving vertically the forces which act on it,

$$(T + W)\cos\theta = P,$$

and, resolving horizontally,

$$T\sin\theta = W\sin\theta, \text{ or } T = W;$$

hence $$2W\cos\theta = P, \quad \cos\theta = \frac{P}{2W} \ldots\ldots\ldots\ldots\ldots (1);$$

but from the geometry we see that

$$b\sin\theta = \tfrac{1}{2}a, \quad \sin\theta = \frac{a}{2b} \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Squaring the equations (1) and (2), and adding the resulting equations, we have

$$1 = \frac{P^2}{4W^2} + \frac{a^2}{4b^2}, \quad \frac{P^2}{4W^2} = \frac{4b^2 - a^2}{4b^2},$$

[1] Whewell's *Philosophy of the Inductive Sciences*, Vol. I. p. 197.

[2] *Correspondance sur l'Ecole Polytechnique*, Tom. I. p. 83, anno 1805.

and therefore $$\frac{P}{W}=\frac{(4b^2-a^2)^{\frac{1}{2}}}{b},$$

which determines the required ratio.

(2) A particle P (fig. 18) is placed on the surface of a smooth prolate spheroid, and is attracted towards the foci S and H with forces varying as SP^m and $HP^{m'}$; to find the position of equilibrium.

Draw a tangent KPL at the point P in the plane passing through the three points S, H, P; let μ, μ', be the absolute forces towards S, H; let $SP=r$, $HP=r'$. Then for the equilibrium of the particle we have, resolving forces parallel to the line KPL,

$$\mu . r^m \cos \angle SPK = \mu' r'^{m'} \cos \angle HPL;$$

but $\angle SPK = \angle HPL$, by the nature of ellipses; hence

$$\mu r^m = \mu' r'^{m'};$$

also, $2a$ denoting the axis of the spheroid, $2a=r+r'$; hence, for the determination of r and r', we have the two equations

$$\mu r^m = \mu' (2a-r)^{m'}, \qquad \mu (2a-r')^m = \mu' r'^{m'}.$$

(3) Two weights m, m', are attached to the points O, O', (fig. 19) of a string $AOO'A'$, suspended from two tacks at A and A' in the same horizontal line; to find the positions of the points that their vertical distances from the horizontal line through A and A' may have given equal values.

Draw $OE, O'E'$, vertical; let $OE=a=O'E'$, $AA'=b$, $c=$ the length of the string; $\angle AOE=\theta$, $\angle A'O'E=\theta'$; $T=$ the tension of the string OO'.

Then for the equilibrium of O we have, by Lami's Principle,

$$\frac{T}{m}=\frac{\sin(\pi-\theta)}{\sin(\frac{1}{2}\pi+\theta)}=\tan\theta;$$

and, for the equilibrium of O',

$$\frac{m'}{T}=\frac{\sin(\frac{1}{2}\pi+\theta')}{\sin(\pi-\theta)}=\cot\theta'.$$

From these two equations we get

$$m\tan\theta = m'\tan\theta' \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Again, from the geometry,

$$EE' = AA' - AE - A'E'$$
$$= b - a(\tan\theta + \tan\theta');$$

but we have also, from the geometry,

$$EE' = OO' = c - AO - A'O'$$
$$= c - a(\sec\theta + \sec\theta');$$

hence $\quad a(\sec\theta - \tan\theta + \sec\theta' - \tan\theta') = c - b$..............(2).

From the equations (1) and (2) the values of θ, θ', are to be determined, and then, EO and $E'O'$ being given, AO, $A'O'$, will be known.

Diarian Repository, p. 627.

(4) A fine string is fixed at two points A and B (fig. 20) in the same horizontal line, and passes over a given set of pegs in the line AB, equal given weights being hung on between every two successive pegs, and also between A and B and the pegs nearest to them: to find the position of equilibrium, and the tension of the string.

Let p, r, be any two successive pegs, pqr the corresponding portion of the string; W the magnitude of each of the weights. Let $\angle qpr = \alpha$, T = the tension of the string, c = the length of the piece pqr of the string, l = the length of the whole string, $AB = a$.

Then, for equilibrium,

$$2T\sin\alpha = W \text{..........................(1)},$$
$$pr = c\cos\alpha \text{.......................(2)}.$$

From (1), since T is the same throughout the string, we see that α is the same for every triangle such as pqr: hence

$$\Sigma(pr) = \cos\alpha\,\Sigma(c), \quad \text{or} \quad a = l\cos\alpha \text{............(3)}.$$

From (1), (2), (3), we have

$$T = \frac{Wl}{2(l^2 - a^2)^{\frac{1}{2}}},$$

and $$c = \frac{l}{a}\cdot pr.$$

(5) A weight W is sustained upon a smooth inclined plane AB (fig. 21), by three forces, each equal to $\frac{1}{3}W$, one acting vertically upwards, another along AB, and the third parallel to the horizontal line AC; to find the inclination of AB to the horizon.

The required angle of inclination $= 2\tan^{-1}\frac{1}{3}$.

(6) Two forces P, Q, of known magnitudes, acting respectively parallel to the base and length of an inclined plane, will each of them singly sustain upon it a particle of weight W; to determine the magnitude of W.

$$W = \frac{PQ}{(P^2 - Q^2)^{\frac{1}{2}}}.$$

(7) Two heavy particles, P and Q, (fig. 22), are connected together by a fine thread passing over a smooth pully at C; P rests on a smooth inclined plane AB, and Q hangs freely; to determine the position of equilibrium and the pressure on the inclined plane.

Let α = the inclination of the plane to the horizon, R = the pressure, and θ = the angle CPB; then

$$\cos\theta = \frac{P\sin\alpha}{Q}, \qquad R = P\cos\alpha - (Q^2 - P^2\sin^2\alpha)^{\frac{1}{2}}.$$

(8) A weight W is supported on an inclined plane AB, (fig. 21), by three forces, each equal to P, one acting vertically upwards, another parallel to the horizontal line AC, and the third along AB: to find the inclination of AB.

The required inclination $= 2\tan^{-1}\left(\frac{P}{W-P}\right)$.

(9) A particle is placed within a thin parabolic tube, (fig. 23), the axis of the parabola being vertical; the particle is acted on by gravity and by a horizontal force, tending from the axis, and equal to μ times the distance of the particle from the axis; to find the condition of equilibrium.

There will be no position of equilibrium unless the latus rectum of the parabola be equal to $\frac{2g}{\mu}$; and, under this condition, every point of the tube will be a position of rest.

SECT. 2. *Friction.*

(1) Two heavy particles P and P' (fig. 24) rest on two inclined planes CA, $C'A$, and are connected together by a fine string passing over a smooth pully at O in the vertical line through A; to determine the positions of P and P' when P is only just sustained.

Let a be the length of the string POP', T its tension, which will be the same throughout; W, W', the weights of the particles P, P'; μ, μ', the coefficients of friction on the planes CA, $C'A$, and R, R', their reactions; α, α', the inclinations of the two planes, and θ, θ', of the two portions of the string, to the vertical.

Then, since by hypothesis the friction on P will be exerted up CA, and that on P' down AC', we have for the equilibrium of P, resolving forces parallel and perpendicular to CA,

$$\mu R + T\cos(\alpha-\theta) = W\cos\alpha \quad\ldots\ldots(1),$$

$$R + T\sin(\alpha-\theta) = W\sin\alpha \quad\ldots\ldots(2);$$

and, in the same way, for the equilibrium of P',

$$\mu' R' + W'\cos\alpha' = T\cos(\alpha'-\theta') \quad\ldots\ldots(3),$$

$$R' + T\sin(\alpha'-\theta') = W\sin\alpha' \quad\ldots\ldots(4).$$

From (1) and (2),

$$T\{\cos(\alpha-\theta) - \mu\sin(\alpha-\theta)\} = W(\cos\alpha - \mu\sin\alpha);$$

and, from (3) and (4),

$$T\{\cos(\alpha'-\theta') + \mu'\sin(\alpha'-\theta')\} = W'(\cos\alpha' + \mu'\sin\alpha').$$

Eliminating T between these last two equations, we obtain

$$W'(\cos\alpha' + \mu'\sin\alpha')\{\cos(\alpha-\theta) - \mu\sin(\alpha-\theta)\}$$
$$= W(\cos\alpha - \mu\sin\alpha)\{\cos(\alpha'-\theta') + \mu'\sin(\alpha'-\theta')\}.$$

Assume $\mu = \tan\epsilon$, $\mu' = \tan\epsilon'$; then this equation becomes

$$W'\cos(\alpha'-\epsilon')\cos(\alpha-\theta+\epsilon) = W\cos(\alpha+\epsilon)\cos(\alpha'-\theta'-\epsilon')\ldots(5).$$

Again, from the geometry, if $OA = k$,

$$OP = \frac{k\sin\alpha}{\sin(\alpha-\theta)}, \qquad OP' = \frac{k\sin\alpha'}{\sin(\alpha'-\theta')},$$

and therefore, since $OP + OP' = a$,

$$a = \frac{k \sin \alpha}{\sin (\alpha - \theta)} + \frac{k \sin \alpha'}{\sin (\alpha' - \theta')} \dots\dots\dots\dots\dots (6).$$

The angles θ, θ', are to be determined from the equations (5) and (6).

(2) Given the semi-sum and semi-difference of the greatest and least angles which the direction of a force, supporting a heavy particle on a rough inclined plane, may make with the plane, and the least elevation of the plane when the particle would, without being supported, slide down it; to determine the angle at which the same force, when inclined to a smooth plane of the same elevation, would support the same particle.

Let ϵ denote the least angle which the force may make with the rough plane to support the particle, P the magnitude of the force, R the reaction of the plane at right angles to itself, μ the coefficient of friction, α the inclination of the plane to the horizon, W the weight of the particle. Then, since the friction must in this case act down the plane, we have for the equilibrium of the particle, resolving forces parallel to the inclined plane,

$$P \cos \epsilon = \mu R + W \sin \alpha;$$

and, resolving forces at right angles to the plane,

$$P \sin \epsilon + R = W \cos \alpha.$$

Eliminating R between these two equations, we get

$$P (\cos \epsilon + \mu \sin \epsilon) = W (\mu \cos \alpha + \sin \alpha) \dots\dots\dots\dots (1).$$

Let ϕ be the least elevation of the plane for the particle without support to slide down it; then $\tan \phi$ will be equal to μ; hence from (1),

$$P = W \frac{\sin (\alpha + \phi)}{\cos (\epsilon - \phi)} \dots\dots\dots\dots\dots\dots\dots\dots (2).$$

If ϵ' denote the greatest angle which P may make with the inclined plane consistently with the equilibrium of the particle, then the friction will act with the greatest force it can exert up

the plane; hence, making μ negative, or putting $-\phi$ for ϕ, we shall have from (2), ϵ' replacing ϵ,

$$P = W\frac{\sin(\alpha-\phi)}{\cos(\epsilon'+\phi)} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Also, if ϵ'' denote the angle of P's inclination in the case of a smooth plane of the same elevation, we have, putting $\phi = 0$ in (2), and replacing ϵ by ϵ'',

$$P = W\frac{\sin\alpha}{\cos\epsilon''} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

From (2) and (3)

$$\cos(\epsilon-\phi)+\cos(\epsilon'+\phi) = \frac{W}{P}\{\sin(\alpha+\phi)+\sin(\alpha-\phi)\},$$

and therefore, if $S = \frac{1}{2}(\epsilon'+\epsilon)$ and $D = \frac{1}{2}(\epsilon'-\epsilon)$, we get

$$2\cos S\cos(D+\phi) = 2\frac{W}{P}\sin\alpha\cos\phi$$

$$= 2\cos\epsilon''\cos\phi, \text{ by (4)};$$

hence $$\cos\epsilon'' = \frac{\cos S}{\cos\phi}\cos(D+\phi).$$

(3) P is the lowest point on the rough circumference of a circle in a vertical plane at which a particle can rest, friction being equal to pressure; to determine the inclination of the radius through P to the horizon.

$$\text{The required angle} = \frac{\pi}{4}.$$

(4) A given force P, acting parallel to the horizon, just sustains a body of given weight W on a rough inclined plane, the angle of which is θ: the same body will just rest without support on a plane of the same material, the inclination of which is α: to determine θ.

The tangent of θ is equal to

$$\frac{P+W\tan\alpha}{W-P\tan\alpha}.$$

(5) A heavy body is to be conveyed to the top of a rough inclined plane, the angle of inclination being α: to determine

whether it will be easier to lift the body or to drag it along the plane.

It will be easier to lift or to drag it accordingly as the coefficient of friction is less or greater than

$$\frac{\sin\left(\dfrac{\pi - 2\alpha}{4}\right)}{\sin\left(\dfrac{\pi + 2\alpha}{4}\right)}.$$

(6) A weight is supported on a rough inclined plane by a force exactly equal to it: to find the direction of the force.

If θ denote the inclination of the force to the inclined plane, α the plane's inclination, and μ the coefficient of friction,

$$\theta = \alpha - 2\beta + \tfrac{3}{2}\pi,$$

where β may have any value from $-\tan^{-1}\mu$ to $+\tan^{-1}\mu$.

CHAPTER III.

EQUILIBRIUM OF A SINGLE BODY.

LET any system of forces act upon a body consisting of a system of points rigidly connected together. Take any three straight lines OA, OB, OC, (fig. 25), in space, no two of which are parallel to each other. Let A, B, C, denote the whole resolved part, parallel to each of these straight lines, of any one of the forces of the system; A, B, C, being positive or negative quantities according as they act in the directions OA, OB, OC, or the opposite ones. Then, $\Sigma(A)$, $\Sigma(B)$, $\Sigma(C)$, denoting the algebraic sums of the resolved parts of all the forces of the system parallel to these three straight lines, it is necessary for the equilibrium of the body that we have

$$\Sigma(A)=0, \quad \Sigma(B)=0, \quad \Sigma(C)=0 \text{..............} \quad \text{(I).}$$

Again, let $O'A'$, $O'B'$, $O'C'$, be any three straight lines in space, no two of which are parallel to each other. Let any force of the system be resolved into two parts, the one at right angles to $O'A'$, and the other parallel to it; let A' be the magnitude of the part which is at right angles to $O'A'$, and a' the perpendicular distance between $O'A'$ and the direction of A'; then $A'a'$ is called the moment of A' about $O'A'$, and the sum of all such moments for all the forces of the system will be denoted by $\Sigma(A'a')$, those moments being considered positive which tend to twist the body about $O'A'$ in one direction, and those which tend to twist it in the opposite direction being considered negative. Similarly the algebraic sums of the moments about $O'B'$, $O'C'$, respectively, will be denoted by $\Sigma(B'b')$, $\Sigma(C'c')$. Then for the equilibrium of the body it is necessary that we have

$$\Sigma(A'a')=0, \quad \Sigma(B'b')=0, \quad \Sigma(C'c')=0 \text{.........} \quad \text{(II).}$$

The three equations (I.), together with the three equations (II.), are universally sufficient and universally necessary for the equilibrium of any body. It may be proper to remark, that any of the lines $O'A'$, $O'B'$, $O'C'$, may be taken to coincide with any of the lines OA, OB, OC, according to convenience.

If all the forces of the system lie in one plane, then, the lines OB, OC, $O'B'$, $O'C'$, being taken within this plane, and the line OA at right angles to it, the six equations of equilibrium will be reduced to the three following,

$$\Sigma(B) = 0, \quad \Sigma(C) = 0, \quad \Sigma(A'a') = 0 \ldots\ldots\ldots\ldots(\text{III.});$$

for it is evident that the three other equations will be identically satisfied.

The basis of the general equations of equilibrium consists in the Theory of the Composition and Resolution of Forces, of which we have treated in the preceding chapter, and in the Theory of Moments. The latter theory, in the case of weights acting at right angles to the arms of a straight lever, was established by Archimedes[1]. In the year 1499, the condition of equilibrium of a force acting obliquely on a lever, and supporting a weight suspended from it, was correctly stated by Leonardo Da Vinci[2], the celebrated painter, to whom must therefore be ascribed the discovery of the theory of oblique action, investigated at a later date by Stevin, in application to the Equilibrium of a Particle. The following elegant geometrical proposition, the application of which to the general Theory of Moments depends upon the Principle of the Parallelogram of Forces, was given by Varignon in his *Nouvelle Mécanique*, sect. I, lem. XVI: "If from any point whatever in the plane of a parallelogram we let fall perpendiculars upon the diagonal and upon the two sides which comprehend this diagonal, the product of the diagonal by its perpendicular is equal to the sum of the products of the two sides by their respective perpendiculars, if the point lie without

[1] *Ἀρχιμήδους Ἐπιπέδων ἰσοῤῥοπικῶν ἢ κέντρα βαρῶν ἐπιπέδων τὸ Α. Πρότ. στ. Ζ.*

[2] Venturi; *Essai sur les Ouvrages Physico-Mathématiques de Léonard da Vinci, avec des Fragmens tirés de ses Manuscrits apportés d'Italie*, Paris, 1797; quoted in Whewell's *History of the Inductive Sciences*, Vol. II. p. 122.

the parallelogram, or to their difference, if it lie within the parallelogram." The six general conditions of equilibrium of a system of rigidly connected points acted on by any forces whatever, were first laid down by D'Alembert, in the second chapter of his *Recherches sur la Précession des Equinoxes*, published in the year 1749.

SECT. 1. *No Friction.*

(1) A beam AB (fig. 26) rests with one end against a horizontal plane at a point A, and with the other against a vertical one at the point B; the vertical plane passing through the beam intersects at right angles the former plane in the line AC, and the latter in the line BC; the beam is attached to the point C by a string EC without weight: to find the tension of the string, E being any assigned point in the beam.

The actions of the horizontal and vertical planes upon the beam at A and B, will be in the directions AR' and BR, which are parallel respectively to CB and CA; let them be denoted by R' and R. Again, let T denote the tension of the string EC. Let G be the centre of gravity of the beam, and W its weight; then, instead of supposing the beam to have weight, we may suppose it to be a rigid rod without weight, provided that we apply the force W vertically downwards at G. Thus we have four forces, R, R', T, W, acting at four points B, A, E, G, rigidly connected together. We proceed to express the equations of equilibrium. Let $\angle ECA = \epsilon$, $\angle BAC = \alpha$, $AG = BG = a$. Then, resolving the forces parallel to the line CA, we have

$$R - T\cos\epsilon = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1);$$

resolving the forces parallel to CB, we have

$$R' - W - T\sin\epsilon = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2);$$

and, taking moments about the point C,

$$R \,.\, 2a\sin\alpha + Wa\cos\alpha - R' \,.\, 2a\cos\alpha = 0,$$

or

$$2R\sin\alpha + W\cos\alpha = 2R'\cos\alpha. \ldots\ldots\ldots\ldots (3).$$

From (1), (2), (3), there is

$$2T\cos\epsilon\sin\alpha + W\cos\alpha = 2W\cos\alpha + 2T\sin\epsilon\cos\alpha,$$

$$2T(\cos\epsilon\sin\alpha - \cos\alpha\sin\epsilon) = W\cos\alpha,$$

$$2T\sin(\alpha-\epsilon) = W\cos\alpha,$$

and therefore $$T = \frac{W\cos\alpha}{2\sin(\alpha-\epsilon)}.$$

If ϵ be equal to α, we have $T=\infty$, which shews that no tension, however great, can sustain the beam in a position of equilibrium. It is easily seen that in this case E coincides with G; and that the length of CE is sufficient to allow the beam to descend continually.

If ϵ be greater than α, T will clearly be negative; and, since the string can pull but not push, the equilibrium is impossible. Thus, in order that the equilibrium may be possible, α must be greater than ϵ.

(2) A smooth beam AB, (fig. 27), rests against two horizontal bars which pierce the vertical plane through the beam at right angles at the points A', A''; the beam passes under the lower and over the higher bar, its lower extremity A being sustained upon a smooth horizontal plane: to determine the pressures upon the two bars and upon the horizontal plane.

The pressures upon the bars and upon the horizontal plane will be equal to their reactions upon the beam; the reactions of the bars upon the beam will be two forces R', R'', at right angles to the beam; and the reaction of the horizontal plane will be a vertical force R. Let G be the centre of gravity of the beam; then, if we suspend its weight W from G, we may, without affecting the circumstances of the equilibrium, conceive the beam to be a rigid rod without weight. Thus we have four forces R, R', R'', W, acting respectively at four points A, A', A'', G, rigidly connected together, so as to produce equilibrium. Let $AG=a$, $A'A''=b$, and $\alpha=$ the inclination of the beam to the horizon.

Then, resolving forces parallel to the beam, we have

$$W\sin\alpha - R\sin\alpha = 0, \text{ and therefore } R = W \,\ldots\ldots\ldots\, (1).$$

Resolving forces at right angles to the beam,

$$R' + W\cos\alpha - R'' - R\cos\alpha = 0,$$

and therefore, by (1),

$$R' = R'' \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Again, taking moments about A,

$$R''.AA'' - R'.AA' - Wa\cos\alpha = 0,$$

and therefore, by (2),

$$R'b = Wa\cos\alpha;$$

whence $$R' = R'' = \frac{Wa\cos\alpha}{b}.$$

(3) A rigid rod AB, (fig. 28), rests upon a fixed point E, while its lower extremity A presses against a vertical line FF'; to find its position of equilibrium and also the pressures at A and E.

Let G be the centre of gravity of the rod, and W its weight; we suppose the whole weight of the rod to be collected at its centre of gravity. Let R be the reaction of the vertical line FF' upon the rod, which will be at right angles to FF'; also let R' be the reaction of the fixed point E, which will be at right angles to the rod. Let EF be at right angles to FF'; and let $EF = c$, $AG = a$, $\angle AEF = \theta$.

Then, resolving forces parallel to the rod,

$$R\cos\theta = W\sin\theta \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

resolving forces at right angles to the rod,

$$R' = W\cos\theta + R\sin\theta \quad\ldots\ldots\ldots\ldots\ldots\ldots (2);$$

and, taking moments about E,

$$\begin{aligned} R.AE\sin\theta &= W.EG\cos\theta \\ &= W(AG - AE)\cos\theta \\ &= W(a\cos\theta - c), \end{aligned}$$

and therefore $$Rc\sin\theta = W(a\cos^2\theta - c\cos\theta) \quad\ldots\ldots\ldots (3);$$

hence, from (1) and (3),

$$Wc\frac{\sin^2\theta}{\cos\theta} = W(a\cos^2\theta - c\cos\theta),$$

$$\frac{c}{\cos\theta} = a\cos^2\theta,\ \cos\theta = \left(\frac{c}{a}\right)^{\frac{1}{3}} \dots\dots\dots\dots\dots\dots (4),$$

which gives the value of θ, and defines the position of the beam.

From (1) and (4) we have

$$R = W\tan\theta = W\frac{\left\{1-\left(\frac{c}{a}\right)^{\frac{2}{3}}\right\}^{\frac{1}{2}}}{\left(\frac{c}{a}\right)^{\frac{1}{3}}} = W\frac{\left(a^{\frac{2}{3}}-c^{\frac{2}{3}}\right)^{\frac{1}{2}}}{c^{\frac{1}{3}}},$$

which determines the pressure on the vertical line.

Also, from (1) and (2),

$$R' = W\cos\theta + W\frac{\sin^2\theta}{\cos\theta} = \frac{W}{\cos\theta},$$

and therefore, by (4), $\quad R' = W\left(\frac{a}{c}\right)^{\frac{1}{3}}$,

which determines the pressure on the fixed point.

If c be greater than a, then we see by (4) that $\cos\theta$ would be greater than unity, which is impossible; thus equilibrium is impossible unless a be at least equal to c.

Fontana; *Memorie della Societa Italiana*, 1802, p. 626, &c.

(4) One end A of a beam AB, (fig. 29), is connected to a fixed point by a hinge, about which the beam is capable of revolving in a vertical plane; the other end B is attached to a weight P by means of a string passing over a pully C in the same vertical plane; to find the position of equilibrium.

Let a horizontal line AD through A meet a vertical line through C in the point D. Let G be the centre of gravity of the beam, at which we shall suppose its whole weight W to be collected. Produce CB to meet AE at right angles to it.

$AG = a$, $BG = b$, $AD = k$, $CD = l$, $\angle BAD = \theta$, $\phi =$ the inclination of CE to the horizon.

Then, taking moments about A,

$$P.AE = W.AF,$$

or $$P(a+b)\sin(\phi-\theta) = Wa\cos\theta \dots\dots\dots\dots (1);$$

again, from the geometry,

$$(a+b)\sin\theta + BC\sin\phi = l,$$
$$(a+b)\cos\theta + BC\cos\phi = k,$$

and therefore, eliminating BC,

$$(a+b)\sin(\theta-\phi)=l\cos\phi-k\sin\phi\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The equations (1) and (2) are sufficient for the determination of θ and ϕ, or of the position of equilibrium.

(5) A weight W (fig. 30) hangs from the end E of a rigid rod BE without weight, which rests on a smooth hinge at B, and is supported by a string CAD, passing through a fixed ring at A in the vertical line through B: the angles ACD, ADC, are equal, $\angle ABC=60^\circ$, and $DE=BC$: to find the direction and magnitude of the pressure on the hinge.

Let X, Y, be respectively the horizontal and vertical components of the pressure exerted by the hinge on the rod, which will be equal and opposite to the components of the pressure exerted by the rod on the hinge. Let T denote the tension of the string.

The resultant of the action of the two portions of the string on the rod will evidently pass through H, the middle point of the rod, at right angles to the rod.

Hence, resolving forces parallel to the rod,

$$X\cos 30^\circ+Y\cos 60^\circ=W\cos 60^\circ,$$

or

$$W-Y=X\sqrt{3}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1):$$

and, taking moments about H,

$$(Y+W)\,.\,\tfrac{1}{2}BE\,.\cos 30^\circ=X\,.\,\tfrac{1}{2}BE\,.\sin 30^\circ,$$

or

$$W+Y=\frac{X}{\sqrt{3}}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

From (1) and (2),

$$X=\frac{\sqrt{3}}{2}W,\quad Y=-\tfrac{1}{2}W:$$

whence, if R denote the resultant action of the hinge on the rod,

$$R=(X^2+Y^2)^{\frac{1}{2}}=W;$$

and, if ϕ denote the inclination of R's direction to BA,

$$\tan\phi=\frac{X}{Y}=-\sqrt{3}=\tan\frac{2\pi}{3},$$

$$\phi=\tfrac{2}{3}\pi.$$

(6) A cylinder rests with its base on a smooth inclined plane; a string, attached to its highest point, and passing over a pully at the top of the inclined plane, hangs vertically and supports a weight; the portion of the string between the cylinder and the pully is horizontal; to determine the conditions of equilibrium.

Let P (fig. 31) be the suspended weight, W the weight of the cylinder, R the resultant action of the inclined plane on the base of the cylinder, M the point of the base through which R passes; C the centre of the base, G the centre of gravity of the cylinder. Draw GK at right angles to BB', KH horizontally to meet the vertical through G.

Let a = the radius of the cylinder, $2b$ = its length, $CM = x$.

The three forces P, W, R, which act on the cylinder, must pass through a single point O.

Resolving forces parallel to the inclined plane,

$$P \cos \alpha = W \sin \alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

perpendicularly to it,

$$R = P \sin \alpha + W \cos \alpha \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Again, from the geometry,

$$GO = GH + OH = a \sin \alpha + b \cos \alpha,$$

$$x = GO \,.\, \sin \alpha = \sin \alpha \, (a \sin \alpha + b \cos \alpha) \ldots\ldots\ldots\ldots\ldots (3).$$

Now, since x cannot be greater than a, we see by (3) that

$$a \text{ is not less than } \sin \alpha \, (a \sin \alpha + b \cos \alpha),$$

$$a \cos^2 \alpha \ \ldots\ldots\ldots\ldots\ldots\ldots \ b \sin \alpha \cos \alpha,$$

$$a \ \ldots\ldots\ldots\ldots\ldots\ldots \ b \tan \alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

The conditions (1) and (4) are sufficient and necessary for equilibrium. By (2) and (3) we know R and x.

(7) To find the force requisite to keep a door in a given position, the post being inclined at a given angle to the vertical; neglecting friction.

Let AB (fig. 32) be the door-post, $ABCD$ the door; Az a vertical line through A; Ax at right angles to Az and in the plane of BAz; E the intersection of the line CD produced with

the horizontal plane through A; join AE. With A as a centre describe a sphere cutting AB, Ax, AE, at the points p, q, r, and join these points by great circles of the sphere.

Let $\angle BAz = \beta$, and $\alpha =$ the inclination of the plane of the door to the plane zAx; $W =$ the weight of the door.

Then, since the angle pqr of the spherical triangle pqr is a right angle, we have, by Napier's rules,

$$\cos prq = \sin qpr \cos pq = \sin \alpha \sin \beta;$$

but, if ϕ denote the angle which W's direction makes with the plane of the door, it is clear that

$$\sin \phi = \cos prq;$$

hence, a denoting the distance of the centre of gravity of the door from the post, the moment of W about AB will be equal to

$$Wa \sin \alpha \sin \beta;$$

let P be the force applied at right angles to the door, at a point distant from the door-post by a space b, sufficient to keep it in its present position; then, by the equation of moments, we have

$$Pb = Wa \sin \alpha \sin \beta.$$

The following solution is rather more brief than the preceding one.

The component of W in the plane zAx at right angles to AB is equal to $W \sin \beta$, and the component of $W \sin \beta$ at right angles to the door is $W \sin \beta \sin \alpha$: hence the moment of W about AB is equal to

$$Wa \sin \alpha \sin \beta,$$

and therefore

$$Pb = Wa \sin \alpha \sin \beta.$$

(8) A uniform bar AB (fig. 33) is placed in the straight line joining two centres of force K, L, which attract with forces varying directly as the distance; to find the position in which the bar will rest.

Let μ, μ', be the absolute forces of the centres K, L; let P be any point in the bar AB; $KA = x$, $LB = y$, $KP = s$, $BP = s'$, $AB = 2a$, $KL = l$; $\rho =$ the density of the bar, $\kappa =$ the

area of a transverse section. Then for the equilibrium of the bar we must have

$$\int_x^{x+2a} \kappa\rho\mu\, s\, ds = \int_y^{y+2a} \kappa\rho\mu' s'\, ds';$$

or, since κ and ρ are supposed to be the same for all points of the bar,

$$\mu \int_x^{x+2a} s\, ds = \mu' \int_y^{y+2a} s'\, ds',$$

$$\mu \{(x+2a)^2 - x^2\} = \mu' \{(y+2a)^2 - y^2\},$$

$$\mu (x+a) = \mu' (y+a) = \mu' (l - a - x),$$

$$(\mu + \mu')\, x = \mu' l - (\mu + \mu')\, a,$$

$$x = \frac{\mu' l}{\mu + \mu'} - a:$$

similarly $$y = \frac{\mu l}{\mu + \mu'} - a.$$

The value of x, or of y, determines the position of equilibrium.

(9) One end A of a uniform beam AB (fig. 34) is placed upon a smooth horizontal plane OA, and the other end B touches a vertical plane OB: the point O is a centre of attractive force, the intensity of the force varying directly as the distance; to determine the position of equilibrium.

Conceive the beam to be inclined at an angle ω to the horizon. Take P any point in the beam and join OP.

Let $OP = r$, $AP = s$, $AG = BG = a$, $\angle POA = \theta$, $\mu =$ the absolute force of attraction, R, R', the reactions of the planes at A, B. Then, resolving forces horizontally, we have

$$R' = \int \mu r ds \cos\theta = \mu \int_0^{2a} ds\, (2a - s) \cos\omega = 2\mu a^2 \cos\omega \text{......(1)}.$$

Resolving forces vertically,

$$R - W = \int \mu r ds \sin\theta = \mu \int_0^{2a} \sin\omega\, s ds = 2\mu a^2 \sin\omega \text{.........(2)}.$$

Taking moments about O,

$$R \, . \, 2a \cos \omega = Wa \cos \omega + R' \, . \, 2a \sin \omega,$$

$$2R \cos \omega = W \cos \omega + 2R' \sin \omega \dots\dots\dots\dots\dots (3),$$

and therefore, substituting in this equation the values of R' and R from (1) and (2), we have

$$2 \cos \omega \, (W + 2\mu a^2 \sin \omega) = W \cos \omega + 2 \sin \omega \, . \, 2\mu a^2 \cos \omega,$$

and therefore $\quad W \cos \omega = 0, \quad \omega = \frac{1}{2}\pi,$

or the beam lies in contact with the vertical plane OB.

It is evident, however, that this is not the only position of equilibrium; the beam will plainly remain at rest if it be placed in contact with the horizontal plane OA with one extremity at O. In writing down the equations (1), (2), (3), it is tacitly assumed that the beam receives no pressure from the planes excepting at its extremities, an hypothesis which holds good in the former position of equilibrium while it evidently does not in the latter: it is for this reason that, in our analytical investigation, out of the two admissible values 0 and $\frac{1}{2}\pi$ for ω we obtained only the latter.

(10) A rigid rod AB (fig. 35), the lower extremity A of which is attached to a hinge about which it can revolve freely, rests against a smooth vertical wall CD: to find the pressure on the wall and hinge.

Let G be the centre of gravity of the rod, at which we may suppose its whole weight W to be collected; let $AG = b$, $AB = a$, $\angle BAC = \alpha$. Also let R denote the reaction of the wall against the rod, which will take place at right angles to CD; and let R', S, be the vertical and horizontal parts of the reaction of the hinge upon the rod. Then

$$R = \frac{Wb}{a \tan \alpha} = S, \qquad R' = W.$$

This problem was first proposed under a vicious form in a work by Stone; where the author proposes to determine the position of AB corresponding to a maximum value of R. In a

review of Stone's work by John Bernoulli[1], the solution given by Stone was declared to be erroneous, and a different one was offered by the reviewer. Bernoulli's solution is, however, essentially vicious. The problem was correctly solved for the first time by Couplet[2]. The opinions however, both of mathematicians and of architects, were for many years divided as to the respective merits of the solutions given by Bernoulli and by Couplet, and even down to very late years numerous memoirs have appeared on the subject by different mathematicians, with various conclusions; several of whom have arrived at results at variance with the solutions both of Bernoulli and of Couplet. The reader who may be curious to examine the various solutions of this problem, which by the aberrations of the learned rather than by any intrinsic difficulty has obtained considerable celebrity, is referred to a memoir by Franchini in the *Memorie della Societa Italiana*, Tom. XVI. parte 1, p. 228; 1813.

(11) A ladder of uniform thickness rests with its lower end upon a smooth horizontal plane, and its upper end on a slope inclined at an angle of 60° to the horizon; the ladder makes an angle of 30° with the horizon: to find the force which must act horizontally at the foot of the ladder to prevent sliding.

If W denote the weight of the ladder,

$$\text{the required force} = \frac{3^{\frac{1}{2}}}{4} W.$$

(12) A sphere rests upon two inclined planes; to find the pressure experienced by each.

Let W be the weight of the sphere; α, α', the inclinations of the inclined planes to the horizon; and R, R', their respective pressures. Then

$$R = \frac{W \sin \alpha'}{\sin(\alpha + \alpha')} \quad R' = \frac{W \sin \alpha}{\sin(\alpha + \alpha')}.$$

Leibnitz; *Opera*, Tom. III. p. 176.

(13) A globe of given uniform density is supported by the rim of a circular hole in the floor: to find the radius of the globe when its whole pressure on the rim is a minimum.

[1] *Opera*, Tom. IV. p. 189. [2] *Mémoires de l'Académie de Paris*, 1781, p. 69.

If a be the diameter of the hole, the required radius of the globe is equal to $\frac{a}{\sqrt{3}}$.

(14) A sphere, of which C is the centre, is supported on an inclined plane AB by a horizontal string CB; to find the tension of CB.

If W be the weight of the sphere, and α the inclination of the plane to the horizon,

$$\text{the tension of the string} = W \tan \alpha.$$

(15) A given weight P is suspended from the rim of a uniform hemispherical bowl placed on a horizontal plane; to find the position in which the bowl will rest.

If W denote the weight of the bowl, r the radius of the sphere, c the distance between its centre and its centre of gravity, and θ the inclination of the axis of the bowl to the vertical,

$$\tan \theta = \frac{Pr}{Wc}.$$

(16) A rigid rod without weight passes through two fixed rings, and is urged by a force P in the direction of its length against a plane to which it is inclined at an angle α: to find the pressure on the plane.

The required pressure is equal to $P \operatorname{cosec} \alpha$.

(17) One end of a beam, the weight of which is W, is placed on a smooth horizontal plane; the other end, to which a string is fastened, rests against another smooth plane, inclined at an angle α to the horizon; the string, passing over a pully at the top of the inclined plane, hangs vertically, supporting a weight P: to find the condition of equilibrium.

If a = the length of the beam, and b = the distance of its centre of gravity from its lower end, the condition of equilibrium is expressed by the equation

$$Pa = Wb \sin \alpha,$$

which shews that, if the beam can rest in any one position, it will rest in all positions.

(18) A uniform beam rests upon two perfectly smooth inclined planes; to find its position and its pressures upon the two planes.

Let α, α', be the inclinations of the two planes to the horizon; R, R', the pressures which they experience; then, supposing the end of the beam which rests against the former plane to be the lower one, and θ to be the inclination of the beam to the horizon, we shall have, W being the weight of the beam,

$$\tan\theta = \frac{\sin(\alpha' - \alpha)}{2\sin\alpha' \sin\alpha}, \quad R = \frac{W\sin\alpha'}{\sin(\alpha + \alpha')}, \quad R' = \frac{W\sin\alpha}{\sin(\alpha + \alpha')}.$$

(19) A uniform beam ABC (fig. 36) is placed with one end A on the inner surface of a fixed hemispherical bowl, the diameter of which is less than the length of the beam, and is in contact with the rim of the bowl at the point B; to find the position in which the beam will rest, the radius OB of the bowl being horizontal.

If r be the radius of the bowl, $2a$ the length of the beam, and θ its angle of inclination to the horizon; then

$$4r\cos^2\theta - a\cos\theta - 2r = 0.$$

(20) To find the position of equilibrium of a uniform beam, one end of which rests against a vertical plane, and the other on the interior surface of a given hemisphere.

Let r be the radius of the hemisphere, c the distance of its centre from the vertical plane, $2a$ the length of the beam; θ the inclination of the beam to the horizon, and ϕ of the radius at the point where the beam presses against the hemisphere. Then the position of equilibrium will depend upon the equations

$$\tan\phi = 2\tan\theta, \quad 2a\cos\theta = r\cos\phi + c.$$

(21) A beam AB (fig. 37) leans against a smooth vertical prop CD, the end A being prevented from sliding along the horizontal plane AD by a string AD fastened at D; to find the tension of the string.

Let G be the centre of gravity of the beam; $AG = a$, $CD = b$, $AD = c$, $W =$ the weight of the beam, $T =$ the tension: then

$$T = \frac{abc}{(b^2 + c^2)^{\frac{3}{2}}} W.$$

(22) A uniform rigid rod AB (fig. 38) rests upon a fixed point E, while its lower end A presses against a vertical line FF'; a weight P is suspended from the extremity B; to find the position of equilibrium of the rod.

Let W = the weight of the rod, b = the perpendicular distance of E from the line FF', $AE = x$, a = the length of the rod; then

$$x = \left(ab^2 \frac{P + \frac{1}{2}W}{P + W}\right)^{\frac{1}{3}}.$$

Fontana; *Memorie della Societa Italiana*, 1802, p. 630.

If we suppose $W = 0$, then we shall have $x = (ab^2)^{\frac{1}{3}}$, whatever be the magnitude of P. This problem is discussed by Euler, *Acad. des Sciences de Berlin*, Tom. VII. p. 196, in illustration of Maupertuis' Principle of Rest.

(23) One end of a beam is connected with a horizontal plane by a hinge about which the beam can revolve freely in a vertical plane; the other end is attached to a weight by means of a string passing over a pully in the same vertical plane; to find the position of equilibrium.

Let a, b, be the distances of the centre of gravity of the beam from its lower and its higher extremities, W its weight, and θ its inclination to the horizon; let ϕ be the inclination of the string to the horizon, and P the weight attached to its extremity; let l be the distance of the pully from the horizontal and k from the vertical line through the hinge. Then the position of equilibrium will depend upon the equations

$$P(a + b) \sin (\phi - \theta) = Wa \cos \theta,$$
$$(a + b) \sin (\phi - \theta) = k \sin \phi - l \cos \phi.$$

(24) A uniform beam rests with one end upon a given inclined plane, the other end being suspended by a string from a fixed point above the plane; to determine the position of equilibrium, the tension of the string, and the pressure on the plane.

Let $2a$ be the length of the beam, θ its inclination to the inclined plane, W its weight, and R the pressure which it exerts on the inclined plane; let T be the tension of the string, c its

length, and ϕ its inclination to the inclined plane; also let b be the distance of the fixed point from the plane; and α the inclination of the plane to the horizon.

Then the position of the beam will depend upon the two equations

$$2 \sin(\phi - \theta) \sin \alpha = \cos \phi \cos(\theta + \alpha),$$

$$c \sin \phi + 2a \sin \theta = b;$$

and then R and T will be given by the formulæ

$$R = \frac{W \cos(\alpha + \phi)}{\cos \phi}, \quad T = \frac{W \sin \alpha}{\cos \phi}.$$

(25) A uniform beam rests with one end against a smooth vertical plane, its other end being supported by a string attached to a fixed point in the plane; to determine the position of the beam, its pressure against the plane, and the tension of the string.

Let b be the length and T the tension of the string; $2a$ the length of the beam, W its weight, and R its pressure against the vertical plane; also let ϕ, θ, be the inclinations of the beam and of the string to the vertical. Then

$$\sin \theta = \left(\frac{16a^2 - b^2}{3b^2}\right)^{\frac{1}{2}}, \quad \sin \phi = \left(\frac{16a^2 - b^2}{12a^2}\right)^{\frac{1}{2}},$$

$$T = \frac{3^{\frac{1}{2}} b W}{(4b^2 - 16a^2)^{\frac{1}{2}}}, \quad R = \left(\frac{16a^2 - b^2}{4b^2 - 16a^2}\right)^{\frac{1}{2}} . W.$$

(26) A weight W hangs from a rod BC (fig. 39), which rests on a fulcrum at B, and is supported by a string DA at right angles to the rod, D being the middle point of BC; to determine the magnitude and direction of the pressure on the fulcrum, the rod being inclined to the horizon at an angle of 30°, and being without weight.

Let $BD = CD = a$; and let X, Y, represent the vertical and horizontal components of the pressure exerted by the rod on the fulcrum; then

$$X = \tfrac{1}{2} W, \quad Y = \frac{3^{\frac{1}{2}}}{2} W;$$

and, if ϕ denote the inclination of the resultant pressure to the vertical, and R its magnitude,

$$R = W, \quad \phi = \tfrac{1}{3}\pi.$$

(27) A uniform beam AB (fig. 40), moveable in a vertical plane about a hinge at A, leans upon a prop CD fixed in the same plane; to determine the normal strain upon the prop CD.

Let $AB = 2a$, $CD = b$, $\angle BAC = \alpha$, $\angle ACD = \beta$. Then the resolved part of the pressure of AB on CD at right angles to CD, which is the normal strain on the prop, will be equal to

$$\frac{Wa \sin 2\alpha \cos(\alpha + \beta)}{2b \sin \beta}.$$

(28) A uniform beam is hung from a fixed point by two unequal strings attached to its extremities: to compare the tension of each string with the weight of the beam.

Let a, b, represent the lengths of the strings, P, Q, their respective tensions, c the length and W the weight of the beam;

then $$\frac{P}{W} = \frac{a}{(2a^2 + 2b^2 - c^2)^{\frac{1}{2}}}, \quad \frac{Q}{W} = \frac{b}{(2a^2 + 2b^2 - c^2)^{\frac{1}{2}}}.$$

(29) An isosceles right-angled triangle rests in a vertical plane with the right angle downwards, between two pegs at a distance a from each other in the same horizontal line; to determine its positions of equilibrium.

Let h = the perpendicular from the right angle on the base, and θ = the inclination of the base to the horizon; then

$$\theta = 0, \text{ or } \theta = \cos^{-1}\left(\frac{h}{3a}\right).$$

(30) A flat board DE (fig. 41), in the form of a square, is supported upon two fixed points P, Q, with its plane vertical, the distance between P, Q, being equal to half a side of the square: to find the positions of equilibrium.

If α be the inclination of PQ and θ of AE to the horizon, the positions of equilibrium are given by the equation

$$\sin^2(2\theta + \alpha) = \sin 2\theta.$$

(31) A uniform rod of given length rests against a peg at the focus of a parabola, the axis of which is vertical and of which the vertex is the lowest point, the lower extremity of the rod being supported on the curve; to determine the angle which the rod makes with the axis of the parabola.

If a be the length of the rod, and $4m$ the latus rectum of the parabola; then

$$\text{the required angle} = 2\cos^{-1}\left(\frac{m}{a}\right)^{\frac{1}{4}}.$$

(32) A uniform rigid rod, of length a, can turn in a horizontal plane about its middle point: at one end a string is tied which passes over a fixed pully, vertically over that end, and at a distance b from it, and is then fastened to a given weight: the rod is then turned through an angle θ, and kept at rest in that position by a horizontal force P perpendicular to the rod through its other end: to find the value of θ when P is a maximum.

The required value of θ is given by the equation

$$\tan^4\frac{\theta}{2} = \frac{b^2}{a^2+b^2}.$$

(33) A uniform isosceles triangle is placed within a smooth hemispherical bowl, its three angular points touching the bowl; to find the position in which it will rest.

Let a = the length of each of the equal sides, h = the altitude of the triangle, r = the radius of the hemisphere, θ = the inclination of the triangle to the vertical; then

$$\tan\theta = \frac{3\,(4r^2h^2 - a^4)^{\frac{1}{2}}}{4h^2 - 3a^2}.$$

(34) A uniform circular lamina is placed with its centre upon a prop; to find at what points on its circumference three weights w_1, w_2, w_3, must be attached that it may remain at rest in a horizontal position.

If θ_1, θ_2, θ_3, be the angles included between the radii of the points of attachment of (w_2, w_3), (w_3, w_1), (w_1, w_2), respectively, then

$$\cos\theta_1 = \frac{w_1^2 - w_2^2 - w_3^2}{2w_2w_3}, \quad \cos\theta_2 = \frac{w_2^2 - w_3^2 - w_1^2}{2w_3w_1},$$

$$\cos\theta_3 = \frac{w_3^2 - w_1^2 - w_2^2}{2w_1w_2}.$$

(35) A hemisphere is fixed with its base on the ground between two parallel vertical planes, both of which touch it, and of which one reaches to a height equal to the diameter: a beam of given length and weight, supported by the hemisphere, rests over the top of the finite plane, one of its ends pressing against the indefinite plane: to find the pressures of the beam on the planes and hemisphere, and to determine the greatest length of the beam for which there can exist any pressure on the hemisphere.

Let r = the radius of the hemisphere, $2a$ = the length of the beam, R = the pressure on the top of the finite plane, S = the pressure on the indefinite plane, T = the pressure on the sphere. Then, W denoting the weight of the beam,

$$R = \frac{32a - 25r}{80r} . W, \quad S = \tfrac{3}{4} W,$$

$$T = \frac{125r - 32a}{80r} . W.$$

SECT. 2. *Friction.*

Statical friction consists in the resistance arising from mutual roughness, which is opposed to the production of relative motion between two substances in contact. If the substances were perfectly smooth, their mutual pressure at every point of the surfaces of contact would take place in some determinate straight line depending upon the forms of the surfaces; if the consideration of roughness be introduced, the force of friction when called into play will exert itself at each point in a direction at

right angles to the mutual pressure corresponding to perfect smoothness. The estimation of the magnitude of friction for assigned substances and for given surfaces of contact, can be effected solely by experiment.

Suppose R to denote the total pressure of two substances, of which the surfaces of contact are two planes, and let F be the greatest force which friction can exert in the prevention of relative motion; then F is taken as the measure of the statical friction. After the performance of numerous experiments, Amontons[1], who was the first to discuss scientifically the subject of friction, was led to conclude that, so long as the substances remain the same, F varies directly as R, and is independent of the magnitude of the area of contact. Thus, μ denoting some constant quantity, the magnitude of which is to be obtained by experiment, we should have for any assigned substances

$$F = \mu R,$$

where μ is called the coefficient of friction. This relation, however, although generally adopted by mathematicians, is probably not quite accurate. Muschenbroek[2] and the Abbé Nolet[3] concluded from experiments that the value of μ depends in some degree upon the magnitude of the area of contact, and that for an assigned area of contact it does not remain invariable for all values of R. Bossut[4] agreed with Amontons in supposing μ to be independent of the area of contact, but considered that its value decreases as R increases. Various experimenters have given their labours to the same subject with very different conclusions. Professor Vince[5] inferred, from the performance of very careful experiments, that the coefficient of friction does really diminish with the increase of R, and that for a given pressure it decreases when the area of contact is diminished.

[1] *Mémoires de l'Académie des Sciences de Paris*, 1699, p. 206.

[2] *Introduct. ad Phil. Nat.* Tom. I. cap. 9, 1762. *Lect. Phys. Exp.* Tom. I. p. 241.

[3] *Leçons de Physique Expérimentale*, Tom. I. p. 280; 1754.

[4] *Traité de Mécanique*, Part I. chap. 4, sect. 1, p. 178.

[5] *Philosophical Transactions* 1785, Part I. p. 165.

It would appear however, from the valuable experiments of Coulomb[1] and Ximenes[2], that the variation of μ, owing to any change in the magnitude of the area of contact, is extremely small and of an irregular character, and that it decreases very slightly as R increases. Bossut[3] has remarked that the statical friction between two substances becomes greater by allowing them to remain in contact for some time before it is called into play, an observation which has been fully confirmed by the experiments of Coulomb.

If the surfaces of contact be not plane areas, the coefficient of friction will on this account receive a change of value; and generally it will depend upon the forms of the surfaces of contact, as well as upon the nature of the substances. The friction of a solid cylinder against a hollow one has been considered by Coulomb and Ximenes, who have found it to be much smaller than between two plane surfaces of the same substance; the coefficient of friction is, however, approximately constant, as in the case of plane surfaces of contact.

The friction of which we have been speaking, is the friction called into play by the *rubbing* of two substances against each other; the roughness of substances, however, exerts force to interrupt the production of relative motion also in the case when one body is urged to *roll* along another without *rubbing;* this may be called the friction of cohesion, depending probably upon the mutual tenacity of the particles of the two bodies. This species of friction was first noticed by Bossut, and afterwards carefully investigated by Ximenes and Coulomb: in the case of a cylinder rolling along a plane, the friction of cohesion is found to vary inversely as the diameter.

The friction which exists between two substances in motion, which may be called their dynamical friction, is very considerably less than their statical friction. The dynamical friction is measured by the force necessary to keep the bodies in motion; the statical friction by the force necessary to set them originally

[1] *Mémoires présent. à l'Académie*, Tom. x. 1785.

[2] *Terria e Pentica delle Resist. de' Sol. ne' loro Attr.* Pisa, 1782.

[3] *Traité de Mécanique*, Part I. chap. 4, sect. 1, p. 178.

in motion. The difference of the magnitudes of statical and dynamical friction was noticed by Camus[1] and Desaguliers[2], and afterwards by various other experimenters. Professor Vince ascertained by experiments, that dynamical friction is a constant force for hard substances, whatever be the velocity of the relative motion; but that in the case of softer bodies it increases considerably with an increase of velocity. The friction of pivots has been fully considered by Coulomb in the *Mémoires de l'Acad. des Sciences de Paris*, 1790. The friction and rigidity of ropes was first investigated experimentally by Amontons in the memoir to which we have alluded above, and afterwards by Coulomb and Ximenes.

(1) A uniform beam AB (fig. 42) rests with one end A upon a rough horizontal plane KL, its other end B being attached to a string which passes over a smooth pully E and supports a weight P; to determine the range of positions in which the beam may be placed consistently with equilibrium.

Let G be the centre of gravity of the beam, and W its weight; θ, ϕ, the angles of inclination of AB, BE, to the horizon for any position of equilibrium; let $AG = BG = a$; let F denote the friction, estimated along LK, which is called into play at A, and which will be at right angles to R, the vertical reaction of the plane on the beam. Suppose the whole weight of the beam to be collected at its centre of gravity.

Then for the equilibrium of the beam we have, resolving the forces horizontally,

$$F = P\cos\phi \quad\text{..............................(1)};$$

resolving vertically,

$$R + P\sin\phi = W \quad\text{........................(2)};$$

and, taking moments about A,

$$Wa\cos\theta = P \,.\, 2a\sin(\phi - \theta),$$

or

$$W\cos\theta = 2P\sin(\phi - \theta) \quad\text{..................(3)}.$$

Assume $F = \lambda R$, where, if μ denote the coefficient of friction

[1] *Traité des Forces Mouvantes.* [2] *Cours de Physique.*

between the end of the beam and the plane, λ may have any value from zero up to μ. Then by (1) we have

$$\lambda R = P \cos \phi \quad \text{.........................(4)}.$$

From (2) and (4), we obtain

$$P \cos \phi + \lambda P \sin \phi = \lambda W,$$

or, putting $\lambda = \tan \epsilon$,

$$P \cos (\phi - \epsilon) = W \sin \epsilon,$$

which determines the angle ϕ in terms of W, P, ϵ; and then θ may be determined from (3). By giving then to ϵ any values from zero up to $\tan^{-1} \mu$, we shall obtain a series of positions of equilibrium.

Suppose for instance λ to be equal to zero; then from (4)

$$P \cos \phi = 0, \text{ and therefore } \phi = \tfrac{1}{2}\pi;$$

hence, by (3), $$W \cos \theta = 2P \cos \theta,$$

and therefore either $W = 2P$, in which case θ remains indeterminate and may have any value whatever, or $\theta = \frac{1}{2}\pi$. Again from (2), since $\phi = \frac{1}{2}\pi$, we have

$$R = W - P \quad \text{.........................(5)},$$

and therefore, if θ be not equal to $\frac{1}{2}\pi$, we must have

$$R = P = \tfrac{1}{2} W.$$

Thus we see that the end B of the beam must be in the vertical line through E; and that, unless AB be placed vertically, the weight P must be equal to half the weight of the beam. If the beam be placed vertically, it is clear from (5) that P may have any value from 0 up to W, but no greater value, because R cannot be negative.

If, instead of taking $\lambda = 0$, we were to give it any other value between 0 and μ, we should have to determine the values of θ and ϕ as in the present case.

(2) A beam AB (fig. 43) is supported on a prop CD by a given force P acting at a given angle of inclination to the horizon; to find the position of the beam when it is upon the point

of sliding past the point C from A towards B, the prop and beam being relatively rough.

Produce BA, PA, to meet the horizontal line KL in the points F, E; let G be the centre of gravity of the beam. Let $AG = a$, $CG = x$, $\angle PEL = \alpha$, $\angle AFE = \theta$, $R =$ the reaction of the prop at right angles to AB, and $\mu =$ the coefficient of friction; then μR will be the friction, of which BA is the direction.

Then for the equilibrium of the beam we have, resolving forces vertically,

$$P \sin \alpha + R \cos \theta = W + \mu R \sin \theta \ldots\ldots\ldots\ldots\ldots (1);$$

resolving horizontally,

$$P \cos \alpha = R \sin \theta + \mu R \cos \theta \ldots\ldots\ldots\ldots\ldots\ldots (2);$$

and, taking moments about C,

$$Wx \cos \theta = P(a + x) \sin (\alpha - \theta) \ldots\ldots\ldots\ldots\ldots (3).$$

From the equations (1) and (2) there is

$$\frac{\cos \theta - \mu \sin \theta}{\sin \theta + \mu \cos \theta} = \frac{W - P \sin \alpha}{P \cos \alpha},$$

and therefore

$$P \cos \alpha (1 - \mu \tan \theta) = (W - P \sin \alpha)(\tan \theta + \mu),$$

$$P(\cos \alpha + \mu \sin \alpha) - \mu W = \{W + P(\mu \cos \alpha - \sin \alpha)\} \tan \theta;$$

assume $\mu = \tan \epsilon$; then, multiplying both sides of the equation by $\cos \epsilon$,

$$P \cos (\epsilon - \alpha) - W \sin \epsilon = \{P \sin (\epsilon - \alpha) + W \cos \epsilon\} \tan \theta,$$

$$\tan \theta = \frac{P \cos (\epsilon - \alpha) - W \sin \epsilon}{P \sin (\epsilon - \alpha) + W \cos \epsilon},$$

which determines the inclination of the beam to the horizon.

Knowing θ we may determine x from the equation (3); and thus the position of the beam will be completely ascertained.

If the beam be on the point of sliding in a direction opposite to that which we have supposed, the quantity μ must be replaced by $-\mu$, or ϵ by $-\epsilon$; and the formulæ for the former case will all become adapted to the latter.

(3) A uniform rectangular board $KLMN$ (fig. 44) is placed upon a rough inclined plane AB: supposing the inclination of the plane AB to the horizon to be gradually increased, to find whether the equilibrium of the board will be disturbed by the commencement of a rolling or of a sliding motion.

First suppose that the board begins to slide; let R be the whole of the reaction of the plane at right angles to itself on the board, μ the coefficient of friction, and ϕ the inclination of the plane at the commencement of sliding. Then, resolving forces parallel to the inclined plane,

$$\mu R = W \sin\phi;$$

and, resolving forces at right angles to it,

$$R = W \cos\phi;$$

hence, eliminating R,

$$\tan\phi = \mu.$$

Next suppose that the board tumbles over the corner K before the commencement of sliding; then the vertical through G will pass through K when ϕ has received the proper value; draw GH at right angles to the plane, let $HK = a$, $GH = b$; then

$$\tan\phi = \tan \angle KGH = \frac{a}{b}.$$

Hence, if μ be less than $\frac{a}{b}$, sliding will take place before rolling; on the contrary, if μ be greater than $\frac{a}{b}$, rolling will take place before sliding; if μ be equal to $\frac{a}{b}$, rolling and sliding will take place simultaneously.

(4) A beam PQ (fig. 45), which is capable of free motion in every direction about a smooth hinge at P, rests with its end Q against a rough vertical plane ABC; to determine the position of the beam when it is bordering on motion.

From P draw PO at right angles to the plane ABC; join OQ; the locus of Q will be a circle in the vertical plane having

O for its centre; let G be the centre of gravity of the beam; PHV be the projection on the horizontal plane through PO of the line PGQ, H and V being the projections of G and Q; draw HK at right angles to PO; let W be the weight of the beam, μ the coefficient of friction between the beam and the vertical plane, and R their mutual pressure; μR will act in the tangent to the locus of Q at the point Q, that is, at right angles to OQ and in the plane ABC, and from A towards B;

let $\quad PG = a,\ QG = b,\ \angle QPO = \alpha,\ \angle QOA = \theta.$

Then for the equilibrium of the beam we have, taking moments about PO,

$$W \,.\, HK = \mu R \,.\, OQ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1);$$

and, taking moments about the horizontal line through P, which is at right angles to PO, it being observed that the vertical resolved part of μR is $\mu R \cos \angle QOV$,

$$W.PK = R \,.\, QV + \mu R \,.\, PO \cos \angle QOV \ldots\ldots\ldots\ldots(2).$$

Now, from the geometry,

$$HK = GK \cos\theta = a \sin\alpha\cos\theta,$$

$$OQ = (a+b)\sin\alpha,\quad PO\cos\angle QOV = (a+b)\cos\alpha\cos\theta,$$

$$PK = a\cos\alpha,\quad QV = OQ\sin\theta = (a+b)\sin\alpha\sin\theta;$$

hence, from the equations (1) and (2),

$$Wa\sin\alpha\cos\theta = \mu R\,(a+b)\sin\alpha,$$

and $\quad Wa\cos\alpha = R\,(a+b)\sin\alpha\sin\theta + \mu R\,(a+b)\cos\alpha\cos\theta:$

dividing the latter of these equations by the former,

$$\frac{\cos\alpha}{\sin\alpha\cos\theta} = \frac{\sin\alpha\sin\theta + \mu\cos\alpha\cos\theta}{\mu\sin\alpha},$$

$$\mu\cos\alpha = \cos\theta\,(\sin\alpha\sin\theta + \mu\cos\alpha\cos\theta),$$

$$\mu\cos\alpha\sin^2\theta = \sin\alpha\sin\theta\cos\theta,$$

$$\tan\theta = \frac{1}{\mu}\tan\alpha.$$

We may solve this problem also in the following manner: taking moments about the vertical line through P we have, since $\mu R \sin\theta$ is the horizontal resolved part of μR,

$$R \,.\, OV = \mu R \,.\, \sin\theta \,.\, PO,$$

and therefore $$OQ\cos\theta = \mu\sin\theta \,.\, PO;$$

but $$OQ = OP\tan\alpha;$$

hence $$\tan\theta = \frac{1}{\mu}\tan\alpha.$$

(5) A beam AB (fig. 46) is placed with one end upon a rough horizontal plane Ox, and rests against a rough plane curve KPL at any point P; supposing that, whatever be the point P against which the beam leans, it is always in an equilibrium bordering on motion, and that the coefficient of friction is the same both for the curve and for the horizontal plane, to find the nature of the curve.

Draw PM at right angles to Ox; let G be the centre of gravity of the beam, W its weight, $AG = a$, $\angle BAx = \theta$, $OM = x$, $PM = y$, $\mu =$ the coefficient of friction; let R and R' be the normal reactions of the curve and of the plane against the beam; in consequence of friction the curve will exert on the beam a force μR along PB, and the horizontal plane a force $\mu R'$ along Ax.

Hence for the equilibrium of the beam, resolving forces parallel to Ox,

$$R\sin\theta = \mu R\cos\theta + \mu R',$$

$$R(\sin\theta - \mu\cos\theta) = \mu R' \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1);$$

resolving forces perpendicularly to Ox,

$$R\cos\theta + \mu R\sin\theta + R' = W,$$

$$R(\cos\theta + \mu\sin\theta) + R' = W \quad\ldots\ldots\ldots\ldots\ldots\ldots (2);$$

and, taking moments about A,

$$R \,.\, AP = Wa\cos\theta, \text{ or } R \,.\, AM = Wa\cos^2\theta \quad\ldots\ldots\ldots\ldots (3).$$

From (1) and (2) we get

$$(1 + \mu^2)\, R\sin\theta = \mu W,$$

and therefore, from (3),

$$(1+\mu^2)\,Wa\sin\theta\cos^2\theta=\mu W.AM,$$

$$(1+\mu^2)\,a\sin\theta\cos^2\theta=\mu.AM;$$

but $$\sin\theta=\frac{dy}{ds},\quad \cos\theta=\frac{dx}{ds},\quad AM=y\frac{dx}{dy};$$

hence we have

$$(1+\mu^2)\,a\frac{dy}{ds}\frac{dx^2}{ds^2}=\mu y\frac{dx}{dy},$$

$$a(1+\mu^2)\frac{dy^2}{dx^2}=\mu y\frac{ds^3}{dx^3};$$

put $\mu=\tan\epsilon$, and this equation becomes

$$\frac{2a}{\sin 2\epsilon}\frac{dy^2}{dx^2}=y\frac{ds^3}{dx^3};$$

which is the differential equation to the curve.

If the friction of the curve and the plane be different, we may obtain the differential equation to the curve with equal ease.

(6) A uniform rod passes over the fixed point A and under the fixed point B, (fig. 47), and is kept at rest by the friction at the points A and B; to determine the circumstances of equilibrium.

Let μR, μS, be the forces of friction at A, B, respectively, R and S being the normal actions of the fixed points on the rods. Let G be centre of gravity of the rod.

Let $AB=a$, $\alpha=$ the inclination of AB to the horizon, $2b$ = the length of the rod, and $AG=x$.

Resolving forces along the rod, we have

$$\mu(R+S)=W\sin\alpha \text{......................}(1);$$

resolving perpendicularly to the rod, we have

$$R=W\cos\alpha+S \text{........................}(2);$$

and, taking moments about G,

$$Rx=S(x+a) \text{........................}(3).$$

From (1) and (2),

$$2\mu S = W(\sin\alpha - \mu\cos\alpha) \dots\dots\dots\dots\dots\dots (4).$$

From (2) and (3) there is

$$aS = xW\cos\alpha \dots\dots\dots\dots\dots\dots\dots\dots (5);$$

and therefore, by (4) and (5),

$$x \,.\, 2\mu\cos\alpha = a(\sin\alpha - \mu\cos\alpha),$$

$$x = \frac{a}{2\mu}(\tan\alpha - \mu) \dots\dots\dots\dots\dots\dots\dots (6).$$

Since S cannot be negative, therefore, by (5), x cannot be negative. Moreover, from the geometry, it is plain that

$$x \angle b - a \dots\dots\dots\dots\dots\dots\dots\dots\dots (7).$$

Let λ be the coefficient of friction. Then μ may have any value between 0 and λ which gives to x, as determined by the equation (6), a positive value consistent with the inequality (7).

If
$$\frac{a}{2\lambda}(\tan\alpha - \lambda) > b - a,$$

or
$$\lambda < \frac{a\tan\alpha}{2b - a},$$

equilibrium is impossible.

(7) A beam rests with its lower extremity on a horizontal, and its higher against a vertical plane; having given its length, the position of its centre of gravity, and the coefficients of the friction of the horizontal and of the vertical plane, to find its position when in a state bordering on motion.

If a, b, be the distances of the centre of gravity of the beam from its lower and higher extremity; μ, μ', the coefficients of friction between the beam and the horizontal, and between the beam and the vertical plane; and θ the inclination of the beam to the horizon; then

$$\tan\theta = \frac{a - \mu\mu' b}{\mu(a+b)}.$$

(8) A uniform and straight plank rests with its middle point upon a rough horizontal cylinder, their directions being

perpendicular to each other; to find the greatest weight which can be suspended from one end of the plank without its sliding off the cylinder.

Let W be the weight of the plank, and P the attached weight; r the radius of the cylinder, $2a$ the length of the plank, $\tan\lambda$ the coefficient of friction. Then P will be given by the relation

$$\frac{P}{W}=\frac{r\lambda}{a-r\lambda}.$$

(9) A uniform rod rests over a smooth peg, its lower end being supported by a rough horizontal plane: to find its position of equilibrium when bordering upon motion.

If $2a$ = the length of the rod, h = the height of the peg above the horizontal plane, and $\tan\epsilon$ = the coefficient of friction: then θ, the inclination of the rod to the horizon in the required position, is determined by the equation

$$a\sin 2\theta \,.\, \sin(\theta+\epsilon)=2h\sin\epsilon.$$

(10) A uniform beam AB, (fig. 48), of which the end B presses against a rough vertical plane CD, is supported by a fine string AC attached to a fixed point C in the plane; to find the position of the beam when bordering upon motion.

Let the point B be on the point of ascending; let μ = the coefficient of friction, a = the length of the beam, $CA=l$, $\angle ACB=\theta$, $\angle ABD=\phi$. Then θ may be found from the equation

$$(4a^2-4l^2-\mu^2l^2)\tan^2\theta-2\mu l^2\tan\theta+4a^2-l^2=0;$$

and then ϕ may be determined by the equation

$$a\sin\phi=l\sin\theta.$$

If B be on the point of sliding downwards, μ must be replaced by $-\mu$.

(11) A uniform rod rests within a rough circle, the plane of which is vertical: to investigate the position of the rod when the friction can only just maintain the equilibrium.

If α denote the angle between the rod and the radius through

either extremity, $\tan\epsilon$ the coefficient of friction, and θ the inclination of the rod to the horizon in the required position,

$$\tan\theta = \frac{\sin 2\epsilon}{\cos 2\epsilon - \cos 2\alpha}.$$

(12) A homogeneous solid hemisphere is capable of rolling on its curve surface upon a horizontal plane, the friction being such as to prevent all sliding; to find the moment of a couple which may keep it at rest with its base inclined at an angle of 30° to the horizon.

If W be the weight and a the radius of the hemisphere, the moment of the couple will be equal to $\frac{3}{16}Wa$.

(13) A sphere of radius a is just supported on a rough plane, inclined at an angle of 45° to the horizon, by a weightless rod, the lower extremity of which is attached by a hinge to the inclined plane, and the higher to the surface of the sphere, at a point where the radius is parallel to the plane; the rod and the centre of the sphere lying in a vertical plane which cuts the inclined plane at right angles. To find the length of the rod, the coefficient of friction being equal to $\tan\epsilon$.

The length of the rod is equal to $a \operatorname{cosec}\epsilon$.

(14) A heavy uniform rough bar is placed over one and under the other of two fixed rough rods which are parallel to each other, and horizontal, in a vertical plane at right angles to them: to find the length of the shortest bar which will rest in such a position.

Let the distance between the two rods be a, and its inclination to the horizon α: then, μ being the coefficient of friction, the required length of the bar is equal to

$$a\left(1 + \frac{1}{\mu}\tan\alpha\right).$$

(15) A square board $ABCD$, (fig. 49), the plane of which is vertical, rests with its side AD in contact with a rough vertical wall, which is perpendicular to the plane of the board; the side AB resting, at a point indefinitely near to B, upon a rough peg:

to find the least value of the coefficient of friction, supposing it to be the same for the wall and for the peg.

The least value of the coefficient of friction is equal to $\sqrt{(2)} - 1$.

(16) An elliptical cylinder, placed between a smooth vertical plane and a rough horizontal one, the major axis of the ellipse being inclined at an angle of 45° to the horizon, is just prevented by friction from sliding; to find the coefficient of friction.

If e be the eccentricity of the ellipse, the coefficient of friction will be equal to $\frac{1}{2}e^2$.

(17) An elliptical board, the plane of which is vertical, rests upon a rough horizontal plane, the coefficient of friction of the plane being μ, and leans against a rough vertical wall, the plane of which is perpendicular to that of the board and of which the coefficient of friction is μ'. The major axis of the ellipse is inclined at an angle of forty-five degrees to the horizon, and the board is just on the point of sliding down: to find the eccentricity of the ellipse.

The eccentricity is equal to the square root of the quantity

$$\frac{2\mu(1+\mu')}{1+\mu\mu'}.$$

(18) A straight uniform beam is placed upon two rough planes, of which the inclinations to the horizon are α and α', and the coefficients of friction $\tan\lambda$ and $\tan\lambda'$; to find the limiting value of the angle of inclination of the beam to the horizon at which it will rest, and the relation between the weight of the beam and each of the two normal pressures upon the planes.

Let θ be the required limiting angle; R, R', the normal pressures on the planes; and W the weight of the beam. Then

$$2\tan\theta = \cot(\alpha'+\lambda') - \cot(\alpha-\lambda),$$

$$\frac{R}{\cos\lambda\sin(\alpha'+\lambda')} = \frac{W}{\sin(\alpha-\lambda+\alpha'+\lambda')} = \frac{R'}{\cos\lambda'\sin(\alpha-\lambda)}.$$

(19) A beam, moveable about a smooth hinge at its lower end, rests with its other extremity on the surface of a fixed rough

sphere, the centre of which is in the same horizontal line with the hinge: to find the limiting positions of equilibrium.

Let A be the lower and B the upper end of the rod, C being the centre of the sphere: let $\angle BAC = \beta$, $\angle BCA = \alpha$: let θ be the inclination of the plane ABC to the horizon in a limiting position of equilibrium: then

$$\tan\theta = \frac{1}{\mu}\cos\alpha\tan\beta.$$

(20) A right cone is placed on its base upon a rough inclined plane, the inclination of which is gradually increased: to investigate the condition that a motion of rolling and of sliding may take place simultaneously.

If β denote the angle of indifference, and 2α the vertical angle of the cone, the required condition is expressed by the equation

$$\tan\beta = 4\tan\alpha.$$

(21) A uniform rectangular plank AB, (fig. 50), of given weight W, is just supported against a rough vertical wall BC by a weight P suspended at one end of a string which passes through a ring at O, vertically above B, and of which the other end is tied to A. To find the least value of the normal pressure on the wall, and the corresponding magnitude of P.

If $\tan\epsilon$ denote the coefficient of friction, the least value of the normal pressure is $\frac{1}{2}W\cot\epsilon$, and the corresponding magnitude of P is $\frac{1}{2}W\operatorname{cosec}\epsilon$.

(22) When a person tries to pull out a two-handled drawer by pulling one of the handles in a direction perpendicular to its front, to find the condition under which the drawer will stick fast.

The drawer will stick fast, whatever be the force employed, if the coefficient of friction be not less than the ratio of the length of either side of the drawer to the distance between its handles.

CHAPTER IV.

EQUILIBRIUM OF SEVERAL BODIES.

IF there be a system of bodies mutually acting on each other by contact, by connecting rods, or in any conceivable way, it will be necessary, in the determination of the circumstances of equilibrium, to represent the unknown actions and reactions by appropriate symbols. We shall then have to write down the equations of equilibrium for each body separately, including among the known forces to which it is subject, the unknown actions which it experiences from its connection with the other bodies of the system. From these different sets of equations, taken conjointly, we shall have to determine the circumstances of equilibrium.

SECT. 1. *No Friction.*

(1) AB (fig. 51) is a uniform beam, capable of motion about its middle point D; a beam CE, moveable about a hinge C in the vertical line through D, presses against the beam AB, from the extremity B of which a weight P is suspended; to determine the positions of equilibrium of the beams, having given that CD is equal to AD or BD.

Let $AD = CD = BD = a$, $\angle ACD = \theta$; $GC = b$, G being the centre of gravity of the beam CE; $R =$ the action and reaction of the two beams at A; $W =$ the weight of the beam CE. Then for the equilibrium of CE, taking moments about C, we have

$$R \,.\, 2a \cos\theta = W \,.\, b \sin\theta\,;$$

and for the equilibrium of AB, taking moments about D,

$$R \,.\, a \cos\theta = P \,.\, a \sin 2\theta, \quad \text{or } R = 2P \sin\theta:$$

from these two equations, by the elimination of R, we get

$$Wb \sin\theta = 2Pa \sin 2\theta = 4Pa \sin\theta \cos\theta,$$

and therefore $\theta = 0$, or $\cos\theta = \dfrac{bW}{4aP}$;

results which determine the required positions of the beams.

(2) Two spheres O and O', (fig. 52), rest upon two smooth inclined planes AC and AC', and press against each other; to determine their position.

Let W, W', be the weights of the spheres O, O'; R their mutual action and reaction; α, α', the inclinations of the planes AC, AC', to the horizon; θ the inclination of the line OO', joining the centres of the spheres, to the horizon.

Then for the equilibrium of the sphere O, resolving forces parallel to AC,

$$R\cos(\alpha+\theta) = W\sin\alpha;$$

and for the equilibrium of the sphere O', resolving forces parallel to AC',

$$R\cos(\alpha'-\theta) = W'\sin\alpha'.$$

Eliminating R between these two equations,

$$W\sin\alpha\cos(\alpha'-\theta) = W'\sin\alpha'\cos(\alpha+\theta),$$

$$W\tan\alpha(1+\tan\alpha'\tan\theta) = W'\tan\alpha'(1-\tan\alpha\tan\theta),$$

and therefore $\tan\theta = \dfrac{W'\cot\alpha - W\cot\alpha'}{W+W'}$.

(3) Three spheres O, O', O'', (fig. 53), are placed in contact within a hollow sphere; a vertical plane through the centre of the hollow sphere being supposed to contain the centres of the three solid spheres; to find their positions of equilibrium.

Let C be the centre of the hollow sphere; O, O', O'', the centres of the solid spheres; join OC, $O'C$, $O''C$; let W, W', W'', be the weights of the three spheres; $CO = r$, $CO' = r'$, $CO'' = r''$; $\angle OCO' = \alpha$, $\angle O'CO'' = \alpha''$; $\theta =$ the inclination of $O'C$ to the horizon.

Then, since the actions of the hollow sphere on the solid ones all three pass through the point C, we have for the equilibrium of the solid spheres, taking moments about C, observing that, if each of the spheres be in equilibrium, they would likewise be at rest if rigidly connected together as a single body,

$$Wr\cos(\theta-\alpha)+W'r'\cos\theta+W''r''\cos(\theta+\alpha'')=0,$$

$$Wr(\cos\alpha+\sin\alpha\tan\theta)+W'r'+W''r''(\cos\alpha''-\sin\alpha''\tan\theta)=0;$$

and therefore $\tan\theta=\dfrac{W''r''\cos\alpha''+W'r'+Wr\cos\alpha}{W''r''\sin\alpha''-Wr\sin\alpha}$.

(4) A hollow cylinder stands upon a horizontal plane, and a rigid imponderable rod, in a vertical plane through the axis of the cylinder, passes over the upper edge of the cylinder and rests against its interior surface: a given weight is attached to the higher extremity of the rod, and the cylinder, which is prevented from slipping by a small obstacle on the plane, is just on the point of turning over. To determine the weight of the cylinder.

Let a = the length of the rod AEB, (fig. 54), $AE=x$; c = the diameter of the cylinder and W = its weight; P = the weight suspended from B, θ = the inclination of AB to the horizon; and let R, S, denote the reactions of the cylinder against the rod.

For the equilibrium of the rod we have, resolving horizontally,

$$R\cos\theta=P\sin\theta \quad\text{.....................(1)},$$

and, taking moments about E,

$$Rx\sin\theta=P(a-x)\cos\theta,$$

or, since $x\cos\theta=c$,

$$Rc\sin\theta=P(a\cos\theta-c)\cos\theta \quad\text{..............(2)}.$$

From (1) and (2),

$$c\sin^2\theta=(a\cos\theta-c)\cos^2\theta,$$

$$\cos\theta=\left(\frac{c}{a}\right)^{\frac{1}{3}} \quad\text{...........................(3)}:$$

hence $(a-x)\cos\theta=c^{\frac{1}{3}}(a^{\frac{2}{3}}-c^{\frac{2}{3}})$.

For the equilibrium of the cylinder and rod, regarded as one system, taking moments about O, we have

$$W.\tfrac{1}{2}c=P(a-x)\cos\theta=Pc^{\frac{1}{3}}(a^{\frac{2}{3}}-c^{\frac{2}{3}}),$$

$$W=2P.\frac{a^{\frac{2}{3}}-c^{\frac{2}{3}}}{c^{\frac{2}{3}}}.$$

COR. From (1) and (3),

$$R = P\left(\frac{a^{\frac{2}{3}} - c^{\frac{2}{3}}}{c^{\frac{2}{3}}}\right)^{\frac{1}{2}},$$

and, resolving vertically for the equilibrium of the rod,

$$P = S\cos\theta,$$

$$S = P\left(\frac{a}{c}\right)^{\frac{1}{3}}.$$

(5) A sphere and cone of given weights are placed in contact on two inclined planes, the intersection of which is a horizontal line; to determine the circumstances of equilibrium.

Let W, W', be the weights of the sphere and the cone, which we may suppose to be applied at their centres of gravity G, G', (fig. 55). Let R be the action of the plane AB upon the sphere, and S the mutual action of the sphere and cone: if ϕ denote the semiangle of the cone, then evidently the line of action of S will make an angle ϕ with the plane AB'. The plane AB' will exert at right angles to itself an action upon every element of the base of the cone; the resultant of all these actions will be some force R' applied at some point E of the base of the cone in the line AB'. Let α, α', be the inclinations of the two planes to the horizon.

For the equilibrium of the sphere we have, resolving forces parallel to the plane AB,

$$W\sin\alpha = S\cos(\alpha + \alpha' - \phi) \dots\dots\dots\dots (1),$$

and, resolving forces at right angles to the plane,

$$R = W\cos\alpha + S\sin(\alpha + \alpha' - \phi) \dots\dots\dots\dots (2):$$

the equation of moments is an identical equation, since all the forces which act upon the sphere pass through its centre.

Again, for the equilibrium of the cone, resolving the forces which act upon it parallel to the plane AB',

$$W'\sin\alpha' = S\cos\phi \dots\dots\dots\dots (3);$$

resolving forces at right angles to the plane AB',

$$R' = W'\cos\alpha' + S\sin\phi \dots\dots\dots\dots (4),$$

and taking moments about G', the lines EH, mG', being represented by x, y,

$$R'x = Sy\cos\phi \dots\dots\dots\dots (5).$$

From the equations (1) and (3),

$$\frac{W \sin \alpha}{W' \sin \alpha'} = \frac{\cos (\alpha + \alpha' - \phi)}{\cos \phi} \quad \text{.................(6)},$$

from which $\tan \phi$ may be readily determined: this relation is the only condition to which the cone and sphere are subject to secure equilibrium; as will be evident when it is observed that the three equations (2), (4), (5), introduce four unknown quantities R, R', x, y, each of the three equations at least one, which are not involved in (1) and (3). From this it is evident that there will be an infinite number of positions of equilibrium, or that if ϕ only have the value given by (6), the cone and sphere will rest in contact in whatever manner they may be placed on the two planes, and whatever be their magnitudes.

The values of ϕ being determined by (6), S will be determined by (1) or (3), and therefore R, R', from (2), (4), respectively. Then from the equation (5) we may determine x, provided that y be given; and y can be given only by our knowing the magnitudes of the cone and sphere, and the particular position of equilibrium in which we may choose to place them.

(6) Two uniform rods AC, $A'C$, of which the lower extremities are situated in the same horizontal plane, and prevented from sliding, lean against each other at the point C, and are in equilibrium; to determine the relation between their angles of inclination to the horizon, the small area of mutual contact at C being vertical.

Let W, W', be the weights of the rods AC, $A'C$, respectively, and ϕ, ϕ', their angles of inclination to the horizon; then

$$W \cot \phi = W' \cot \phi'.$$

Franchini; *Memorie della Societa Italiana*, Tom. XVI. P. I. p. 237; 1813.

(7) An inextensible string binds tightly together two smooth cylinders of given radii; to find the ratio of the mutual pressure between the cylinders to the tension by which it is produced.

If R be the mutual pressure, T the tension of the string, r, r', the radii of the cylinders; then

$$\frac{R}{T} = \frac{4(rr')^{\frac{1}{2}}}{r+r'}.$$

(8) A sphere of given weight and radius is suspended by a string of given length from a fixed point, to which point also is attached another given weight by a string so long that the weight hangs below the sphere; to find the angle which the string, to which the sphere is attached, makes with the vertical.

If P denote the weight, Q the weight and sphere together, a the radius of the sphere, and b the distance of its centre from the point of suspension; then the required angle will be equal to

$$\sin^{-1}\left(\frac{Pa}{Qb}\right).$$

(9) A heavy sphere is placed upon three spheres, each equal to itself, which rest in contact on a horizontal plane: to find the pressure on each, and also the horizontal force which must be applied to each to preserve the equilibrium.

If W = the weight of each sphere, R = the pressure on each, and F = the required horizontal force; then

$$R = \frac{W}{\sqrt{6}}, \quad F = \frac{W}{3\sqrt{2}}.$$

(10) A sphere, of which C is the centre, is attached to a point O by a fine string and touches a uniform rod OB moveable in a vertical plane about a hinge at O: to find the position of equilibrium.

Let W = the weight of the sphere, W' = the weight of the rod, r = the radius of the sphere, $2a$ = the length of the rod, b = the distance between O and C, and θ = the inclination of OC to the vertical: then

$$\cot\theta = \frac{Wb^2}{W'ar} + \left(\frac{b^2}{r^2} - 1\right)^{\frac{1}{2}}.$$

(11) A rod AB (fig. 56) is fixed at a given angle of inclination to the vertical; a rod CD is attached to AB by connec-

tions at the points B, C, a weight W being suspended from the extremity D; to determine the pressures exerted by AB upon CD, the weight of CD being neglected.

Let F, G, denote the resolved parts of the pressures at B, C, on CD, estimated along its length; and R, S, the pressures at right angles to the former; let $CD = b$, $CB = c$; then, α being the inclination of the rods to the vertical,

$$R = \frac{b}{c} W \sin \alpha, \quad S = \frac{b-c}{c} W \sin \alpha,$$

$$F + G = W \cos \alpha,$$

the single value of F or G being indeterminate.

(12) A uniform rod OA, moveable about a smooth hinge at O, rests tangentially against a smooth sphere, of which C is the centre, and which is placed upon a smooth horizontal plane passing through O: the sphere is tied to O by a string. To find the tension of the string.

If $a =$ the length of the rod, $W =$ its weight, $r =$ the radius of the sphere, $c =$ the distance of C from O, and $T =$ the tension of the string,

$$T = W \cdot \frac{ar}{c^3} \cdot \frac{c^2 - 2r^2}{(c^2 - r^2)^{\frac{1}{2}}}.$$

(13) A beam AB (fig. 57) is moveable in a vertical plane about its middle point G: another beam, hanging by a string, attached to its higher end, from a point in the same plane, rests with its lower end C upon GB. To determine the position of a point E in AG at which a given weight W must be suspended so as to preserve equilibrium.

If $W =$ the weight of AB, and $P =$ that of the other beam, then

$$GE = \frac{P}{2W} \cdot CG.$$

(14) Two equal and uniform rods, equally inclined to the horizon and connected by a smooth hinge at their higher ends,

pass through two small fixed rings in a horizontal line: to find the inclination of either rod, when the rods are in a position of equilibrium.

If a be the length of each rod, b the distance between the two rings, and θ the inclination of either rod to the horizon,

$$\cos\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}.$$

(15) Two spheres A, B, (fig. 58), of equal weights and volumes, support a third sphere C, the weight of which is equal to that of A or B; the spheres A, B, being attached by equal strings to a fixed point O: to find the condition of equilibrium.

If α denote the inclination of either string, and β of either AC or BC to the vertical,

$$\tan\beta = 3\tan\alpha.$$

(16) Two equal uniform rods, equally inclined to the horizon, support a sphere which rests against their higher extremities: the lower ends of the rods are fixed to hinges in a horizontal line: to find the inclination of either rod to the horizon.

If $2a =$ the length and $W =$ the weight of each rod, $r =$ the radius and $W' =$ the weight of the sphere, and $2c =$ the distance between the two hinges, then θ, the required angle, is determined by the equation

$$(c - 2a\cos\theta)^2 . \{(W + W')^2 \cos^2\theta + W'^2 \sin^2\theta\}$$
$$= r^2 (W + W')^2 . \cos^2\theta.$$

(17) Two equal uniform rods AOB, $A'OB'$, (fig. 59), in a vertical plane, are connected together by a smooth hinge at their middle point O: their lower ends B, B', rest on a smooth horizontal plane, and their upper ends A, A', are tied together by a fine string: a sphere C is placed between them: to find the tension of the string.

If r denote the radius and W the weight of the sphere; $2a$ the length and P the weight of each rod; α the inclina-

tion of each rod to the vertical, and T the tension of the string; then

$$T = \frac{Wr\cos\alpha + (2P + W)\,a\sin^3\alpha}{2a\sin^2\alpha\cos\alpha}.$$

(18) Six thin uniform rods of equal lengths and equal given weights are connected by smooth hinge-joints at their extremities so as to constitute the six edges of a tetrahedron: one face of the tetrahedron rests on a smooth horizontal plane: to find the longitudinal strain of each of the rods of the lowest face.

If W be the weight of each rod, the required strain is equal to $\dfrac{W}{2\sqrt{6}}$.

(19) A heavy ring is suspended from a point by any number of equal strings attached to it symmetrically; and another ring, of the same weight but of smaller radius, is in equilibrium when resting on the strings at their middle points: to compare the depths of the rings below the point of suspension.

The depths are in the ratio of 2 to 3.

(20) Three equal rods are at rest, the higher ends of two of the rods being attached to fixed hinges, at an unknown distance from each other, in a horizontal line, and their lower ends to hinges at the respective ends of the third rod: to find the greatest inclination of one of the higher rods to the direction of the pressure on its higher hinge.

The required inclination is equal to $\sin^{-1}\left(\dfrac{1}{5}\right)$.

(21) Two equal balls, (fig. 60), are placed within a hollow vertical cylinder, open at both ends, which rests upon a horizontal plane: the weight of each ball is W and radius r, the radius of the cylinder being r': to find the least value of the weight of the cylinder in order that it may not be upset by the balls.

If W' = the least weight,

$$W' = 2W\left(1 - \frac{r}{r'}\right).$$

(22) A paraboloid of revolution is placed with its vertex downwards and its axis vertical, between two planes equally inclined to the horizon; to find the greatest ratio which the length of the paraboloid may have to its latus rectum, so that, if the solid be divided by a plane through its axis and the line of intersection of the inclined planes, the two parts may remain in equilibrium.

Let α = the inclination of either plane to the vertical, h = the greatest length of the axis of the paraboloid, and l = its latus rectum: then

$$\left(\frac{h}{l}\right)^{\frac{1}{2}} = \frac{15\pi}{64} \cdot \frac{1 + \sin^2\alpha}{\sin^3\alpha} \cdot \cos\alpha.$$

SECT. 2. *Friction.*

(1) The higher extremities K, K', of two equal uniform beams AK, AK', (fig. 61), which are capable of revolving in a vertical plane about a fixed point A to which their lower extremities are attached, are connected by a string KK'; a heavy sphere is placed between the two beams: supposing the string to contract, to determine its tension when the sphere is just going to be forced upwards, the friction between the sphere and each of the beams being given.

It is plain that the two beams must make equal angles with the vertical line AL which passes through A, because the centre of gravity of the system consisting of the two beams and the sphere must lie in this line.

Let R, R', denote the actions of the beams upon the sphere at right angles to their lengths, and F, F', their actions along their lengths which are due to roughness. Let 2α be the angle at which the two beams are inclined to each other, T the tension

of the string KK'; W the weight of the sphere, W' of each of the beams, and $2a$ the length of each.

Then for the equilibrium of the sphere we have, resolving forces parallel to LA,

$$(F+F')\cos\alpha + W = (R+R')\sin\alpha \;\ldots\ldots\ldots\; (1);$$

resolving at right angles to LA,

$$(F'-F)\sin\alpha = (R-R')\cos\alpha \;\ldots\ldots\ldots\ldots\ldots\; (2);$$

and taking moments about O, the centre of the sphere,

$$F.OE = F'.OE', \quad \text{or } F = F' \;\ldots\ldots\ldots\;\ldots\ldots\ldots (3).$$

From (2) and (3) we have

$$R' = R \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (4).$$

Now supposing the sphere to be on the point of being disturbed by the contraction of the string, one or both of the points E, E', of the sphere must be on the point of sliding along the corresponding beams. Suppose that sliding is on the point of taking place at E.

Then, μ being the coefficient of friction between the sphere and the beam AK, we have

$$F = \mu R;$$

and therefore from (1), (3), (4),

$$2\mu R\cos\alpha + W = 2R\sin\alpha,$$

and therefore, putting $\mu = \tan\epsilon$,

$$R = \frac{W}{2(\sin\alpha - \mu\cos\alpha)} = \frac{W\cos\epsilon}{2\sin(\alpha-\epsilon)} \ldots\ldots\ldots\ldots\ldots (5).$$

Also, from (3) and (4),

$$\frac{F'}{R'} = \frac{F}{R} = \mu,$$

and therefore $\qquad F' = \mu R'.$

Hence we see that, if μ' be the coefficient of friction between the sphere and the beam AK', μ is not greater than μ', since the greatest value of F' will be $\mu' R'$. If μ be less than μ', the sphere would, with the slightest increase in the tension of KK', begin

to roll along AK' without sliding; and, if μ be equal to μ', the sphere would begin to slide at both points simultaneously.

Again, for the equilibrium of AK we have, taking moments about A, it being remembered that the actions and reactions between the sphere and the beams are equal and opposite,

$$R \,.\, AE + W' \,.\, a \sin \alpha = T \,.\, 2a \cos \alpha;$$

and therefore, r being the radius of the sphere,

$$Rr \cot \alpha + W'a \sin \alpha = 2 Ta \cos \alpha;$$

hence, putting for R its value given in (5),

$$\frac{Wr \cos \epsilon \cos \alpha}{2 \sin \alpha \sin (\alpha - \epsilon)} + W'a \sin \alpha = 2 Ta \cos \alpha,$$

and therefore

$$T = \frac{Wr \cos \epsilon}{4a \sin \alpha \sin (\alpha - \epsilon)} + \tfrac{1}{2} W' \tan \alpha.$$

(2) AB (fig. 51) is a uniform beam, capable of motion about its middle point D; CE is a beam, moveable about a hinge C in the vertical line through D, and pressing against the beam AB, from the extremity B of which a weight P is suspended; CD, AD, BD, are equal lines; from observing the magnitude of the angle ACD when the end A of the beam AB is on the point of sliding in the direction CE, to find the coefficient of friction between the two beams.

Let G be the centre of gravity of the beam CE; μ the coefficient of friction; R the mutual action of the two beams at right angles to CE; $\angle ACD = \beta = \angle CAD$; $AD = a = BD$; $CG = b$, $4Q$ the weight of the beam CE.

Then for the equilibrium of CE we have, taking moments about C,

$$R \,.\, 2a \cos \beta = 4Q \,.\, b \sin \beta,$$

or

$$aR \cos \beta = 2b\, Q \sin \beta \qquad \text{(1)};$$

and for the equilibrium of AB, taking moments about D,

$$R \,.\, a \cos \beta = \mu R \,.\, a \sin \beta + P \,.\, a \sin 2\beta,$$

$$R (\cos \beta - \mu \sin \beta) = P \sin 2\beta \qquad \text{(2)}.$$

From (1) and (2) there is

$$\frac{2bQ\sin\beta}{a\cos\beta}(\cos\beta-\mu\sin\beta)=2P\sin\beta\cos\beta;$$

and therefore $$bQ(1-\mu\tan\beta)=aP\cos\beta,$$

$$\mu=\frac{bQ-aP\cos\beta}{bQ\tan\beta}.$$

(3) A weight W (fig. 62) is suspended from the middle point of a rigid rod without weight, connecting the centres O, O', of two equal heavy wheels, which rest on a rough inclined plane: the wheel O is locked: to find the greatest inclination of the plane which is consistent with the equilibrium of the carriage.

Let P be the weight, and r the radius of each of the wheels; let $OO'=2a$, $\phi=$ the inclination of the plane to the horizon; let R, R', be the reactions of the plane on the wheels at right angles to itself; μR the friction on the wheel O, μ being the coefficient of friction; F the action of the plane on the wheel O' at right angles to R'; X, Y, the resolved parts, parallel and perpendicular to the plane, of the action of the wheel O' on the rod OO'; and X', Y', the similarly resolved parts of the reaction.

For the equilibrium of the wheel O and the rod OO', regarded as one system, we have, resolving forces parallel to the inclined plane,

$$\mu R=X+(P+W)\sin\phi \text{..................} (1);$$

resolving forces at right angles to the plane,

$$R+Y=(P+W)\cos\phi \text{..................} (2);$$

and, taking moments about O,

$$\mu Rr+2aY=Wa\cos\phi \text{..................} (3).$$

Again, for the equilibrium of the wheel O', we have, taking moments about the point of contact of this wheel with the plane,

$$X'r=Pr\sin\phi, \quad \text{or } X'=P\sin\phi \text{..................} (4).$$

From the equations (1) and (4), observing that X' is by the nature of action and reaction equal to X, we get

$$\mu R=(2P+W)\sin\phi \text{..........................} (5).$$

Again, from (2) and (3),

$$\mu r R + 2a\,(P + W)\cos\phi - 2aR = Wa\cos\phi,$$

$$(2a - \mu r)\,R = a\,(2P + W)\cos\phi \text{..............}(6).$$

From (5) and (6) we obtain for the required inclination of the plane,

$$\tan\phi = \frac{\mu a}{2a - \mu r}.$$

COR. Having ascertained ϕ, we know R from (5) and X' or X from (4), and therefore Y from (2); also, F being the only force acting on the wheel O' which does not pass through its centre, it is evident that F must be equal to zero.

(4) Two equal beams AC, BC, connected by a smooth hinge at C, are placed in a vertical plane, their lower extremities A and B resting on a rough horizontal plane; from observing the greatest value of the angle ACB for which equilibrium is possible, to determine the coefficient of friction at the ends A and B.

If β be the greatest value of $\angle ACB$, and μ be the coefficient of friction at each of the ends; then

$$\mu = \tfrac{1}{2}\tan\tfrac{1}{2}\beta.$$

(5) Two equal semicylinders are placed horizontally at the same vertical altitude, their flat faces, which are rough, resting against two vertical and parallel plane surfaces the distance between which is infinitesimally greater than the diameter of either cylinder: a smooth wedge, the vertex of which is downwards, rests between the two semi-cylinders on their curved surfaces: to find the vertical angle of the wedge, supposing the cylinders to be on the point of slipping downwards.

Let W be the weight of either cylinder, W' that of the wedge, μ the coefficient of friction, and 2θ the vertical angle of the wedge: then

$$\tan\theta = \frac{\mu W'}{2W + W'}.$$

SECT. 3. *Systems of Beams.*

(1) Two uniform rods AC, BC, (fig. 63), are connected together by a smooth hinge-joint at C, their other ends being fastened to two smooth fixed hinges A, B, in a vertical line: to find the magnitudes and directions of the pressures on the hinges and of the mutual action of the rods at the joint.

It is frequently convenient in problems of this class, to make use of diagrams in which the several members of the system are represented to the eye in a state of slight detachment; the actions and reactions being indicated by arrowed lines not running into each other. The student will thereby escape falling into errors of sign in writing down the equations of equilibrium, to which he is liable from confounding together actions and reactions. In fact, the problem thereby resolves itself into the consideration of the equilibrium of several distinct bodies.

Let $AC=2a$, $BC=2b$, and let $\tan\alpha$, $\tan\beta$, be represented by m, n, respectively. The horizontal and vertical components of the actions and reactions on the rods are indicated in the diagram, as well as the weights of the rods.

For the equilibrium of AC there is, resolving horizontally,

$$X+X''=0 \quad\quad (1),$$

vertically,

$$Y+Y''=P \quad\quad (2),$$

and, taking moments about C,

$$X\,.\,2a\cos\alpha+Y\,.\,2a\sin\alpha=P\,.\,a\sin\alpha,$$

or

$$2X+2mY=mP \quad\quad (3).$$

In like manner, for the equilibrium of BC,

$$X'=X'' \quad\quad (4),$$

$$Y'=Q+Y'' \quad\quad (5),$$

and

$$Y'\,.\,2b\sin\beta=X'\,.\,2b\cos\beta+Q\,.\,b\sin\beta,$$

or

$$2nY'=2X'+nQ \quad\quad (6).$$

From (1), (2), (3), there is

$$2X''+2mY''=mP \quad\quad (7).$$

From (4), (5), (6), there is

$$2X'' - 2nY'' = nQ \text{ .... } (8).$$

From (7) and (8) we have

$$Y'' = \tfrac{1}{2} \cdot \frac{mP - nQ}{m+n} \text{ .... } (9).$$

Also, from (7) and (8), we have

$$X'' = \tfrac{1}{2} \cdot \frac{mn}{m+n} \cdot (P + Q) \text{ .... } (10).$$

Hence also, by (1) and (4),

$$X = -\tfrac{1}{2} \frac{mn}{m+n} (P + Q) \text{ .... } (11),$$

$$X' = \tfrac{1}{2} \frac{mn}{m+n} (P + Q) \text{ .... } (12).$$

From (2) and (9),

$$Y = \frac{mP + n(2P + Q)}{2(m+n)} \text{ .... } (13).$$

From (5) and (9),

$$Y' = \frac{nQ + m(2Q + P)}{2(m+n)} \text{ .... } (14).$$

The two components of the pressures exerted at A, C, B, upon each rod having been ascertained, the required directions and magnitudes of these pressures are therefore known.

(2) At the middle points of the sides of any polygon $ABCDE$...... (fig. 64), and at right angles to them, are applied a series of forces P, Q, R, ..., respectively proportional to the sides; the sides of the polygon are perfectly rigid, and capable of moving freely about the angular points A, B, C, D, ...; to determine the form of the polygon that it may be in equilibrium, the lengths of the sides being given.

Let p, q, r, s, ... denote the mutual actions of the sides of the polygon at the angles A, B, C, D, ..., of which the directions will lie in certain straight lines $bB\beta$, $cC\gamma$, $dD\delta$, ...

For the equilibrium of the side BC we have, resolving forces at right angles to it,

$$Q = q \sin \angle CB\beta + r \sin \angle BCc \quad \text{..............(1)};$$

resolving forces parallel to BC,

$$q \cos \angle CB\beta = r \cos \angle BCc \quad \text{..................(2)};$$

and, taking moments about the middle point of BC,

$$q \sin \angle CB\beta = r \sin \angle BCc \quad \text{..................(3)}.$$

Dividing (3) by (2), we have

$$\tan \angle CB\beta = \tan \angle BCc,$$

and therefore $\angle CB\beta = \angle BCc$ ....(4);

hence also, from (2) or (3), $q = r$ ....(5).

Again, from (1) and (3), we have

$$Q = 2r \sin \angle BCc;$$

in precisely the same manner we may find that

$$R = 2r \sin \angle DC\gamma,$$

and therefore $\dfrac{Q}{R} = \dfrac{\sin \angle BCc}{\sin \angle DC\gamma}$;

but, by the hypothesis,

$$\frac{Q}{R} = \frac{BC}{DC} = \frac{\sin \angle BDC}{\sin \angle CBD};$$

hence $\dfrac{\sin \angle BCc}{\sin \angle DC\gamma} = \dfrac{\sin \angle BDC}{\sin \angle CBD}$;

but from the geometry it is evident that

$$\angle BCc + \angle DC\gamma = \angle BDC + \angle CBD;$$

hence we readily see that

$$\angle BCc = \angle BDC \quad \text{.....................(6)}.$$

In just the same way we might prove that

$$\angle CB\beta = \angle BAC,$$

and therefore, by (4),

$$\angle BDC = \angle BAC \quad \text{....................(7)}.$$

From this relation (7) it is plain that a circle passing through the three points A, B, C, must pass likewise through the point D; similarly we might shew that this circle, since it passes through B, C, D, must likewise pass through E, and so on indefinitely; hence we see that when the sides of the polygon are arranged consistently with equilibrium, all its angular points must be situated in the circumference of a single circle.

From (5) we gather that

$$p = q = r = s \ldots\ldots,$$

or that the mutual pressures at all the angular points are equal. It is evident also from the relation (6), that all the lines $a\alpha$, $b\beta$, $c\gamma$, $d\delta$, ... are tangents to the circle passing through A, B, C, D, ...

The value of the mutual pressure at each of the angular points is easily obtained: thus, as we have shewn,

$$Q = 2r \sin \angle BCc;$$

but since $\angle BCc$ is equal to half the angle subtended by BC at the centre of the circle circumscribing the polygon, it is clear that

$$\sin \angle BCc = \frac{\frac{1}{2}BC}{\text{radius}};$$

hence
$$r = \text{radius} \times \frac{Q}{BC},$$

and therefore,
$$p = q = r = s \ldots\ldots = k\rho,$$

where ρ denotes the radius and k the ratio between any one of the forces and the corresponding side of the polygon.

Fuss; *Mémoires de St Pétersb.* 1817, 1818, p. 46.

The following is a different solution of the same problem:—Let the forces P, Q, R,...... be represented in magnitude by the lines $2AB$, $2BC$, $2CD$, ..., to which they are proportional. Instead of the force $2AB$ acting at the middle point of the side AB, apply two forces, each equal to AB, one at the end A and the other at the end B of the side AB; each of these forces being at right angles to the side AB. Again, instead of the force $2BC$ acting at the middle point of BC, apply a force BC

at C, and a force BC at the extremity B of the side AB, (which we are at liberty to do, because the point B of AB is rigidly attached to the point B of BC,) each of these forces being at right angles to BC. Now, according to this distribution of the forces, the only force which could twist BC about C, is the action of the rod AB upon the end B of BC; and therefore for the equilibrium of BC it is necessary that this action should take place exactly along BC. Hence conversely the action of CB upon BA will take place entirely in the direction CB. Let this action be denoted by R.

Thus, the line AB is acted upon at the point B by a force AB at right angles to AB, a force BC at right angles to BC, and a force R in the direction CB: but, by the principle of the parallelogram of forces, the forces AB and BC at B are equivalent to a single force AC acting at right angles to AC; hence for the equilibrium of AB we have, taking moments about A,

$$R \,.\, AB \,.\, \sin \angle ABC = AC \,.\, AB \cos \angle BAC,$$

or
$$R \sin \angle ABC = AC \cos \angle BAC.$$

Similarly, for the equilibrium of the side CD,

$$R \sin \angle BCD = BD \cos \angle BDC;$$

and therefore
$$\frac{\sin \angle ABC}{\sin \angle BCD} = \frac{AC}{BD} \frac{\cos \angle BAC}{\cos \angle BDC}.$$

But, by the geometry,

$$\frac{\sin \angle BAC}{\sin \angle BDC} = \frac{\dfrac{BC}{AC} \sin \angle ABC}{\dfrac{BC}{BD} \sin \angle BCD} = \frac{BD}{AC} \frac{\sin \angle ABC}{\sin \angle BCD}.$$

Hence from these two relations we have

$$\frac{\sin \angle BAC}{\sin \angle BDC} = \frac{\cos \angle BAC}{\cos \angle BDC},$$

$$\tan \angle BAC = \tan \angle BDC, \quad \angle BAC = \angle BDC;$$

which shews, as in the former solution, that the sides of the polygon must be so arranged that its angular points may all lie in the circumference of a single circle.

(3) A quadrilateral $ABCD$, (fig. 65), consists of four rigid rods, which are capable of free motion about the angular points A, B, C, D; supposing the points A, C, and B, D, to be attached together by strings AC and BD in given states of tension, to determine the geometrical conditions necessary for the equilibrium of the quadrilateral.

Let P, Q, represent the tensions of the strings AC, BD. Let K, L, M, N, denote the actions and reactions between the four pairs of points (A, B), (B, C), (C, D), (D, A).

The force P acting upon the point A in the direction AC, is equivalent to a force, in the direction AB,

$$=P\frac{\sin CAD}{\sin BAD}=P\frac{\sin ADB}{\sin BAD}\cdot\frac{DO}{AO}=P\frac{OD.AB}{BD.OA};$$

and to some force (F suppose) in AD.

Similarly, the force Q acting upon the point B in the direction BD, is equivalent to

$$\text{a force, in } BA, = Q\frac{OC.AB}{AC.OB},$$

and some force (G suppose) in BC.

Hence clearly the point A is solicited by a force $F-N$ in AD, and a force

$$P\frac{OD.AB}{BD.OA}-K \text{ in } AB \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

and therefore for its equilibrium we have

$$F-N=0, \text{ and } P\frac{OD.AB}{BD.OA}-K=0.$$

Similarly for the equilibrium of the point B there is

$$G-L=0, \text{ and } Q\frac{OC.AB}{AC.OB}-K=0 \ldots\ldots\ldots (2).$$

From (1) and (2) we have

$$P\frac{OD.AB}{BD.OA}=Q\frac{OC.AB}{AC.OB}$$

and therefore

$$\frac{P.OD}{BD.OA}=\frac{Q.OC}{AC.OB},$$

which is the condition for the equilibrium of the quadrilateral.

Euler; *Act. Acad. Petrop.* 1779, P. II. p. 106.

The following is a different solution of the same problem.

For the equilibrium of the rod AB there is, taking moments about B,

$$N.BD.\sin\angle BDA=P.BO.\sin\angle BOC;$$

and for the equilibrium of the rod CD, taking moments about C,

$$N.CA.\sin\angle CAD=Q.CO.\sin\angle BOC:$$

hence obviously

$$\frac{BD\sin\angle ODA}{CA\sin\angle OAD} \text{ or } \frac{BD.AO}{AC.DO}=\frac{P.BO}{Q.CO}.$$

(4) Four rigid rods AB, BC, CD, DA, (fig. 66), are so joined together that they are capable of revolving freely about the angular points of the quadrilateral which they form; these rods are attached together, two and two, viz. those which are contiguous, by strings $a\alpha$, $b\beta$, $c\gamma$, $d\delta$, in given states of tension; to determine the form of the quadrilateral which shall correspond to the equilibrium of the rods.

Let A, B, C, D, denote the tensions of the strings $a\alpha$, $b\beta$, $c\gamma$, $d\delta$. Then the force A in $a\alpha$ upon the point a is equivalent to a force, in BA,

$$=A\frac{\sin\alpha aD}{\sin AaD}=A\frac{\sin aD\alpha.\dfrac{D\alpha}{a\alpha}}{\sin ADa.\dfrac{DA}{Aa}}=A\frac{Aa.D\alpha}{a\alpha.DA};$$

and to a force, in aD,

$$=A\frac{\sin Aa\alpha}{\sin AaD}=A\frac{\sin aA\alpha.\dfrac{A\alpha}{a\alpha}}{\sin aAD.\dfrac{AD}{aD}}=A\frac{A\alpha.Da}{a\alpha.DA}$$

$$=A' \text{ suppose.}$$

But the force A' in aD is equivalent to

$$\text{a force in } AD, = A' \frac{\sin aDB}{\sin ADB} = A' \frac{\sin aBD \,.\, \frac{Ba}{Da}}{\sin ABD \,.\, \frac{BA}{DA}}$$

$$= A' \frac{Ba \,.\, DA}{Da \,.\, BA} = A \frac{A\alpha \,.\, Ba}{a\alpha \,.\, BA};$$

and to a force in BD, $= A' \dfrac{\sin ADa}{\sin ADB}$

$$= A' \frac{\sin DAa \,.\, \frac{Aa}{Da}}{\sin DAB \,.\, \frac{AB}{DB}} = A' \frac{Aa \,.\, BD}{AB \,.\, Da} = A \frac{A\alpha \,.\, Aa \,.\, BD}{a\alpha \,.\, DA \,.\, BA}.$$

Thus we see that the force A, acting upon the point a in the direction $a\alpha$, is equivalent to the three forces

$$A \frac{aD \,.\, Aa}{AD \,.\, a\alpha} \text{ in } BA \text{ upon } A, \quad A \frac{A\alpha \,.\, aB}{AB \,.\, a\alpha} \text{ in } AD \text{ upon } A,$$

and
$$\frac{aA \,.\, A\alpha \,.\, BD}{AB \,.\, AD \,.\, a\alpha} \text{ in } BD \text{ upon } B.$$

Similarly, the force A acting upon the point α in the direction αa, is equivalent to

$$A \frac{\alpha B \,.\, A\alpha}{AB \,.\, a\alpha} \text{ in } DA \text{ upon } A, \quad A \frac{Aa \,.\, \alpha D}{AD \,.\, a\alpha} \text{ in } AB \text{ upon } A,$$

and
$$A \frac{\alpha A \,.\, Aa \,.\, DB}{AD \,.\, AB \,.\, a\alpha} \text{ in } DB \text{ upon } D.$$

Now these three forces are equal and opposite to the three former, and therefore the string $a\alpha$ with a tension A produces the same effect, and may therefore be replaced by a string BD with a tension

$$A \frac{aA \,.\, A\alpha \,.\, BD}{AB \,.\, AD \,.\, a\alpha}.$$

In the same way we may shew that the tension of $c\gamma$ is equivalent to a string BD of which the tension is equal to

$$C\frac{cC\,.\,C\gamma\,.\,BD}{CB\,.\,CD\,.\,c\gamma}.$$

Hence the tensions of $a\alpha$, $c\gamma$, together, are equivalent to a string BD with a tension

$$A\frac{Aa\,.\,A\alpha\,.\,BD}{BA\,.\,DA\,.\,a\alpha}+C\frac{Cc\,.\,C\gamma\,.\,BD}{BC\,.\,DC\,.\,c\gamma}.$$

Similarly it may be shewn that the tensions $b\beta$, $d\delta$, are equivalent to a string AC with a tension

$$B\frac{Bb\,.\,B\beta\,.\,CA}{AB\,.\,CB\,.\,b\beta}+D\frac{D\delta\,.\,Dd\,.\,AC}{AD\,.\,CD\,.\,d\delta}.$$

Hence, by the result of the preceding problem, the condition of equilibrium is expressed by the relation

$$\frac{OB\,.\,OD}{BD^2}\left(\frac{B\,.\,Bb\,.\,B\beta}{AB\,.\,CB\,.\,b\beta}+\frac{D\,.\,D\delta\,.\,Dd}{AD\,.\,CD\,.\,d\delta}\right)$$

$$=\frac{OA\,.\,OC}{AC^2}\left(\frac{A\,.\,Aa\,.\,A\alpha}{BA\,.\,DA\,.\,a\alpha}+\frac{C\,.\,Cc\,.\,C\gamma}{BC\,.\,DC\,.\,c\gamma}\right).$$

Euler; *Act. Acad. Petrop.* 1779, P. 2, p. 106.

(5) Two equal uniform beams AB, AC, moveable about a hinge at A, are placed upon the convex circumference of a circle in a vertical plane; to find their inclination to each other when they are in their position of equilibrium.

Let $2a$ = the length of each beam, 2θ = their inclination to each other, and r = the radius of the circle. Then θ will be determined by the equation

$$r\cos\theta = a\sin^3\theta.$$

(6) A and C (fig. 67) are fixed points in the same vertical line: beams AB, CD, are freely moveable about these points by hinge joints; AB, from the end B of which a weight is suspended, is supported in a horizontal position by CD, with which it is connected by a hinge joint at D: to find the pressure at C, the weights of the beams being neglected.

Let H and V be the horizontal and vertical pressures at C, and P the weight suspended from B. Then

$$H = P.\frac{AB}{AC}, \quad V = P.\frac{AB}{AD},$$

and therefore the whole pressure at C is equal to

$$P.AB.\left(\frac{1}{AC^2} + \frac{1}{AD^2}\right)^{\frac{1}{2}}.$$

(7) Two uniform rods AC, BD, (fig. 68), are connected together by a hinge at D: their ends A, B, resting on a smooth horizontal plane, are tied together by a string. To find the tension of the string.

If $AC = 2a$, $BD = 2b$, $AB = c$, $\angle\, CAB = \alpha$, $\angle\, DBA = \beta$, and P, Q, denote the weights of AC, BD, respectively, the tension of the string will be equal to

$$\frac{Pa \sin \alpha + Qb \sin \beta}{c \tan \alpha \tan \beta}.$$

(8) Three uniform beams AB, BC, CD, of the same thickness, and of lengths l, $2l$, l, respectively, are connected by hinges at B and C, and rest on a perfectly smooth sphere, the radius of which is equal to $2l$, so that the middle point of BC and the extremities of AB, CD, are in contact with the sphere; to compare the pressure at the middle point of BC, and the pressures at A and D, with the weight of the three beams.

Let W be the weight of the three beams taken together; R the pressure at each of the points A and D; and R' the pressure at the middle point of BC. Then

$$\frac{R}{W} = \frac{3}{40}, \quad \frac{R'}{W} = \frac{91}{100}.$$

(9) Four equal uniform beams AB, BC, CD, DE, (fig. 69), connected together by joints at their extremities, rest in equilibrium in a vertical plane; the distances AE and CF, of which the latter is perpendicular to AE and vertical, are given; to determine the conditions of equilibrium.

If α, β, be the inclinations of AB and ED, BC and DC, to the horizon; we must have

$$\tan\alpha = 3\tan\beta.$$

Draw BK at right angles to AE; let $CF = a$, $AF = b$, $FK = x$, $BK = y$; then from the equation in α and β, and the geometry of the figure, we may get

$$x = \frac{a^2 + 2b^2 - (a^4 + a^2b^2 + b^4)^{\frac{1}{2}}}{2b}, \quad y = \frac{2a^2 + b^2 - (a^4 + a^2b^2 + b^4)^{\frac{1}{2}}}{2a}.$$

These values of x and y are obtained by Couplet in his *Recherches sur la Construction des Combles de Charpente*, in the *Mémoires de l'Académie des Sciences de Paris*, 1731, p. 69.

CHAPTER V.

EQUILIBRIUM OF FLEXIBLE STRINGS.

THE form of equilibrium assumed by a uniform flexible string sustained at its two extremities and acted on by gravity, attracted the attention of Galileo[1], who, from a want of sufficient examination, concluded it to be a parabola; this mistake may have arisen from the fact that, in the immediate neighbourhood of its lowest point, it approximates very nearly to the parabolic form. The inaccuracy of Galileo's conclusion was experimentally ascertained by Joachim Jungius[2]. This subject having been at last successfully investigated by James Bernoulli[3], he proposed the problem of the chaînette, the name which he gave to the required curve, as a trial of skill to the mathematicians of the day. The four mathematicians who succeeded in arriving at correct solutions of the problem were, James Bernoulli, by whom it had been proposed, his brother John, Leibnitz, and Huyghens: their four solutions appeared without analysis in the *Acta Eruditorum* for the year 1691, Jun. pp. 273—282. A demonstration of the results of these four illustrious mathematicians was first published by David Gregory, in the *Philosophical Transactions* for the year 1697.

The form of equilibrium of the chaînette or catenary, of which the thickness is supposed to be uniform, having been thoroughly discussed, James Bernoulli[4] next directed his attention to more complicated problems of the same character; he investigated the form of equilibrium when the thickness varies

[1] *Mechanica;* Dialogo 2, p. 131.
[2] *Geometria Empyrica.*
[3] *Acta Eruditorum*, Lips. 1690, Mai. p. 217; *Opera*, Tom. I. p. 424.
[4] *Acta Eruditorum*, Lips. 1691, Jun. p. 289; *Opera*, Tom. I. p. 449.

from point to point according to any assigned law, and, conversely, determined the law of its variation that the string may hang in assigned curves: he likewise considered the problem of the catenary when the string is extensible, the extension of each element being assumed, according to the law established experimentally by Hooke[1], to vary as the tension. The analysis of these problems, of which the solutions only were published by James Bernoulli, was supplied by John Bernoulli[2]. The consideration of the general conditions of the equilibrium of flexible strings was first attempted by Hermann[3], whose investigations, however, were not free from error; a more accurate analysis was furnished by John Bernoulli[4], who has particularly examined various cases of the equilibrium of strings acted on by central forces.

Among the numerous mathematicians who afterwards discussed the theory of the equilibrium of flexible strings, may be mentioned Euler[5], Clairaut[6], Krafft[7], Legendre[8], Fuss[9], Venturoli[10], and Poisson[11].

SECT. 1. *Free Inextensible String; general Conditions of Equilibrium.*

To investigate the conditions for the equilibrium of an inextensible string, of which the density and thickness vary from point to point according to any assigned law; the accelerating forces which act upon the string being any whatever.

[1] *De Potentia Restitutiva, or Spring.*
[2] *Lectiones Mathematicæ in usum Hospitalii*, *Opera*, Tom. IV. p. 387.
[3] *Phoronomia*, lib. I. cap. 3, and Append. § V.
[4] *Opera*, Tom. IV. p. 234.
[5] *Comment. Petrop.* Tom. III.; *Nov. Comment. Petrop.* Tom. XV. and Tom. XX.
[6] *Miscellanea Berolinensia*, Tom. VII. p. 270, 1743.
[7] *Nov. Comment. Petrop.* Tom. V. p. 143; 1754 and 1755.
[8] *Mém. Acad. Par.* 1786, p. 20.
[9] *Nova Acta Petrop.* Tom. XII. p. 145, 1794.
[10] *Elements of Mechanics*, by Cresswell, Part I. p. 62.
[11] *Traité de Mécanique*, Tom. I. p. 564, seconde édition.

Let APB (fig. 70) be any portion of the string in a position of rest; Pp being a small element of its length; x, y, z, and $x+\delta x$, $y+\delta y$, $z+\delta z$, the co-ordinates of P and p respectively; s the length of the string reckoned from some assigned point up to P, and $s+\delta s$ the length up to p; t the tension of the string at P.

The resolved parts, parallel to the axes of x, y, z, of the force exerted upon the point P of the element Pp by the portion AP of the string, will evidently be

$$-t\frac{dx}{ds},\ -t\frac{dy}{ds},\ -t\frac{dz}{ds};$$

and therefore, since each of these three forces must be some function of s, it is plain by Taylor's theorem that the resolved parts of the force exerted on the element Pp by the portion pB of the string, will be

$$t\frac{dx}{ds}+\frac{d}{ds}\left(t\frac{dx}{ds}\right)\delta s,$$

$$t\frac{dy}{ds}+\frac{d}{ds}\left(t\frac{dy}{ds}\right)\delta s,$$

$$t\frac{dz}{ds}+\frac{d}{ds}\left(t\frac{dz}{ds}\right)\delta s.$$

Again, let X, Y, Z, be the sums of the resolved parts of the accelerating forces which act upon the element Pp; ρ the density of the string at P, and k the area of a section at right angles to its length at that point. Then the mass of the portion Pp of the string will be $k\rho\delta s$, which therefore, for a constant value of δs, will vary as $k\rho$; the product $k\rho$, which we will represent by m, may be called "the mass of the string at the point P." The resolved parts, parallel to the co-ordinate axes, of the moving force of the element Pp, will be

$$mX\delta s,\quad mY\delta s,\quad mZ\delta s.$$

Hence for the equilibrium of Pp we must have, equating to zero the sum of the resolved forces which act upon it parallel

to each of the three axes, and dividing the three resulting equations by δs,

$$\left.\begin{aligned}\frac{d}{ds}\left(t\frac{dx}{ds}\right) + mX = 0,\\ \frac{d}{ds}\left(t\frac{dy}{ds}\right) + mY = 0,\\ \frac{d}{ds}\left(t\frac{dz}{ds}\right) + mZ = 0,\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(a);$$

which three equations constitute the conditions of equilibrium of the entire string.

By the elimination of t we readily obtain the three following equations,

$$dy\int mZds = dz\int mYds,$$

$$dz\int mXds = dx\int mZds,$$

$$dx\int mYds = dy\int mXds;$$

any two of which will be differential equations to the required curve of equilibrium.

COR. 1. From the equations (a) we have also

$$t\frac{dx}{ds} = -\int mXds, \quad t\frac{dy}{ds} = -\int mYds, \quad t\frac{dz}{ds} = -\int mZds;$$

squaring and adding these equations, and observing that

$$\frac{dx^2}{ds^2} + \frac{dy^2}{ds^2} + \frac{dz^2}{ds^2} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (b),$$

we obtain for the value of the tension at any point,

$$t^2 = \left(\int mXds\right)^2 + \left(\int mYds\right)^2 + \left(\int mZds\right)^2.$$

We may obtain also another expression for the tension: differentiating (b) with respect to s, we get

$$\frac{dx}{ds}\frac{d^2x}{ds^2} + \frac{dy}{ds}\frac{d^2y}{ds^2} + \frac{dz}{ds}\frac{d^2z}{ds^2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (c);$$

hence, multiplying the three equations (*a*) by dx, dy, dz, in order, and adding the resulting equations, we have, by the aid of (*b*) and (*c*),

$$t = C - \int m\,(Xdx + Ydy + Zdz),$$

where C is an arbitrary constant.

COR. 2. If the whole string lie entirely within one plane, let the plane of xy be so chosen as to coincide with this plane; then the three differential equations to the string will be reduced to the single one

$$dx \int mYds = dy \int mXds \text{..................} (d);$$

and the two formulæ for the tension will become

$$t^2 = \left(\int mXds\right)^2 + \left(\int mYds\right)^2,$$

$$t = C - \int m\,(Xdx + Ydy).$$

These two formulæ for the tension, and also the differential equation (*d*) to the string, coincide with those given by Fuss; *Mémoires de St. Pétersbourg*, 1794, pp. 150, 151.

SECT. 2. *Parallel Forces.*

(1) A flexible string fixed at any two points A and B, (fig. 71), is acted on by gravity; supposing the mass of the string to vary according to any assigned law as we pass from one point to another, to find the equation to the catenary of rest; and conversely, the curve being known, to determine the law of the mass of the string.

Let the axis of y extend vertically upwards, and let the axis of x be horizontal, the plane xOy coinciding with the plane which contains the catenary. Then, since

$$X = 0, \quad Y = -g,$$

we have, by the first two of the equations (*a*) of section (1),

$$\frac{d}{ds}\left(t\frac{dx}{ds}\right)=0 \quad\text{..........}\quad (a),$$

$$\frac{d}{ds}\left(t\frac{dy}{ds}\right)=mg \quad\text{..........}\quad (b).$$

Integrating the equation (*a*), we get

$$t\frac{dx}{ds}=C,$$

where C is a constant quantity: let τ denote the tension at the lowest point of the curve: then evidently $\tau = C$, and therefore

$$t\frac{dx}{ds}=\tau \quad\text{..........}\quad (c).$$

From (*b*) and (*c*), we have

$$\tau\frac{d}{ds}\frac{dy}{dx}=mg,$$

and therefore $$\tau\frac{dy}{dx}=\int mg\,ds;$$

but at the lowest point of the catenary $\frac{dy}{dx}=0$, and therefore, supposing a to be the value of s at the lowest point,

$$\tau\frac{dy}{dx}=g\int_a^s m\,ds \quad\text{..........}\quad (d).$$

If m be given in terms of the variables x, y, s, the form of the catenary may be determined from (*d*).

Again, differentiating (*d*), we obtain

$$m=\frac{\tau}{g}\frac{\dfrac{d^2y}{dx^2}}{\dfrac{ds}{dx}} \quad\text{..........}\quad (e),$$

a formula by which m may be computed for every point of the string when the form of the catenary is given. Also from (*c*) we get

$$t = \tau \frac{ds}{dx} \text{ ....} (f),$$

which gives the tension at any point of the catenary when its form is known.

John Bernoulli; *Lectiones Mathematicæ*, Lect. 38, 39, 40; *Opera*, Tom. III.

(2) A flexible string AOB, (fig. 72), fixed at two points A and B, is acted on by gravity; the mass at any point P varies inversely as the square root of the length OP measured from the lowest point O; to find the equation to the catenary.

Let the origin of co-ordinates be taken at O, x being horizontal, and y vertical, and the plane of xy coinciding with the plane of the catenary; also let O be the origin of s.

Then, if μ be the mass at the end of a length c from the lowest point,

$$m = \mu \frac{c^{\frac{1}{2}}}{s^{\frac{1}{2}}},$$

and therefore by (1, d), α being in the present case zero, we have

$$\tau \frac{dy}{dx} = g\mu c^{\frac{1}{2}} \int_0^s \frac{ds}{s^{\frac{1}{2}}} = 2g\mu c^{\frac{1}{2}} s^{\frac{1}{2}};$$

hence, putting for the sake of brevity

$$\frac{2g\mu c^{\frac{1}{2}}}{\tau} = \frac{1}{\beta^{\frac{1}{2}}},$$

we get

$$\frac{dy}{dx} = \left(\frac{s}{\beta}\right)^{\frac{1}{2}}, \quad \frac{dy^2}{dx^2} = \frac{s}{\beta},$$

$$\beta \frac{d}{dx} \frac{dy^2}{dx^2} = \frac{ds}{dx} = \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}},$$

$$\frac{\beta \dfrac{d}{dx} \dfrac{dy^2}{dx^2}}{\left(1 + \dfrac{dy^2}{dx^2}\right)^{\frac{1}{2}}} = 1;$$

integrating with respect to x we obtain

$$2\beta \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} = x + C;$$

but $x = 0$, $\frac{dy}{dx} = 0$, simultaneously; hence $C = 2\beta$, and therefore

$$2\beta\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} = x + 2\beta \ldots\ldots\ldots\ldots\ldots\ldots (a);$$

squaring and transposing,

$$4\beta^2 \frac{dy^2}{dx^2} = (x + 2\beta)^2 - 4\beta^2,$$

$$2\beta\, dy = \{(x + 2\beta)^2 - 4\beta^2\}^{\frac{1}{2}} dx;$$

integrating we have

$$C + 2\beta y = \tfrac{1}{2}(x + 2\beta)(x^2 + 4\beta x)^{\frac{1}{2}} - 2\beta^2 \log\{x + 2\beta + (x^2 + 4\beta x)^{\frac{1}{2}}\};$$

but $x = 0$, $y = 0$, simultaneously; hence

$$C = -2\beta^2 \log(2\beta);$$

hence, eliminating C,

$$2\beta y = \tfrac{1}{2}(x + 2\beta)(x^2 + 4\beta x)^{\frac{1}{2}} - 2\beta^2 \log \frac{x + 2\beta + (x^2 + 4\beta x)^{\frac{1}{2}}}{2\beta},$$

which is the required equation to the catenary.

COR. From (a) we get

$$\frac{ds}{dx} = \frac{x + 2\beta}{2\beta},$$

and therefore, by $(1, f)$,

$$t = \tau \frac{ds}{dx} = \frac{\tau}{2\beta}(x + 2\beta),$$

which gives the tension at any point of the curve.

John Bernoulli; *Lect. Math.*, *Opera*, Tom. III. p. 497.

(3). To find the law of variation of the mass of a catenary acted on by gravity that it may hang in the form of a semicircle with its diameter horizontal.

The notation remaining the same as in (2), the equation to the catenary will be

$$x^2 = 2ay - y^2,$$

where a denotes the radius of the semicircle: hence

$$a^2 - x^2 = (a - y)^2, \quad y = a - (a^2 - x^2)^{\frac{1}{2}};$$

$$\frac{dy}{dx}=\frac{x}{(a^2-x^2)^{\frac{1}{2}}},\qquad \frac{d^2y}{dx^2}=\frac{a^2}{(a^2-x^2)^{\frac{3}{2}}};$$

also $$\frac{ds^2}{dx^2}=1+\frac{dy^2}{dx^2}=\frac{a^2}{a^2-x^2},\qquad \frac{ds}{dx}=\frac{a}{(a^2-x^2)^{\frac{1}{2}}};$$

and therefore, by (1, e),

$$m=\frac{\tau}{g}\frac{\frac{d^2y}{dx^2}}{\frac{ds}{dx}}=\frac{\tau}{g}\frac{a}{a^2-x^2}=\frac{\tau a}{g\,(a-y)^2};$$

or the mass at any point varies inversely as the square of its depth below the horizontal diameter of the semicircle.

COR. By (1, f) we have for the tension at any point

$$t=\tau\frac{ds}{dx}=\frac{\tau a}{(a^2-x^2)^{\frac{1}{2}}}=\frac{\tau a}{a-y}.$$

John Bernoulli; *Opera*, Tom. III. p. 502.

(4) To find the length of a uniform chain ALB, (fig. 73), suspended from two points A and B in the same horizontal line, when the stress on each point of support is equal to the whole weight of the chain; to find also the depth of the lowest point L of the chain below the line AB, and the direction of its tangent at A or B.

Let $yCLO$ be vertical, OL being equal to a length of the chain of which the weight is equal to the tension at the lowest point L, Ox horizontal; PM at right angles to Ox. Let $OM=x$, $PM=y$, $OL=c$, $ALB=l$, $AC=BC=a$.

Then the equation to the curve will be

$$y=\tfrac{1}{2}c\,(\epsilon^{\frac{x}{c}}+\epsilon^{-\frac{x}{c}})\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;(1),$$

and also $$l=c\,(\epsilon^{\frac{a}{c}}-\epsilon^{-\frac{a}{c}})\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;(2).$$

Let m denote the mass at any point of the chain, which is the same at all its points; then the tension at P will be equal to

$$mgy=\tfrac{1}{2}mcg\,(\epsilon^{\frac{x}{c}}+\epsilon^{-\frac{x}{c}}),$$

and therefore at B to $\quad \tfrac{1}{2}mcg\,(\epsilon^{\frac{a}{c}}+\epsilon^{-\frac{a}{c}})$;

but by the hypothesis the tension at B is equal to mgl, and therefore by (2) to

$$mcg\,(\epsilon^{\frac{a}{c}} - \epsilon^{-\frac{a}{c}});$$

hence $$\tfrac{1}{2}mcg\,(\epsilon^{\frac{a}{c}} + \epsilon^{-\frac{a}{c}}) = mcg\,(\epsilon^{\frac{a}{c}} - \epsilon^{-\frac{a}{c}});$$

$$\tfrac{1}{2}\epsilon^{\frac{a}{c}} = \tfrac{3}{2}\epsilon^{-\frac{a}{c}},$$

$$\epsilon^{\frac{2a}{c}} = 3,\quad \frac{2a}{c} = \log_\epsilon 3,\quad \frac{a}{c} = \tfrac{1}{2}\log_\epsilon 3 \text{..........} (3).$$

Hence from (2) we have

$$l = \frac{2a}{\log_\epsilon 3}\left(3^{\frac{1}{2}} - \frac{1}{3^{\frac{1}{2}}}\right) = \frac{4a}{3^{\frac{1}{2}}\log_\epsilon 3},$$

which gives the length of the chain.

Again, putting $x = a$, we have from (1),

$$OC = \tfrac{1}{2}c\,(\epsilon^{\frac{a}{c}} + \epsilon^{-\frac{a}{c}}),$$

and therefore $$CL = (\tfrac{1}{2}\epsilon^{\frac{a}{c}} + \tfrac{1}{2}\epsilon^{-\frac{a}{c}} - 1)\,c$$

$$= \left(\frac{3^{\frac{1}{2}} + 3^{-\frac{1}{2}}}{2} - 1\right)\frac{2a}{\log_\epsilon 3}, \text{ from (3)},$$

$$= \left(\frac{2}{3^{\frac{1}{2}}} - 1\right)\frac{2a}{\log_\epsilon 3} = \frac{2a}{\log_\epsilon 3}\,\frac{2 - 3^{\frac{1}{2}}}{3^{\frac{1}{2}}},$$

which gives the depth of the lowest point of the chain below the line AB.

Again, from (1) we have

$$\frac{dy}{dx} = \tfrac{1}{2}\,(\epsilon^{\frac{x}{c}} - \epsilon^{-\frac{x}{c}}),$$

and therefore, ϕ denoting the inclination of the chain at B to the horizon,

$$\tan\phi = \tfrac{1}{2}\,(\epsilon^{\frac{a}{c}} - \epsilon^{-\frac{a}{c}}) = \tfrac{1}{2}\left(3^{\frac{1}{2}} - \frac{1}{3^{\frac{1}{2}}}\right) = \frac{1}{3^{\frac{1}{2}}};$$

hence $$\phi = \frac{\pi}{6}.$$

(5) A uniform string $A'ALBB'$ (fig. 74) is placed over two supports A and B in the same horizontal line, so as to remain in equilibrium; having given the length of the string, and the distance between the points of support, to find the pressure which they have to bear.

Let L be the lowest point of the curve ALB, OLy a vertical line through L, where OL is equal to a length of the chain, the weight of which is equal to the tension at L; Ox horizontal. Then, Ox, Oy, being taken as the axes of co-ordinates, we shall have for the equation to the curve ALB, putting $OL = c$,

$$y = \tfrac{1}{2}c\,(\epsilon^{\frac{x}{c}} + \epsilon^{-\frac{x}{c}}) \dots\dots\dots\dots (1);$$

and, if m be the mass at each point of the string, the tension at P will be equal to

$$mgy \text{ or } \tfrac{1}{2}\,mcg\,(\epsilon^{\frac{x}{c}} + \epsilon^{-\frac{x}{c}});$$

hence, if $AC = BC = a$, the tension at B will be equal to

$$\tfrac{1}{2}mcg\,(\epsilon^{\frac{a}{c}} + \epsilon^{-\frac{a}{c}}) \dots\dots\dots\dots (2).$$

But the tension at B is evidently equal to the weight of BB', and therefore, if $BB' = s$, to the expression mgs; hence

$$mgs = \tfrac{1}{2}\,mcg\,(\epsilon^{\frac{a}{c}} + \epsilon^{-\frac{a}{c}}),$$

or

$$s = \tfrac{1}{2}c\,(\epsilon^{\frac{a}{c}} + \epsilon^{-\frac{a}{c}}) \dots\dots\dots\dots (3).$$

Suppose that the length of the whole string $A'ALBB'$ is $2l$; then the length of the portion LBB' will be l, and $l - s$ will be the length of BL. Hence, by the nature of the catenary,

$$l - s = \tfrac{1}{2}c\,(\epsilon^{\frac{a}{c}} - \epsilon^{-\frac{a}{c}}) \dots\dots\dots\dots (4).$$

Adding together the equations (3) and (4), we obtain

$$l = c\epsilon^{\frac{a}{c}},$$

whence c is made to depend upon the known quantities a and l: hence the expression (2) for the tension at B is known.

Differentiating (1), we get

$$\frac{dy}{dx} = \tfrac{1}{2}\left(\epsilon^{\frac{x}{c}} - \epsilon^{-\frac{x}{c}}\right):$$

but, if $LP = s'$,

$$s' = \tfrac{1}{2}c\left(\epsilon^{\frac{x}{c}} - \epsilon^{-\frac{x}{c}}\right), \quad \frac{ds'}{dx} = \tfrac{1}{2}\left(\epsilon^{\frac{x}{c}} + \epsilon^{-\frac{x}{c}}\right);$$

hence

$$\frac{dy}{ds'} = \frac{\epsilon^{\frac{x}{c}} - \epsilon^{-\frac{x}{c}}}{\epsilon^{\frac{x}{c}} + \epsilon^{-\frac{x}{c}}},$$

and therefore, if ϕ denote the angle between the line BB' and the curve BL at B,

$$\cos\phi = \frac{\epsilon^{\frac{a}{c}} - \epsilon^{-\frac{a}{c}}}{\epsilon^{\frac{a}{c}} + \epsilon^{-\frac{a}{c}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Let P denote the pressure on the point B, and τ the tension of the string at B; then

$$\begin{aligned} P^2 &= 2\tau^2 + 2\tau^2\cos\phi \\ &= 2\tau^2(1 + \cos\phi); \end{aligned}$$

and therefore, from (2) and (5),

$$\begin{aligned} P^2 &= \tfrac{1}{2}m^2c^2g^2\left(\epsilon^{\frac{a}{c}} + \epsilon^{-\frac{a}{c}}\right)^2 \left\{1 + \frac{\epsilon^{\frac{a}{c}} - \epsilon^{-\frac{a}{c}}}{\epsilon^{\frac{a}{c}} + \epsilon^{-\frac{a}{c}}}\right\} \\ &= m^2c^2g^2\left(\epsilon^{\frac{a}{c}} + \epsilon^{-\frac{a}{c}}\right)\epsilon^{\frac{a}{c}} \\ &= m^2c^2g^2\left(1 + \epsilon^{\frac{2a}{c}}\right), \\ P &= mcg\left(1 + \epsilon^{\frac{2a}{c}}\right)^{\frac{1}{2}}, \end{aligned}$$

which gives the required value of the pressure, c having been previously determined.

(6) A uniform chain ABC (fig. 75) is suspended from a point A above an inclined plane RS: having given the angle which the chain at the point of suspension and which the plane makes with the horizon, and also the length of the whole chain, to find the length of the portion BC which is in contact with the plane.

Let $ABLA'$ denote the catenary, of which AB is an arc, L being the lowest point. Let P be any point in the curve AL; ϕ the inclination of the curve at P to the horizon, t the tension at B; α, β, the values of ϕ at A, B, respectively; c the length of chain of which the weight is equal to the tension at L; m the mass of the chain at any point; $LP=s$, $ABC=l$, $BC=l'$.

Then, by the nature of the catenary,

$$t\cos\beta = mcg \text{..........................} (1),$$

$$s = c\tan\phi \text{......................} (2).$$

Now it is evident that the tension at B is equal to $mgl'\sin\beta$; hence, from (1),

$$mgl'\sin\beta\cos\beta = mcg, \quad c = l'\sin\beta\cos\beta \text{.........} (3).$$

Again, from (2), we have

$$LBA = c\tan\alpha, \quad LB = c\tan\beta,$$

and therefore $$l - l' = c(\tan\alpha - \tan\beta);$$

hence, from (3),

$$l - l' = l'\sin\beta\cos\beta(\tan\alpha - \tan\beta),$$

$$l\cos\alpha = l'(\cos\alpha + \sin\alpha\sin\beta\cos\beta - \sin^2\beta\cos\alpha)$$

$$= l'\cos\beta . \cos(\alpha - \beta)$$

$$l' = \frac{l\cos\alpha}{\cos\beta\cos(\alpha-\beta)}.$$

(7) To find the form of equilibrium of a heavy chain the density of which varies inversely as the square of the length of the chain reckoned from a given point in it.

For the equilibrium of the chain we have, a being a constant, the axis of x being horizontal, that of y vertical,

$$\frac{d}{ds}\left(t\frac{dx}{ds}\right) = 0, \quad \frac{d}{ds}\left(t\frac{dy}{ds}\right) = \frac{a}{s^2}.$$

Integrating these equations and then eliminating t, we have

$$p = \frac{dy}{dx} = b - \frac{a}{s} \text{.....................} (1),$$

a and b being constants.

Let ρ be the radius of curvature at any point: then

$$\rho = \frac{(1+p^2)^{\frac{3}{2}}}{\frac{dp}{dx}} = \frac{1+p^2}{\frac{dp}{ds}},$$

and therefore, by (1),

$$a\rho = (1+b^2)\,s^2 - 2abs + a^2.$$

Put $s = s_1 + \mu$, and determine μ so that in the result the coefficient of the first power of s shall vanish. Then we see that $\mu = \frac{ab}{1+b^2}$, and that

$$\rho = \frac{1+b^2}{a}\,s_1^2 + \frac{a}{1+b^2}$$

$$= \frac{s_1^2}{c} + c,$$

where c is a constant. Let $d\phi$ be the angle of contingence: then $\rho = \frac{ds_1}{d\phi}$, and therefore

$$d\phi = \frac{c\,ds_1}{c^2 + s_1^2}:$$

integrating and supposing that $\phi = 0$ when $s_1 = 0$, we have

$$s_1 = c \tan \phi.$$

Let x_1, y_1, be the co-ordinates referred to a system of axes such that the tangent is parallel to that of x_1 when $\phi = 0$: then

$$s_1 = c\frac{dy_1}{dx_1} = cp_1,$$

$$(1+p_1^2)^{\frac{1}{2}} = c\frac{dp_1}{dx_1},$$

$$\frac{x_1}{c} + \epsilon = \log\{p_1 + (1+p_1^2)^{\frac{1}{2}}\},$$

ϵ being a constant. Suppose that the origin is such that $x_1 = 0$ when $p_1 = 0$: then $\epsilon = 0$, and we have

$$p_1 + (1+p_1^2)^{\frac{1}{2}} = e^{\frac{x_1}{c}},$$

$$-p_1 + (1+p_1^2)^{\frac{1}{2}} = e^{-\frac{x_1}{c}},$$

whence
$$2p_1 = e^{\frac{x_1}{c}} - e^{-\frac{x_1}{c}},$$
$$y_1 = \frac{c}{2}\left(e^{\frac{x_1}{c}} + e^{-\frac{x_1}{c}}\right),$$

supposing the origin such that $y_1 = c$ when $x_1 = 0$.

When $s_1 = 0$, $s = \mu = \frac{ab}{1+b^2}$, and therefore, by (1), $p = -\frac{1}{b}$: hence the axis of the catenary is inclined to the horizon at an angle $\tan^{-1} b$.

Haton de la Goupillière; *Nouvelles Annales de Mathématiques*, 2me Série, Tome IX. p. 554.

(8) A uniform chain hangs in equilibrium over two smooth pegs which are in a horizontal line and at a given distance from each other: to find the depths of the ends of the chain below the pegs, when the tension at its middle point is equal to the weight of its curvilinear portion.

If a be the distance between the pegs, each of the required depths is equal to

$$\frac{a\sqrt{5}}{4\log\left(\frac{1+\sqrt{5}}{2}\right)}.$$

(9) A uniform heavy string passes through two small smooth rings, which rest on a fixed horizontal bar: if, one of the rings being kept stationary, the other be held at any other point of the bar, to find the locus of the position of equilibrium of that end of the string which is the farther from the stationary ring.

Let l be the whole length of the string: then, the fixed ring being taken as origin of co-ordinates, the axis of x being vertical and that of y horizontal, the equation to the required locus is

$$y = (lx)^{\frac{1}{2}} . \log\left(\frac{l}{4x}\right).$$

(10) AOB (fig. 72) is a flexible string acted on by gravity, and is in a position of rest; the mass at any point varies as

the cosine of the angle at which an element of the curve at the point is inclined to the horizon; to find the equation to the catenary.

Assuming $m = \beta \dfrac{dx}{ds}$, where β is some constant quantity, the equation to the catenary will be

$$x^2 = \frac{2\tau}{\beta g} y;$$

which shews that the catenary is the common parabola.

James Bernoulli; *Act. Erudit.* Lips. Jun.; *Opera*, Tom. I. p. 449. John Bernoulli; *Opera*, Tom. III. p. 501.

(11) To find the equation to the catenary when the mass at any point varies as $x \cos\phi$, where ϕ is the angle of inclination of the element of the curve at any point to the horizon.

Assuming $m = \beta x \dfrac{dx}{ds}$, the required equation will be.

$$6\tau y = g\beta x^3,$$

which belongs to a cubical parabola.

James Bernoulli; *Ib.* John Bernoulli; *Ib.*

(12) To find the equation to the catenary when the mass at any point varies as $x^{\frac{1}{2}} \cos\phi$.

Assuming $m = \beta x^{\frac{1}{2}} \dfrac{dx}{ds}$, the equation will be

$$16 g^2 \beta^2 x^5 = 225 \tau^2 y^2.$$

James Bernoulli; *Ib.* John Bernoulli; *Ib.*

(13) To find the equation to the catenary when the mass at any point varies as $y^n \sin\phi$, where n is any positive quantity.

If the origin of co-ordinates be so chosen that the axis of x passes through the lowest point of the catenary, and that $y = \infty$ when $x = 0$, the required equation will be

$$xy^n = -\frac{(n+1)\tau}{ng\beta}.$$

James Bernoulli; *Ib.* John Bernoulli; *Ib.*

(14) To find the mass at any point when the catenary is the common parabola.

The construction and notation being the same as in (2),

$$m = \frac{2\tau}{g\,(a^2 + 4x^2)^{\frac{1}{2}}},$$

where a is the latus rectum of the parabola.

John Bernoulli; *Opera*, Tom. III. p. 504.

(15) A chain, suspended at its extremities from two tacks in the same horizontal line, forms itself into a cycloid; to find the mass at any point of the string and the weight of the arc between this and the lowest point.

Let w denote the weight of the arc; then, taking the ordinary equations to the cycloid

$$x = a\,(\theta + \sin\theta), \quad y = a\,(1 - \cos\theta),$$

we shall have

$$m = \frac{\tau\,(\sec\frac{1}{2}\theta)^3}{4ag}, \quad w = \tau \tan\tfrac{1}{2}\,\theta.$$

(16) One end of a uniform chain is attached to a fixed point A, and the other to a weight which is placed on a rough horizontal plane passing through A, and the chain hangs through a slit in the horizontal plane; to find the greatest distance of the weight from A, in order that the equilibrium may be possible.

If a be the length of the chain, x the greatest distance of the weight from A, μ the coefficient of friction, and n twice the ratio between the given weight and that of the chain,

$$\epsilon^{\frac{x}{a}} = \left[\frac{1 + \{1 + \mu^2\,(1+n)^2\}^{\frac{1}{2}}}{\mu\,(1+n)}\right]^{\mu\,(1+n)}.$$

(17) A uniform chain is suspended from two tacks in the same horizontal line at a distance $2a$ from each other; to determine the length of the chain that the stress on the tacks may be a minimum.

Let c denote a length of the chain of which the weight is equal to the tension at the lowest point; and let l denote the required length of the chain: then

$$\epsilon^{\frac{a}{c}} = \left\{\frac{\frac{a}{c}+1}{\frac{a}{c}-1}\right\}^{\frac{1}{2}}, \quad l = c\,(\epsilon^{\frac{a}{c}} - \epsilon^{-\frac{a}{c}}):$$

from the former equation $\frac{a}{c}$ and therefore c is to be determined, and then l will be given by the latter.

If for instance $2a = 10$ feet, then $c = 4.168$ feet nearly, and $l = 12.578$ feet nearly.

Diarian Repository, p. 644.

(18) A chain acted on by gravity hangs in the form of a curve, of which $a^3y = x^4$ is the equation; to find the point at which the mass is a maximum, and its maximum value.

When m is a maximum, x and y being the co-ordinates of the point,

$$x = \frac{a}{2^{\frac{1}{2}}}, \quad y = \tfrac{1}{4}a, \quad m = \frac{2\,.\,3^{\frac{1}{2}}\tau}{ag}.$$

The law of the mass of the chain is erroneously investigated in the *Lady's and Gentleman's Diary* for the year 1745; see also *Diarian Repository*, p. 435.

(19) A uniform chain of length $2l$ is suspended from two points in a horizontal line, the distance $2a$ between which is given: to investigate an equation for the determination of the inclination of the curve to the horizon at either point of support.

If ϕ represent the required angle,

$$a = l \cot\phi \log\left(\tan\frac{\pi + 2\phi}{4}\right).$$

(20) A uniform chain $ABCDE$, (fig. 76), passes over a smooth pully B, the portions BA, DE, of the chain hanging freely, while the portion CD rests upon a smooth table. Sup-

posing CD to be half the length of the whole chain, and AB to be equal to twice DE, to compare the length of the chain with the height of B above CD.

The required ratio is equal to

$$2(3+\sqrt{3}).$$

(21) A chain of variable thickness hangs in the form of a curve defined by the equation

$$\frac{y}{b} = \log\left(\sec\frac{x}{b}\right),$$

the axis of x being horizontal: to find the law of variation of the thickness and of the tension.

The tension of the string at any point varies as the area of the section of the string, each varying as $\sec\left(\frac{x}{b}\right)$.

(22) The mass of a string is so distributed that, when it is suspended by its extremities from two points, the tension varies as the density: to find the form of the string in its position of equilibrium.

Let the origin be at the lowest point of the string and let the axis of x be a tangent at this point: then, c being a constant, the equation to the form will be

$$y = c\log\left(\sec\frac{x}{c}\right).$$

(23) A uniform chain of length l hangs over two fixed points, which are in a horizontal line: from its middle point is suspended by one end another chain of equal thickness and of length l'. Supposing each of the two tangents of the former chain at its middle point to make an angle θ with the vertical, to find the distance between the two fixed points, and to shew that θ can never exceed a certain value.

The distance between the two fixed points is equal to

$$l'\tan\theta \, . \log\left\{\frac{l+l}{l'} \cdot \frac{\tan\frac{\theta}{2}}{\tan\theta}\right\}.$$

The value of θ can never exceed that given by the equation

$$\tan^2\frac{\theta}{2}=\frac{l-l'}{l+l'}.$$

(24) A chain of variable density, suspended from two points, hangs in the form of a curve the intrinsic equation to which is $s=f(\phi)$, the origin of s being at the lowest point of the curve and ϕ being the inclination of the tangent at any point to the vertical: to find the law of the density.

The density at any point varies inversely as $(\sin\phi)^2.f'(\phi)$.

(25) A chain of variable thickness, but of the same material throughout, is suspended from two points: to find the law of the thickness and the form of the curve, when the tension at different points of the chain varies as the strength of the chain at those points[1].

Let the origin of co-ordinates be at the lowest point, the axis of y being horizontal and that of x vertical: let s be the length of a uniform chain the thickness of which is equal to that at the lowest point, and of which the weight is equal to that of the length of the chain from the lowest point to the point (x, y); and let c be the length of a uniform chain of the thickness at the lowest point and of which the weight is equal to the tension at the lowest point. Then, κ denoting the thickness at the lowest point and κ' at the point (x, y),

$$\kappa'=\frac{\kappa}{c}(c^2+s^2)^{\frac{1}{2}},\quad x=c\log\frac{(c^2+s^2)^{\frac{1}{2}}}{c},\quad s=c\tan\frac{y}{c}.$$

Sir Davies Gilbert: *Philosophical Transactions* for 1826.

SECT. 3. *Central Forces.*

(1) To find the equation to a flexible string in a position of equilibrium under the action of any central attractive force.

Let APB (fig. 77) be any portion of the string; S the centre of force; P any point in the string, PT a tangent at this point; SY a perpendicular from S upon PT; p a point of the string

[1] The curve is called the Catenary of Uniform Strength.

indefinitely near to P, and pk a tangent at p. Also let OP, Op, be the normals at P, p, O being therefore the centre and OP the radius of curvature at P; let OP produced meet pk in k.

Let $OP=\rho$, $SP=r$, $SY=p$, $\angle SPT=\phi$, $\angle kOp=\psi$, $m=$ the mass at P; $t=$ the tension at P and $t+dt$ at p; $Pp=ds$, $F=$ the central force at P.

Then for the equilibrium of the element Pp we have, resolving the forces which act upon it at right angles to PT,

$$Fmds \sin\phi = (t+dt)\cos pkO = (t+dt)\sin\psi,$$

or, retaining infinitesimals of the first order,

$$Fmds \sin\phi = t\psi = t\frac{ds}{\rho};$$

and therefore

$$Fm \sin\phi = \frac{t}{\rho} \dots\dots\dots\dots\dots (a);$$

and resolving forces parallel to PT we have

$$Fmds \cos\phi = (t+dt)\sin Okp - t$$
$$= (t+dt)\cos\psi - t,$$

or, retaining infinitesimals of the first order only,

$$Fmds\cos\phi = dt;$$

and therefore, $ds\cos\phi$ being equal to dr,

$$Fmdr = dt \dots\dots\dots\dots\dots (b).$$

From the equation (a), since

$$\rho = -r\frac{dr}{dp} \text{ and } \sin\phi = \frac{p}{r},$$

we have
$$Fmdr + \frac{dp}{p}t = 0;$$

and therefore from (b)

$$\frac{dp}{p} + \frac{dt}{t} = 0,$$

$$\log(pt) = \log C,$$

where C is an arbitrary constant; and therefore

$$p = \frac{C}{t} = \frac{C}{\int Fmdr}, \dots\dots\dots\dots\dots (c),$$

which is the equation to the catenary in p and r when the form of F is known.

Let θ be the angle between SP and any fixed line; then

$$p=\frac{r^2\,d\theta}{(dr^2+r^2d\theta^2)^{\frac{1}{2}}},$$

and therefore from (c), putting $\int Fmdr=R$,

$$Rr^2\,d\theta=C\,(dr^2+r^2\,d\theta^2)^{\frac{1}{2}},$$

$$R^2r^4\,d\theta^2=C^2\,(dr^2+r^2\,d\theta^2),$$

and therefore $$d\theta=\frac{Cdr}{r\,(R^2r^2-C^2)^{\frac{1}{2}}}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(d),$$

the differential equation to the catenary between r and θ. This is the form in which the solution is given by John Bernoulli[1].

The value of the tension at any point of the catenary is given by (b), when the expression for F in terms of r is known.

The relations at which we have arrived may be deduced from the general equations of equilibrium of section (1); the method however of the tangential and normal resolution is more convenient in the case of central forces.

If the central force be repulsive instead of attractive, we must replace F by $-F$, wherever it occurs in the above formulæ.

(2) To find the form of the catenary when the central force is attractive and varies inversely as the square of the distance; the mass at every point being the same.

Let AOB (fig. 78) be the catenary; S the centre of force; SO the radius vector which meets the curve at right angles.

Let $\tau=$ the tension at O, and $SO=c$.

Then, if k denote the attraction at the distance c,

$$R=\int Fmdr=\int k\frac{c^2}{r^2}\,mdr=C'-\frac{mkc^2}{r},$$

[1] *Opera*, Tom. IV. p. 238.

where C' is an arbitrary constant: but, by (1, b), $t = R$, and therefore

$$\tau = C' - mkc;$$

hence $$t = R = \tau + mkc - \frac{mkc^2}{r} \dots\dots\dots\dots\dots\dots (a).$$

Hence, from (1, d), we have

$$d\theta = \frac{Cdr}{r\left\{\left(\tau + mkc - \frac{mkc^2}{r}\right)^2 r^2 - C^2\right\}^{\frac{1}{2}}};$$

but from (1, c), since $p = c$ and $t = \tau$ at the point O, we see that $C = c\tau$; therefore

$$d\theta = \frac{c\tau dr}{r\left\{\left(\tau + mkc - \frac{mkc^2}{r}\right)^2 r^2 - c^2\tau^2\right\}^{\frac{1}{2}}}.$$

For the sake of simplicity put $\tau = nmkc$; then

$$d\theta = \frac{ncdr}{r\left\{\left(n + 1 - \frac{c}{r}\right)^2 r^2 - n^2c^2\right\}^{\frac{1}{2}}}$$

$$= \frac{ncdr}{r\left\{(n+1)^2 r^2 - 2(n+1)cr + c^2 - n^2c^2\right\}^{\frac{1}{2}}};$$

the equation to the catenary resulting from the integration of this differential equation will be of three different forms according as n is greater than, equal to, or less than unity.

First, suppose that n is greater than unity; then the integral of the equation will be, supposing that $\theta = 0$ when $r = c$,

$$r = \frac{(n-1)c}{n\cos\left\{\left(n^2 - 1\right)^{\frac{1}{2}}\frac{\theta}{n}\right\} - 1}.$$

Secondly, suppose that $n = 1$; then the equation to the catenary will be, if $\theta = 0$ when $r = c$,

$$r = \frac{c}{1 - \theta^2}.$$

Thirdly, let n be less than unity; then, if as before $\theta = 0$ when $r = c$, the equation will be

$$\epsilon^{(1-n^2)^{\frac{1}{2}}\frac{\theta}{n}} + \epsilon^{-(1-n^2)^{\frac{1}{2}}\frac{\theta}{n}} = \frac{2}{nr}\{r - (1-n)\,c\}.$$

Again, from (1, c) we have, since $C = c\tau$,

$$p = \frac{c\tau}{t},$$

and therefore by (a)

$$p = \frac{c\tau}{\tau + mkc - \dfrac{mkc^2}{r}} = \frac{nc}{n + 1 - \dfrac{c}{r}};$$

hence, putting $r = \infty$, we have $p = \dfrac{nc}{n+1}$, which shews that the three catenaries, corresponding to the three values of n, have all of them asymptotes passing within a distance $\dfrac{nc}{n+1}$ from the centre of force. Put $r = \infty$ in the equations to the three curves, and we get for the inclinations of the pair of asymptotes of each to the line SO,

$$\frac{n}{(n^2-1)^{\frac{1}{2}}}\cos^{-1}\frac{1}{n}, \text{ and } \frac{n}{(1-n^2)^{\frac{1}{2}}}\log\frac{1+(1-n^2)^{\frac{1}{2}}}{n}.$$

John Bernoulli; *Opera*, Tom. IV. p. 240.
Whewell's *Mechanics*, 3rd edit. p. 183.

(3) To find the equation to a uniform catenary AOB (fig. 78), acted on by a central force tending to S, the intensity of which varies as the μ^{th} power of the distance; the tension at O being equal to the $(1+\mu)^{\text{th}}$ part of the weight of a length SO of the string, each element of which length is supposed to be acted on by a constant force equal to that at O and towards S.

The notation remaining the same as in (2), the equation to the catenary will be

$$\left(\frac{c}{r}\right)^{\mu+2} = \cos(\mu+2)\,\theta.$$

(4) To find the equation to a uniform catenary $SAOB$, (fig. 79), acted on by a central repulsive force emanating from S, at which the two ends of the string are fastened, the intensity of this force varying inversely as the μ^{th} power of the distance; the tension at O being equal to the $(\mu - 1)^{th}$ part of the weight of a length SO of the string, each element of which length is supposed to be acted on by a constant force equal to that at O and from S.

The notation remaining the same as before, the equation to the curve will be

$$\left(\frac{r}{c}\right)^{\mu-2} = \cos(\mu - 2)\,\theta.$$

SECT. 4. *Constrained Equilibrium.*

(1) A flexible string ab, (fig. 80), acted on by gravity, rests on the arc of a curve APB in a vertical plane; to find the tension of the string and the pressure on the curve at any point.

Let P, p, be any two points of the curve very near to each other; PO, pO, normals at these points, the point O being the centre of curvature when p approaches indefinitely near to P: let ax, ay, be the axes of x, y, the former being horizontal, the latter vertical; $aP = s$, $Pp = ds$; $t =$ the tension at P and $t + dt$ at p; $R =$ the pressure on the curve at P, $m =$ the mass at any point of the string, $\angle POp = \phi$, $PO = \rho$.

Then, resolving forces, which act on the element Pp of the string, parallel to the tangent at P, we have

$$(t + dt)\cos\phi - t = mgds \cdot \frac{dy}{ds},$$

or, neglecting infinitesimals of higher orders than the first,

$$dt = mgdy\,;$$

integrating and observing that t is equal to zero when $y = 0$, we get

$$t = mgy \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

which gives the tension at any point of the string.

Again, resolving the forces on the element Pp parallel to the normal OP,

$$mgds \cdot \frac{dx}{ds} + (t + dt) \sin \phi = Rds,$$

or, neglecting infinitesimals of orders higher than the first,

$$mg \frac{dx}{ds} + t \frac{\phi}{ds} = R;$$

but $\frac{\phi}{ds}$ is equal to $\frac{1}{\rho}$; hence we have for the pressure on the curve at any point

$$R = mg \frac{dx}{ds} + \frac{t}{\rho}$$

$$= mg \left(\frac{dx}{ds} + \frac{y}{\rho}\right).$$

(2) Two equal weights Q, Q, are suspended at the extremities of a flexible string hanging over a smooth curve in a vertical plane; to find the pressure at any point of the curve, the weight of the string being reckoned inconsiderable.

Let APB (fig. 81) be the curve; OP, Op, normals at two consecutive points P, p; θ the inclination of OP to some assigned line in the plane of the curve, $POp = d\theta$; $PO = \rho$, $AP = s$, $Pp = ds$; $p =$ the pressure at the point P; $t =$ the tension of the string at P, $t + dt =$ the tension at p.

Then for the equilibrium of the element Pp of the string we have, resolving forces at right angles to the tangent at P,

$$(t + dt) \sin d\theta = pds = p\rho d\theta,$$

and therefore, retaining infinitesimals of the first order,

$$t\, d\theta = p\rho d\theta, \quad t = p\rho \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Again, resolving forces parallel to the tangent at P,

$$(t + dt) \cos d\theta - t = 0,$$

and therefore, retaining infinitesimals of the first order,

$$dt = 0, \ t = \text{constant};$$

but evidently at A the tension is equal to Q; hence $t = Q$. Hence from (1) we have

$$Q = p\rho, \quad p = \frac{Q}{\rho}.$$

COR. The whole pressure on the curve AB is equal to

$$\int p ds = Q \int \frac{ds}{\rho} = Q \int_{\theta_1}^{\theta_2} d\theta = Q(\theta_2 - \theta_1).$$

If the tangents at the points where the string leaves the curve be vertical, we have πQ for the whole pressure along the curve; if they be not vertical there will of course be pressures at the points A, B, in addition to the pressure along the curve.

Euler; *Nov. Comment. Petrop.* 1775, p. 307.
Poisson; *Traité de Mécanique*, Tom. I. ch. 3.

(3) To find the pressure on a curve AB, (fig. 82), when two weights Q, R, balance each other over it by means of a fine string, the friction between the string and the curve being taken into account; and the weight Q being considered as much greater than R as is consistent with equilibrium.

Let μ be the coefficient of friction; the rest of the notation being the same as in the preceding problem. Then the friction on the element Pp will be $\mu p ds$, and will act nearly in the direction of the tangent at P. Hence, resolving forces on the element Pp parallel to PO, we have

$$(t + dt) \sin d\theta = p ds = p\rho d\theta;$$

and therefore in the limit

$$t d\theta = p\rho d\theta, \quad t = p\rho \text{.....................(1)};$$

again, resolving forces parallel to the tangent at P,

$$(t + dt) \cos d\theta - t + \mu p ds = 0,$$

and therefore in the limit

$$dt + \mu p ds = 0,$$

and consequently by (1)

$$dt + \frac{\mu t}{\rho} ds = 0;$$

integrating, we get

$$\log t = -\mu \int \frac{ds}{\rho} = -\mu \int d\theta = C - \mu\theta \ldots\ldots\ldots(2);$$

hence, the values of t at A and B being Q and R,

$$\log Q = C - \mu\theta_1, \quad \log R = C - \mu\theta_2 \ldots\ldots\ldots\ldots(3),$$

and therefore $\log \dfrac{Q}{R} = \mu(\theta_2 - \theta_1)$, $\dfrac{Q}{R} = e^{\mu(\theta_2 - \theta_1)}$,

which expresses the relation which must subsist between Q and R under the circumstances of the problem.

Also, from (2) and (3),

$$\log \frac{t}{R} = \mu(\theta_2 - \theta), \quad t = Re^{\mu(\theta_2 - \theta)} \ldots\ldots\ldots\ldots(4);$$

hence the whole pressure along the curve is equal to

$$\int p\,ds = \int \frac{ds}{\rho}\,t, \text{ from (1),}$$

$$= \int t\,d\theta = R\int e^{\mu(\theta_2 - \theta)}\,d\theta = C - \frac{R}{\mu}e^{\mu(\theta_2 - \theta)};$$

but, when $\theta = \theta_1$, it is clear that the pressure along the curve is zero; hence

$$0 = C - \frac{R}{\mu}e^{\mu(\theta_2 - \theta_1)},$$

and therefore the whole pressure from θ_1 to θ_2 is equal to

$$\frac{R}{\mu}\{e^{\mu(\theta_2 - \theta_1)} - 1\}.$$

In addition to this pressure along the curve there are the pressures at the extremities A and B.

Cor. If the curve be a semicircle $\theta_2 - \theta_1 = \pi$, and we have

$$\frac{Q}{R} = e^{\mu\pi}.$$

Euler; *Nov. Comment. Petrop.* 1775, p. 316.
Poisson; *Traité de Mécanique*, Tom. I. ch. 3.

(4) A heavy uniform string rests on a complete cycloid, the axis of which is vertical and vertex upwards, the whole length of the string exactly coinciding with the whole arc of the cycloid: to find the law of the pressure at any point of the cycloid.

The pressure at any point varies inversely as the curvature.

(5) A heavy chain ABC (fig. 83), is fixed at its highest end to the circumference of a circular section of a rough horizontal cylinder, moveable about its axis: having given the lengths of the two portions AB, BC, of the chain, to determine the moment of a force about the axis of the cylinder which shall maintain the equilibrium.

Let $AB=a$, $BC=b$, $r=$ the radius of the cylinder, and $m=$ the mass of a unit of the chain's length. Then the required moment is equal to

$$mrg\left(r \sin \frac{a}{r} + b\right).$$

(6) Two equal weights P, P', are connected by a string which passes over a rough fixed horizontal cylinder: to compare the forces required to raise P, accordingly as P is pushed up or P' pulled down.

If p be the force in the former and p' in the latter case,

$$\frac{p'}{p} = e^{\mu\pi}.$$

(7) If a weight P attached to one end of a fine cord, which is laid over a rough horizontal cylinder, can support a weight nP attached to the other end, to determine the weight which it can support when the cord is wrapped r times round the cylinder.

The required weight is equal to

$$n^{2r+1} . P.$$

(8) A string is wrapped round a smooth elliptic cylinder in a plane perpendicular to its axis and is acted on by two forces

which tend from the foci, vary inversely as the square of the distance, and are equal at equal distances: to compare the tensions of the string at the ends of the major and minor axes.

The tension at an end of the major is to that at an end of the minor axis in the ratio of $1+e^2$ to 1, where e is the eccentricity of a transverse section of the cylinder.

(9) The extremities of a light thread, the length of which is $7a$, are fastened to those of a uniform heavy rod, the length of which is $5a$; and, when the thread is passed over a thin round peg, it is found that the rod will hang at rest provided that the point of support be anywhere within a space a in the middle of the thread: to determine the coefficient of friction between the thread and the peg; and, when the rod hangs in a position bordering upon motion, to find its inclination to the horizon and the tensions of the two parts of the string.

If μ represent the coefficient of friction, W the weight of the rod, S, T, the tensions of the longer and shorter parts of the string respectively, and θ the inclination of the rod to the horizon;

$$\mu = \frac{2}{\pi} \log \tfrac{4}{3}, \quad S = \tfrac{4}{5} W, \quad T = \tfrac{3}{5} W, \quad \cos\theta = \tfrac{24}{25}.$$

(10) A thin inextensible cord, in which the density of the material increases in geometric as the distance from one extremity increases in arithmetic progression, is laid directly across a rough horizontal cylinder, the circumference of a vertical section of which is equal to twice the length of the cord: to determine the coefficient of friction, supposing the cord to be only just supported when its two extremities are both in the horizontal plane through the axis of the cylinder.

Taking, in accordance with the hypothesis, $q\epsilon^{\frac{s}{k}}$ as the expression for the density at a distance s from one end of the string, and denoting the radius of the cylinder by r,

$$\mu = \frac{r}{2k}.$$

SECT. 5. *Extensible Strings.*

If a uniform extensible string of given length be stretched by any force, it is found by experiment that the extension of the string beyond its natural length is proportional to the force. From this it is easily seen that, if the string be of variable length, the extension will vary as the product of the force and the natural length of the string. Hence, if a denote the natural length of the string, and a' the length under the action of a stretching force P, we shall have

$$a' = a\left(1 + \frac{P}{\lambda}\right),$$

where λ is a constant quantity depending upon the quality of the string, called the modulus of elasticity.

This theory was first announced by Hooke, in the form of an anagram, among a list of inventions at the end of his *Descriptions of Helioscopes*, published in the year 1676. The anagram is *ceiiinosssttuu*, from which may be extracted the proposition, "ut tensio sic vis." He afterwards published a work entitled *De Potentia Restitutiva or Spring*, in which the theory was developed at large with experimental illustrations. Hooke's theory forms the basis of a memoir by Leibnitz, in the *Acta Eruditorum* for the year 1684, entitled *Demonstrationes Novæ de Resistentia Solidorum*. For additional information on the subject the student is referred to s'Gravesande's *Element. Physic.* Lib. I. c. 26.

(1) An elastic string AC (fig. 84) is suspended from its extremity A, and a weight is attached to it at a point B; the natural lengths of AB, BC, being given, to find the length of the string AC in its present circumstances.

Let m denote the mass of a unit of length of the string in its natural state; a, b, the natural lengths of AB, BC, and a', b', their lengths under the circumstances of the problem; c the length of a portion of the natural string, the weight of which

is equal to the weight attached to B; let P be any point in AB, and p an adjacent point; let $AP = x$, $Pp = dx$; let t be the tension at P and $t + dt$ at p.

Then, by Hooke's Principle, the weight of Pp will be

$$\frac{mgdx}{1 + \frac{t}{\lambda}},$$

and therefore, for the equilibrium of Pp,

$$t + dt + \frac{mgdx}{1 + \frac{t}{\lambda}} - t = 0,$$

$$\left(1 + \frac{t}{\lambda}\right) dt + mgdx = 0:$$

integrating we get

$$t\left(1 + \frac{t}{2\lambda}\right) + mgx = C;$$

but it is evident that

$$t = mg\,(a + b + c), \quad \text{when } x = 0,$$

and

$$t = mg\,(b + c), \quad \text{when } x = a';$$

hence we obtain

$$(a + b + c)\left\{1 + \frac{mg}{2\lambda}(a + b + c)\right\} = (b + c)\left\{1 + \frac{mg}{2\lambda}(b + c)\right\} + a',$$

and therefore

$$a' = a + \frac{mg}{2\lambda}\{(a + b + c)^2 - (b + c)^2\}$$

$$= a + \frac{mg}{2\lambda}(a^2 + 2ab + 2ac)$$

$$= a\left\{1 + \frac{mg}{2\lambda}(a + 2b + 2c)\right\} \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Again, if Q be any point in BC, $BQ = y$, and $\tau =$ the tension at Q, we shall have, as before,

$$\tau\left(1 + \frac{t}{2\lambda}\right) + mgy = C;$$

but evidently

$$\tau = mgb, \quad \text{when } y = 0,$$

and

$$\tau = 0, \quad \text{when } y = b';$$

hence we have

$$b' = b\left(1 + \frac{mgb}{2\lambda}\right) \quad\text{.......................(2).}$$

Hence, from (1) and (2), if l' denote the whole length of the string AC, we find that

$$l' = a + b + \frac{mg}{2\lambda}\{a(a + 2b + 2c) + b^2\}.$$

(2) An elastic string, of which the unstretched length is a, is fixed at one end to the summit of a smooth inclined plane the length of which is also equal to a; to find the length which will hang over the plane, the string being stretched by its own weight.

Let ACD (fig. 85) be the string, A the summit of the inclined plane AC; P any point in AC, and p an adjacent point; let t be the tension at P, T at A, and τ at C; let $AP = x$, $Pp = dx$; m = the mass at any point of the string when unstretched, α = the inclination of AC to the horizon.

Then, by virtue of Hooke's Principle, the mass of Pp will be

$$\frac{mdx}{1 + \frac{t}{\lambda}};$$

and therefore, $t + dt$ being the tension at p, we have, for the equilibrium of Pp,

$$dt + \frac{mg\sin\alpha}{1 + \frac{t}{\lambda}}dx = 0,$$

$$\left(1 + \frac{t}{\lambda}\right)dt + mg\sin\alpha\, dx = 0:$$

integrating we obtain

$$t\left(1 + \frac{t}{2\lambda}\right) + mgx\sin\alpha = C;$$

hence, τ being the value of t when $x = a$ and T when $x = 0$, we have

$$\tau\left(1 + \frac{\tau}{2\lambda}\right) + mga\sin\alpha = T\left(1 + \frac{T}{2\lambda}\right),$$

$$(\tau - T)\left\{1 + \frac{1}{2\lambda}(\tau + T)\right\} + mga\sin\alpha = 0 \quad\text{...............(1).}$$

Let s be the natural length of CD and therefore $a-s$ the natural length of AC. It is evident that

$$\tau = mgs, \quad T = mg\{s + (a-s)\sin\alpha\};$$

hence, by (1),

$$(a-s)\left[1 + \frac{mg}{2\lambda}\{2s + (a-s)\sin\alpha\}\right] = a,$$

$$\frac{mg}{2\lambda}(a-s)\{2s + (a-s)\sin\alpha\} = s,$$

whence s may be determined by the solution of a quadratic equation.

If s' be the actual length of the portion CD of the string, we may shew that

$$s' = s\left(1 + \frac{mgs}{2\lambda}\right),$$

and therefore s and s' are both known.

(3) A slightly extensible string $A\alpha$ (fig. 86) is attached to the upper extremity A of the vertical radius AO of a circular arc AB along which it rests; having given its natural length, to find its length as it rests on the arc.

Let P, p, be any two adjacent points in the string $A\alpha$; draw the lines PO, pO; let $AP = s$, $Pp = ds$, $\angle AOP = \phi$, $\angle POp = d\phi$, $AO = a$; $m =$ the mass of a unit of length of the string when unstretched; let t, $t + dt$, be the tensions at P, p; s', ds', the natural lengths of AP, Pp.

Then, by Hooke's Principle,

$$ds = \left(1 + \frac{t}{\lambda}\right) ds' \qquad\qquad (1).$$

Again, for the equilibrium of the portion Pp of the string, we have, resolving forces parallel to the tangent at P,

$$(t + dt)\cos(d\phi) + mgds'\sin\phi = t;$$

and therefore, retaining infinitesimals of the first order,

$$dt + mgds'\sin\phi = 0;$$

hence, by the aid of (1), we have

$$\left(1+\frac{t}{\lambda}\right)dt + mg \sin\phi\, ds = 0;$$

or, since $s = a\phi$,

$$\left(1+\frac{t}{\lambda}\right)dt + mag \sin\phi\, d\phi = 0:$$

integrating we get

$$t\left(1+\frac{t}{2\lambda}\right) - mag\cos\phi = C.$$

Let β be the angle subtended at O by the arc Aa; then it is clear that, when $\phi = \beta$, t will be equal to zero; hence

$$-mag\cos\beta = C,$$

and therefore $\quad t\left(1+\dfrac{t}{2\lambda}\right) = mag(\cos\phi - \cos\beta)$ ....(2).

From (1) we have, putting $a\phi$ for s,

$$ad\phi = \left(1+\frac{t}{\lambda}\right)ds',$$

and, λ being by the hypothesis a large quantity,

$$ds' = ad\phi\left(1-\frac{t}{\lambda}\right) = ad\phi - \frac{at}{\lambda}d\phi \text{ ....(3).}$$

Now from (2) we get approximately

$$t = mag(\cos\phi - \cos\beta),$$

and therefore, substituting this value of t in the small term of the equation (3),

$$ds' = ad\phi - \frac{ma^2g}{\lambda}(\cos\phi - \cos\beta)\,d\phi.$$

Integrating we get

$$s' + C = a\phi - \frac{ma^2g}{\lambda}(\sin\phi - \phi\cos\beta);$$

but, when $\phi = 0$, it is evident that $s' = 0$; hence $C = 0$, and we have

$$s' = a\phi - \frac{ma^2g}{\lambda}(\sin\phi - \phi\cos\beta);$$

let $a\beta'$ be equal to the natural length of $A\alpha$; then evidently

$$a\beta' = a\beta - \frac{ma^2g}{\lambda}(\sin\beta - \beta\cos\beta),$$

$$\beta' = \beta - \frac{mag}{\lambda}(\sin\beta - \beta\cos\beta);$$

but, since $\beta = \beta'$ nearly, we may substitute β' for β in the coefficient of the small quantity $\frac{1}{\lambda}$; thus we obtain

$$\beta = \beta' + \frac{mag}{\lambda}(\sin\beta' - \beta'\cos\beta'),$$

which determines the required length $A\alpha$.

(4) Two weights P, Q (fig. 87), resting on two smooth inclined planes CA, CB, are connected by a given elastic string PQ; to find their position of equilibrium.

Let θ be the inclination of QP to the horizon; α, β, the inclinations to the horizon of the planes CA, CB; a the natural length of the string PQ. Then the position of equilibrium will be defined by the two equations,

$$\tan\theta = \frac{P\cot\beta - Q\cot\alpha}{P+Q}, \quad PQ = a\left\{1 + \frac{P\sin\alpha}{\lambda\cos(\alpha-\theta)}\right\}.$$

(5) Two equal weights P, Q (fig. 88), are connected by a fine elastic string PQ, of which the horizontal line BC is the natural length; to find the nature of the curves BP, CQ, on which they will always remain in equilibrium when the string is parallel to the horizon, the plane of the curves being vertical.

Bisect BC in A, and draw AM vertical; let $AB = a = AC$, $AM = x$, $MP = y = MQ$; then the equation to each of the curves will be

$$\lambda(y-a)^2 = 2aPx,$$

and therefore BP, CQ, are two semi-parabolas of which B, C, are the vertices.

(6) An elastic ring BC is placed horizontally round a cone the axis of which is vertical; to find the position of equilibrium of the ring.

Let A (fig. 89) be the vertex of the cone, O the centre of the

ring in its position of equilibrium; let $\angle OAB = \alpha$, $2\pi a =$ the natural length of the ring, and $W =$ its weight; then

$$BO = a\left(1 + \frac{W \cot \alpha}{2\pi\lambda}\right),$$

which determines the required position.

Earnshaw; *Statics*, 1st edition, p. 256.

(7) A heavy uniform elastic ring rests horizontally on a portion of a surface of revolution, of which the axis is vertical, in every position; to find the nature of the generating curve.

The generating curve is a parabola of which the axis of revolution is a diameter.

(8) A fine elastic string is tied round two equal cylinders, the surfaces of which are in contact and axes parallel, the string being just so long as not to be stretched beyond its natural length; one of the cylinders is turned through two right angles, so that the axes are again parallel: to find the tension of the string, supposing that a weight of one pound would stretch it to twice its natural length.

The tension of the string $= \dfrac{\pi - 2}{\pi + 2}$, where one pound is the unit.

(9) A heavy uniform rod AB is attached by a hinge at A to an upright rod AC; the points B and C are connected together by a fine elastic string which, when unstretched, forms the hypotenuse of an isosceles right-angled triangle, A being the right angle: to find the position of equilibrium of AB.

If θ denote either of the angles at B or C, W the weight of AB, and λ the modulus of elasticity, the oblique position of equilibrium of AB is defined by the equation

$$\cos \theta = \frac{1}{\sqrt{(2)} - \dfrac{W}{\lambda}}.$$

There is therefore no oblique position unless

$$\lambda > W(1 + \sqrt{2}).$$

(10) A fine elastic string, passing over a smooth tack, supports a uniform rod, to the extremities of which the ends of the

string are attached. Supposing the increase of the length of the string, when stretched in a straight line by a force equal to the weight of the rod, to be equal to twice the length of the rod, to determine the position of equilibrium under the present circumstances.

If $2a =$ the natural length of the string, $2b =$ the length of the rod, and $\theta =$ the angle included between the two parts of the string,

$$\sin\theta = \frac{2b}{a^2}\{(a^2 + b^2)^{\frac{1}{2}} - b\}.$$

(11) Two equal rigid rods AC, BC, without weight, are connected together by a smooth hinge at C and rest in a vertical plane, their lower extremities, which are tied together by an elastic string AB, being placed upon a smooth horizontal plane. If α be the inclination of each rod to the horizon, when a weight W is fixed to the middle point of each, and α' the inclination when a weight W' is so fixed; to find the natural length of the string.

If a be the length of each rod, the natural length of the string is equal to

$$2a \cdot \frac{W\sin\alpha' - W'\sin\alpha}{W\tan\alpha' - W'\tan\alpha}.$$

(12) Two fine strings, slightly elastic, are fastened to the middle points of the sides of a uniform rectangular board, thus crossing the board parallel to its sides, and intersecting in the centre. Supposing the board to be suspended from the intersection of the strings, to find approximately the distance at which it will hang below the point of suspension.

If W be the weight of the board, $2a$, $2b$, the lengths of its sides, and λ the modulus of the elasticity of the strings, the required distance is equal to

$$\frac{\left(\frac{W}{\lambda}\right)^{\frac{1}{3}}}{\left(\frac{1}{a^3} + \frac{1}{b^3}\right)^{\frac{1}{3}}}.$$

(13) Six equal rods are connected together by hinges at their ends so as to form a hexagon, and, one of the rods being supported in a horizontal position, the opposite one is fastened to it by a fine elastic string joining their middle points. Supposing the modulus of elasticity to be equal to the weight of each rod, to find the natural length of the string in order that the hexagon may be equiangular in the position of equilibrium.

If a = the length of each rod, and l = the natural length of the string,

$$l = \frac{\sqrt{3}}{4} \cdot a.$$

(14) A heavy elastic string is laid upon a smooth double inclined plane in such a manner as to remain at rest: to find how much the string is stretched.

If W = the weight of the string, c = its natural length, and α, α', denote the inclinations of the planes; then the required extension is equal to

$$\frac{Wc}{2\lambda} \cdot \frac{\sin \alpha \sin \alpha'}{\sin \alpha + \sin \alpha'}.$$

(15) An elastic string, the upper end of which is fixed, rests on a rough inclined plane in the direction of greatest slope: to determine the limits of the extension of the string beyond its natural length.

Let α be the inclination of the plane; l the natural length of the string, and l' that of a portion of it the weight of which is equal to its modulus of elasticity; ϵ the angle of indifference: then the extension of the string will lie between the limits

$$\frac{l^2 \sin (\alpha + \epsilon)}{2l' \cos \epsilon}, \quad \frac{l^2 \sin (\alpha - \epsilon)}{2l' \cos \epsilon}.$$

CHAPTER VI.

VIRTUAL VELOCITIES.

THE Principle of Virtual Velocities consists in the following general proposition:

"If a material system, acted on by any forces whatever, be in equilibrium; and we conceive this system to experience, consistently with its geometrical relations, any indefinitely small arbitrary displacement; the sum of the forces, multiplied each of them by the resolved part, parallel to its direction, of the space described by its point of application, will be equal to zero; this resolved part being considered positive when it lies in the direction of its corresponding force, and negative when in an opposite direction."

The resolved parts of the spaces described by the points of application of the forces are called their Virtual Velocities. Let P, Q, R,... denote any system of forces acting on a system of points consistently with equilibrium; and let α, β, γ,.... denote their virtual velocities; then, as far as the first powers of α, β, γ,...... are concerned,

$$P\alpha + Q\beta + R\gamma + S\delta + \ldots\ldots = 0 \ldots\ldots\ldots (\text{A}).$$

The Principle of Virtual Velocities was first detected by Guido Ubaldi[1] as a property of the equilibrium of the lever and of moveable pullies. Its existence was afterwards recognized by Galileo[2] in the inclined plane, and the machines depending upon it. The expression 'moment' of a force or weight acting on any machine, was used by Galileo to denote its energy or effort to set the machine in motion, who accordingly declared that for the equilibrium of a machine, acted on by two forces,

[1] *Mechanicorum Liber; De Libra, De Cochlea.*

[2] *Della Scienza Mecanica, Opera*, Tom. I. p. 265; Bologna, 1655.

it is necessary that their moments should be equal, and should take place in opposite directions; he shewed moreover that the moment of a force is always proportional to the force multiplied by its virtual velocity. The word 'moment' was used in the same sense by Wallis[1], who adopted Galileo's principle of the equality of moments as the fundamental principle of Statics; and deduced from it the conditions for the equilibrium of the principal machines. Descartes[2] has likewise reduced the whole science of Statics to a single principle, which virtually coincides with that of Galileo; it is presented however under a less general aspect. The principle is, that it requires precisely the same force to raise a weight P through an altitude a, as a weight Q through an altitude b, provided that P is to Q as b to a. From this it follows, that two weights attached to a machine will be in equilibrium when they are disposed in such a manner that the small vertical paths which they can simultaneously describe are reciprocally as the weights.

Torricelli[3] is the author of another principle which may be immediately deduced from the principle of virtual velocities: the principle is, that when any two weights rigidly connected together are so placed that their centre of gravity is in the lowest position which it can assume consistently with the geometrical conditions to which they are subject, they will be in equilibrium. The principle of Torricelli has given birth to the following more general one, viz.—that any system whatever of heavy bodies will be in equilibrium when their centre of gravity is in its lowest or highest position.

John Bernoulli was the first to announce the principle of virtual velocities under its most general aspect in the form which we have given above, in a letter to Varignon[4], dated Bâle, Jan. 26, 1717. The striking value of the principle, as an instrument of analytical generalization, has been splendidly exhibited by Lagrange in his *Mécanique Analytique.*

[1] *Mechanica, sive de Motu, Tractatus Geometricus.*

[2] *Lettre* 73, Tom. I. 1657; *de Mechanica Tractatus, Opuscula Posthuma.*

[3] *De Motu gravium naturaliter descendentium*, 1644.

[4] *Nouvelle Mécanique*, Tom. II. sect. 9.

From the principle of virtual velocities may be immediately deduced the principle which was proposed by Maupertuis in the *Mémoires de l'Académie des Sciences de Paris* for the year 1740, under the name of the *Loi de Repos;* and which Euler has developed at large in the *Mémoires de l'Académie de Berlin* for the year 1751. Suppose that any number of forces $P, Q, R,...$ tending towards fixed centres and functional of their distances $p, q, r,...$ from the centres, act on a system of points rigidly connected together. Then, supposing the system of points to be slightly displaced, so that $p, q, r,...$ receive increments $dp, dq, dr,...$we shall have, by the principle of virtual velocities,

$$Pdp + Qdq + Rdr + ... = 0.$$

Let $d\Pi$ denote the left-hand member of this equation; then

$$d\Pi = 0 \ldots\ldots\ldots (B).$$

From this it appears that, if the system be so placed that Π may have a maximum or a minimum value, there will be equilibrium: this proposition constitutes Maupertuis' Principle of Rest. It does not however follow conversely that, whenever the system is at rest, Π shall have a maximum or minimum value, since by the principles of the differential calculus we know that the equation (B), although a necessary, is not the only condition for the existence of such a value. Lagrange[1] has shewn that, if Π be a minimum, the equilibrium will be stable, and, if a maximum, unstable.

As an example of this theory, it is evident that, if any system be in equilibrium under the action of gravity, there will be stable or unstable equilibrium accordingly as the centre of gravity is in the lowest or highest position which is compatible with the geometrical relations to which the system is subject.

The principle of equilibrium developed by Courtivron[2] is likewise grounded upon the principle of virtual velocities; Courtivron's Principle asserts that, if a system of bodies be in motion under the action of any forces varying according to any

[1] *Mécanique Analytique. Première Partie,* sect. 5.
[2] *Mémoires de l'Académie des Sciences de Berlin,* 1748, 1749.

assigned laws, a position of the system corresponding to a maximum or minimum value of the *vis viva* will be a position of equilibrium; a maximum value of the *vis viva* corresponding to stable, and a minimum to unstable equilibrium.

SECT. 1. *Equilibrium.*

(1) A particle P (fig. 90) is attracted towards two centres of force A and B; to find the position of the particle that it may be in equilibrium.

Let A, B, denote the two forces; let $AP=r$, $BP=s$, $AB=a$; draw PM at right angles to AB, and let $AM=x$, $PM=y$. Then, supposing P to receive some slight arbitrary displacement, the decrements dr, ds, of r, s, will be the virtual velocities of the forces A, B; hence, by the formula (A),

$$Adr + Bds = 0 \dots\dots\dots\dots (1).$$

But
$$r = (x^2+y^2)^{\frac{1}{2}}, \quad s = \{(a-x)^2+y^2\}^{\frac{1}{2}},$$

$$dr = \frac{xdx + ydy}{(x^2+y^2)^{\frac{1}{2}}}, \quad ds = \frac{-(a-x)\,dx + ydy}{\{(a-x)^2+y^2\}^{\frac{1}{2}}};$$

and therefore, by (1),

$$A\,\frac{xdx+ydy}{(x^2+y^2)^{\frac{1}{2}}} + B\,\frac{-(a-x)\,dx+ydy}{\{(a-x)^2+y^2\}^{\frac{1}{2}}} = 0;$$

but, since dx and dy are independent quantities, whatever be the small variation in the position of P, we have, equating their coefficients to zero,

$$\frac{Ax}{(x^2+y^2)^{\frac{1}{2}}} - \frac{B(a-x)}{\{(a-x)^2+y^2\}^{\frac{1}{2}}} = 0 \dots\dots\dots\dots (2),$$

$$\frac{Ay}{(x^2+y^2)^{\frac{1}{2}}} + \frac{By}{\{(a-x)^2+y^2\}^{\frac{1}{2}}} = 0 \dots\dots\dots\dots (3).$$

From (3) we have $y=0$, and therefore from (2) we see that $A=B$; thus it appears that if any particle be acted on by two forces tending towards two fixed centres, the conditions for its

equilibrium are, first, that it shall lie in the straight line joining the two centres, and, secondly, that the two forces shall be equal.

Euler; *Mémoires de l'Académie de Berlin,* 1751, p. 184.

(2) A rigid rod AB (fig. 91), without weight, rests over a peg O, and against a smooth wall CD, and is acted on by a weight P suspended from the extremity A; to determine its position of equilibrium and the pressures on the wall and the peg.

Draw EOF horizontally; let $AB = a$, $OB = x$, $OE = b$, $AF = y$. Let R, S, denote the reactions of the wall and peg against the rod, of which the former will be horizontal, and the latter at right angles to AB. Conceive the rod AB to be slightly displaced from its position of rest by making its end B slide along CD, the peg O still touching the rod; then it is evident that the point B will have no motion parallel to R, and that the motion of the point O of the rod resolved parallel to S will be an infinitesimal of the second order. Hence of the three forces P, S, R, P alone will have a virtual velocity. We have then, by the principle of virtual velocities,

$$Pdy = 0, \text{ or } dy = 0 \quad\ldots\ldots\ldots\ldots (1),$$

Now, by similar triangles AFO, BEO, there is

$$AF = AO \cdot \frac{BE}{BO},$$

and therefore

$$y = \frac{a - x}{x} (x^2 - b^2)^{\frac{1}{2}};$$

differentiating this equation and performing obvious simplifications, we shall have

$$dy = \frac{ab^2 - x^3}{x^2 (x^2 - b^2)^{\frac{1}{2}}} dx;$$

and therefore, by (1),

$$ab^2 - x^3 = 0, \quad x = (ab^2)^{\frac{1}{3}} \quad\ldots\ldots\ldots\ldots (2),$$

which defines the position of equilibrium.

In order to determine S, conceive each point of the rod to receive the same vertical displacement β, the point B thus sliding along CD and the rod moving parallel to itself.

Then, putting $\angle\, BOE = \phi$, the virtual velocities of P, R, S, will be $-\beta$, 0, $\beta \cos \phi$, respectively, and therefore

$$S.\beta \cos \phi = P.\beta, \quad S \cos \phi = P;$$

but $$\cos \phi = \frac{b}{x} = \left(\frac{b}{a}\right)^{\frac{1}{3}}, \text{ by (2)};$$

hence $$S = P\left(\frac{a}{b}\right)^{\frac{1}{3}}.$$

In order to find R, conceive the rod to be displaced along its length through a space β; then, the virtual velocities of P, R, S, being $-\beta \sin \phi$, $\beta \cos \phi$, 0, respectively, we have

$$R\beta \cos \phi = P\beta \sin \phi, \quad R = P \tan \phi;$$

but $$\tan \phi = \frac{(x^2 - b^2)^{\frac{1}{2}}}{b} = \frac{(a^{\frac{2}{3}} - b^{\frac{2}{3}})^{\frac{1}{2}}}{b^{\frac{1}{3}}}, \text{ by (1)};$$

therefore $$R = P\frac{(a^{\frac{2}{3}} - b^{\frac{2}{3}})^{\frac{1}{2}}}{b^{\frac{1}{3}}}.$$

Euler; *Mémoires de l'Académie de Berlin*, 1751, p. 196.

(3) A particle is placed upon a smooth inclined plane AB (fig. 92), at a point O, and acted on by a force P in a given direction; to determine the magnitude of P, that the particle may be at rest, and the pressure on the plane.

Let W be the weight of the particle, R the reaction of the plane, $\angle\, POB = \epsilon$, $\alpha =$ the inclination of AB to the horizon AC.

Conceive the particle O to receive a displacement β along the plane AB; then, the virtual velocity of R being zero, the virtual velocities of P, W, will be $\beta \cos \epsilon$, $-\beta \sin \alpha$, respectively. Hence, by the principle of virtual velocities,

$$P\beta \cos \epsilon = W\beta \sin \alpha, \quad P \cos \epsilon = W \sin \alpha \;\ldots\ldots\; (1),$$

which determines the value of P.

Next, displace the particle parallel to AC through a space β; then, the virtual velocity of W being zero, the virtual velocities of P, R, will be respectively $\beta \cos(\alpha + \epsilon)$, $-\beta \sin \alpha$, and therefore

$$P\beta \cos(\alpha + \epsilon) = R\beta \sin \alpha,$$

whence $$R = \frac{P\cos(\alpha+\epsilon)}{\sin\alpha} = \frac{W\cos(\alpha+\epsilon)}{\cos\epsilon}, \text{ by (1).}$$

Euler; *Mémoires de l'Académie de Berlin*, 1751, p. 191.

(4) A rigid rod OA (fig. 93), without weight, is acted on by a weight P hanging from its extremity A; the end O of the rod is fixed; EF is a spring in the form of a circular arc to a centre O, of which the force of contraction varies as the angle AOB, OB being a horizontal line; to find the position of the rod that it may be at rest.

Let $OA = a$, $\angle AOB = \phi$, $OF = b$; let α be the value of ϕ when the force of the spring's contraction is equal to E; then, corresponding to the angle ϕ, the force of contraction will be equal to

$$\frac{E\phi}{\alpha}.$$

Let AO be displaced slightly through an angle $d\phi$ into the position Oa; draw ap at right angles to AP, and let f be the new position of F; then, by the principle of virtual velocities,

$$P.Ap - \frac{E}{\alpha}\phi.Ff = 0 \text{ .... (1)};$$

but, since Ap, Aa, are respectively at right angles to OB, OA, it is clear that $\angle aAp = \phi$, and therefore

$$Ap = Aa\cos\phi = ad\phi.\cos\phi;$$

also $$Ff = bd\phi:$$

hence from (1) we have

$$Pa\cos\phi - \frac{E}{\alpha}\phi b = 0;$$

or $$\frac{\cos\phi}{\phi} = \frac{Eb}{Pa\alpha},$$

the required condition of equilibrium, from which ϕ is to be determined.

Euler; *Mémoires de l'Académie de Berlin*, 1751, p. 196.

(5) A smooth rod AB (fig. 94) rests against two horizontal bars which pierce the vertical plane through the rod at right angles at the points A', A''; the rod passes under the lower and over the higher bar, its lower extremity A being sustained upon

a smooth horizontal plane; to determine the pressures upon the two bars, and upon the horizontal plane.

The pressures upon the bars and upon the horizontal plane will be equal to their reactions upon the rod; the reactions of the bars upon the rod will be two forces R', R'', at right angles to the rod; and the reaction of the horizontal plane will be a vertical force R. Let G be the centre of gravity of the rod, at which point we will suppose its whole weight to be collected. Thus we have four forces R, R', R'', W, acting respectively at the four points A, A', A'', G, rigidly connected together, so as to produce equilibrium.

Let $AG = a$, $A'A'' = b$, and $\alpha =$ the inclination of the rod to the horizon.

Conceive the rod to receive a small displacement of such a character that it still remains in contact with the two bars; then evidently the virtual velocities of W and R will be equal, the one being a positive and the other a negative velocity, and therefore, α denoting the magnitude of the virtual velocity of each, we have

$$R\alpha - W\alpha = 0,$$

and therefore $$R = W \dots\dots\dots\dots\dots\dots\dots\dots (1).$$

Next, conceive the rod to receive a slight displacement, as in (fig. 95), by revolving through a small angle ω about the point A'' which is supposed to be kept stationary; the points a, a', g, being the new positions of A, A', G; from a draw am at right angles to the vertical line through A, and from g draw gn at right angles to the vertical Gn through G. Then, by the principle of virtual velocities,

$$R \,.\, Am - R' \,.\, A'a' - W \,.\, Gn = 0,$$

and therefore, by (1),

$$W(Am - Gn) - R' \,.\, A'a' = 0 \dots\dots\dots\dots (2);$$

but $$Am = Aa \cos\alpha = AA'' \,.\, \omega \,.\, \cos\alpha,$$

and $$Gn = Gg \cos\alpha = A''G \,.\, \omega \,.\, \cos\alpha;$$

and therefore $$Am - Gn = AG \,.\, \omega \cos\alpha = a\omega \cos\alpha;$$

also $$A'a' = A'A'' \,.\, \omega = b\omega.$$

Hence from (2) we have, substituting for $Am - Gn$ and $A'a'$ their values, $$Wa\omega \cos\alpha - b\omega R' = 0,$$

and therefore $$R' = \frac{Wa\cos\alpha}{b} \qquad\text{.......................(3).}$$

Again, conceive the rod, as in fig. 96, to be slightly displaced into the position $aa'a''b$, parallel to its original position, and still touching the horizontal plane; a, a', a'', b, being the new positions of A, A', A'', B. Then, the virtual velocities of R' and R'' being equal and opposite, we have

$$R'' = R' = W\frac{a\cos\alpha}{b}, \text{ by (3).}$$

(6) A string of given length passes over a given pully; to the two extremities of the string are attached two weights, one of which is capable of sliding freely on a given curve; to determine the curve on which the other ought to slide in order that, in every position of the two weights, they may be in equilibrium.

Let P, P' (fig. 97), denote the two weights in any position; A the pully; and let AB be a vertical line through A; let $AP = \rho$, $AP' = \rho'$. Draw PM, $P'M'$, at right angles to AB; let $AM = x$, $AM' = x'$, $\angle PAB = \phi$, $\angle P'AB = \phi'$.

Then, supposing the two weights to receive displacements along two small arcs of their corresponding curves of constraint, we have, by the principle of virtual velocities,

$$Pdx + P'dx' = 0;$$

and, since this relation is true for all corresponding points of the two curves, we have, integrating,

$$Px + P'x' = c,$$

where c is some constant quantity; and therefore, in polar coordinates,

$$P\rho\cos\phi + P'\rho'\cos\phi' = c \qquad\text{..................(1).}$$

Again, if l denote the length of the string,

$$\rho + \rho' = l \qquad\text{..........................(2).}$$

Supposing the curve on which P' moves to be the given one, we have $$f(\rho', \phi') = 0 \qquad\text{..........................(3),}$$

where $f(\rho', \phi')$ denotes some known function of ρ', ϕ'.

Eliminate from the equations (1), (2), (3), the quantities ρ', ϕ', and we shall get for the equation to the required curve

$$\chi(\rho, \phi) = 0,$$

$\chi(\rho, \phi)$ denoting some function of ρ, ϕ.

John Bernoulli; *Act. Erudit*, 1695, Febr. p. 59. Leibnitz; *Ib.* April, p. 184. L'Hôpital; *Act. Erudit. Suppl.* Tom. II. sect. 6, p. 289. Fuss; *Nova Acta Acad. Petrop.* 1788, p. 197.

(7) Four uniform beams AB, BC, CD, DE (fig. 98), connected together by smooth hinges, are placed in a position of equilibrium, the ends A and E being attached to two smooth hinges in the same horizontal line AE; the beam AB is equal to the beam ED, and the beam BC to the beam CD; to compare the angles BAE and CBD.

Let $AB = 2a = DE$, $BC = 2b = CD$, $\angle BAE = \alpha$, $\angle CBD = \beta$, $AE = c$; m = the weight of each of the lower and n = that of each of the higher beams. Then, by the principle of virtual velocities,

$$2md(a \sin \alpha) + 2nd(2a \sin \alpha + b \sin \beta) = 0,$$

and therefore

$$0 = (m + 2n) a \cos \alpha d\alpha + nb \cos \beta d\beta \dots\dots\dots\dots\dots(1).$$

Again, it is evident by the geometry that

$$c = 4a \cos \alpha + 4b \cos \beta,$$

and therefore $$0 = a \sin \alpha d\alpha + b \sin \beta d\beta \dots\dots\dots\dots\dots\dots(2).$$

Multiply (1) by $\sin \alpha \sin \beta$, and then, by (2), we have

$$(m + 2n) \cos \alpha \sin \beta \,.\, b \sin \beta d\beta = n \cos \beta \sin \alpha \,.\, b \sin \beta d\beta,$$

and therefore $$(m + 2n) \cos \alpha \sin \beta = n \cos \beta \sin \alpha,$$

$$\tan \alpha = \frac{m + 2n}{n} \tan \beta.$$

If $m = n$, we have

$$\tan \alpha = 3 \tan \beta.$$

(8) A beam AB (fig. 99), one end A of which is in contact with a smooth vertical plane OK, touches a smooth curve $\alpha\beta$; the plane of the beam and the curve being at right angles to the plane OK; to determine the nature of the curve, that the beam may rest in any position.

From any point O in the section OK of the vertical plane draw OL horizontal; from the point of contact P corresponding to any position of the beam draw PM vertical; let G be the centre of gravity of the beam; draw GH vertical. Let $OM=x$, $PM=y$, $AG=a$. Then, from the geometry, we see that

$$GH=y+PG.\frac{dy}{ds}=y+\left(a-x\frac{ds}{dx}\right)\frac{dy}{ds}=y-x\frac{dy}{dx}+a\frac{dy}{ds}.$$

Now since, for the equilibrium of a material system acted on by gravity, it is necessary that the differential of the altitude of its centre of gravity be zero, it is clear that in the present problem, equilibrium being possible for every position of the beam, GH must be of invariable magnitude. Hence, if $p=\dfrac{dy}{dx}$,

$$C=y-xp+\frac{ap}{(1+p^2)^{\frac{1}{2}}};$$

differentiating with respect to x, and putting $\dfrac{dp}{dx}=q$,

$$0=-xq+\frac{aq}{(1+p^2)^{\frac{3}{2}}};$$

hence $\qquad q=0, \quad \text{or } x=\dfrac{a}{(1+p^2)^{\frac{3}{2}}}:$

the former of these solutions gives a straight line for the locus of P; if we integrate the second equation, we shall get, for the equation to the locus of P,

$$C=(a^{\frac{2}{3}}-x^{\frac{2}{3}})^{\frac{3}{2}}+y.$$

Suppose the origin of co-ordinates to be so chosen that $y=0$ when $x=a$, in which case O will be the intersection of the beam,

in its horizontal position, with the line OK; then $C=0$, and the equation will be

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

(9) A particle O is acted upon by three forces A, B, C, passing through three points A, B, C; to determine the conditions for the equilibrium of the particle by the principle of virtual velocities.

The three points A, B, C, must all lie in a single plane containing the particle; also the relative magnitudes of the forces A, B, C, are given by any two of the three proportions,

$$B : C :: \sin COA : \sin AOB,$$
$$C : A :: \sin AOB : \sin BOC,$$
$$A : B :: \sin BOC : \sin COA.$$

Euler; *Mémoires de l'Académie de Berlin*, 1751, p. 185.

(10) A particle is acted on by any number of forces; to find the conditions to which their magnitudes and directions must be subject that the particle may be at rest.

From the particle draw straight lines representing the forces in magnitude and in direction; then, that the particle may be in equilibrium, its position must coincide with the centre of gravity of a number of equal particles placed at the extremities of the straight lines.

This celebrated theorem for the equilibrium of a particle is due to Leibnitz[1]: Euler[2] gave a demonstration by the aid of Maupertuis' *Loi de Repos*, and Lagrange[3] by the principle of Virtual Velocities. See also Poisson, *Traité de Mécanique*, Tom. I. No. 67. A more general theorem of forces, which comprehends this of Leibnitz as a particular case, has been given by Chasles[4]: see *Bulletins de l'Académie des Sciences et Belles-Lettres de Bruxelles*, 1840, 2me partie, p. 261.

[1] *Journal des Savans*, 1693; *Opera*, Tom. III. p. 283.
[2] *Mémoires de l'Académie de Berlin*, 1751.
[3] *Mécanique Analytique*, Tom. I. p. 106.
[4] *Correspondance Mathématique*, Tom. V. p. 106—108; 1829.

(11) A little ring, moveable along an elliptic wire, is attracted towards a centre of force which varies directly as the distance: to find the positions of equilibrium of the ring.

The required positions lie in an hyperbola, the asymptotes of which are parallel to the axes of the ellipse.

(12) A string of given length passes over a fixed point; to its extremities are attached two weights, one of which is capable of sliding freely along an inclined plane passing through the point; to determine the curve on which the other must be placed in order that, in every position of the two weights, they may be in equilibrium.

Let the angle which the inclined plane makes with the vertical be α; then, the notation remaining the same as in (6), the equation to the required curve will be

$$(P \cos \phi - P' \cos \alpha)\, \rho = c - P'l \cos \alpha,$$

which belongs to a conic section.

Fuss; *Nova Acta Acad. Petrop.* 1788.

(13) A beam PQ (fig. 100) rests against a smooth vertical plane AB and a smooth curve AP; to find the nature of the curve in order that the beam may be at rest in all positions.

Let G be the centre of gravity of the beam; draw PM horizontal; let $PQ = a$, $GP = c$, $AM = x$, $PM = y$; then the equation to the curve will be

$$\frac{(c-x)^2}{c^2} + \frac{y^2}{a^2} = 1,$$

which is the equation to an ellipse, the centre of which coincides with Q when PQ is horizontal.

(14) A uniform beam AB (fig. 101) rests upon a smooth horizontal plane Ca, and against a smooth vertical plane Cb; a string ACP, one end of which is attached to the end A of the beam, hangs through a small ring at C, a weight P being attached to the other extremity of the string; to find the position of the beam when at rest.

If W denote the weight of the beam, and θ the angle BAC, then

$$\tan\theta = \frac{W}{2P}.$$

(15) A plane equilateral pentagon is formed of five equal rods AB, BC, CD, DE, EA, loosely jointed together: the plane of the pentagon is vertical, the point C being uppermost: the angular points B, D, of the pentagon are capable of sliding on a smooth fixed horizontal rod: to find the relation between the inclinations of the rods AB, BC, to the horizon, when there is equilibrium.

If θ, ϕ, be the respective inclinations of AB, BC,

$$\tan\theta = 2\tan\phi.$$

(16) A plane figure, bounded by a parabola, rests in a vertical plane, on two points in the same horizontal line, the centre of gravity of the figure being in the axis of the parabola at a given distance from the vertex; to find the position of equilibrium.

Let $2a$ be the distance between the two points, $4m$ the latus rectum, h the distance of the centre of gravity from the vertex, and θ the inclination of the axis to the vertical in the position of equilibrium; then the equation

$$\sin\theta\,\{3a^2\cos^4\theta - 4m\,(h-m)\cos^2\theta + 4m^2\} = 0$$

will give the positions of equilibrium.

(17) A particle is attracted towards each of two fixed centres of force varying inversely as the square of the distance; to find the equation to the surface on which it may remain at rest in every position.

If μ, μ', be the absolute forces of attraction; r, r', simultaneous distances of the particle from the centres; and a, a', given values of r, r'; the equation to the surface will be

$$\frac{\mu}{r} + \frac{\mu'}{r'} = \frac{\mu}{a} + \frac{\mu'}{a'}.$$

(18) To the extremity B of a rod AB (fig. 102), which is able to revolve freely about A, is attached an indefinitely fine thread BCM, passing over a point C vertically above A, and sustaining

a heavy particle at M on a smooth curve CMN in the vertical plane BAC; to determine the nature of the curve in order that, for all positions of the rod and particle, the system may be in equilibrium.

Let $AB = 2a$, $AC = b$, $l =$ the length of the thread BCM, $\rho =$ the straight line CM, $\theta = \angle ACM$, $m =$ the mass of the particle, $m' =$ the mass of the rod. Then the equation to the curve will be

$$m'\rho^2 + 2(2mb\cos\theta - m'l)\rho = c,$$

where c is a constant quantity.

This problem was proposed by Sauveur to L'Hôpital, by whom a solution was published in the *Acta Eruditorum*, 1695, Febr. p. 56. The curve was shewn by John Bernoulli, *Ib.* p. 59, to be an Epitrochoid. See also James Bernoulli, *Ib.* p. 65.

(19) A rod, passing through a fixed ring in the axis, which is vertical, of a surface of revolution, rests in all positions with one end on the surface: to find the nature of the generating curve.

Let P be any point of the curve, O the ring, θ the inclination of OP to the vertical line drawn downwards from O; then, if $OP = r$, and $2a =$ the length of the rod, the equation to the curve is

$$r = a + c\sec\theta,$$

c being an arbitrary constant.

(20) The plane of a parabola is vertical and its axis horizontal: two weights are placed on the curve, being attached to the ends of a fine string which passes over a pully at the focus: to find the condition of equilibrium.

If P, P', be the weights and y, y', their distances below the axis, the condition of equilibrium is expressed by the equation

$$\frac{P}{P'} = \frac{y}{y'}.$$

(21) A uniform rod of length l rests between the concave arc of an ellipse and the axis minor, which is vertical, the axes of

the ellipse being $2l$ and l: to determine the position of the rod's equilibrium.

The rod will be in equilibrium at all inclinations to the horizon.

(22) Two weights, P, P', are attached to the ends of a string placed upon a parabola of which the axis is vertical; to find the condition of equilibrium.

If x, x', represent the distances of the weights P, P', below the vertex, and $4m$ denote the latus rectum, the weights will rest if

$$\frac{P^2}{x'} - \frac{P'^2}{x} = \frac{P'^2 - P^2}{m}.$$

(23) To find a point on the surface

$$\frac{x^3}{a^3} + \frac{y^3}{b^3} + \frac{z^3}{c^3} = 1,$$

where a particle, attracted towards the origin by any force, will remain at rest.

The point required is given by the equations

$$\frac{x}{a^3} = \frac{y}{b^3} = \frac{z}{c^3} = \frac{1}{(a^6 + b^6 + c^3)^{\frac{1}{3}}}.$$

(24) A right cylinder on an elliptic base, the semiaxes of which are a and b, rests between two smooth planes inclined at right angles to each other, the line of intersection of the planes being parallel to the cylinder's axis, which is horizontal: to determine its positions of equilibrium, (1) when the inclination of one of the planes is greater than $\tan^{-1}\frac{a}{b}$, (2) when the inclination of each plane is less than $\tan^{-1}\frac{a}{b}$.

Let α be the inclination of one of the planes to the horizon, and ϕ the inclination of the major axis of a transverse section of the cylinder to the other plane. Then, under the hypothesis (1), there will be two positions of equilibrium, viz. when the major-

axis of the section is parallel to either of the two planes. Under the hypothesis (2), there will be three positions of equilibrium, viz. two the same as under the former hypothesis, and one as defined by the equation

$$\cos 2\phi = -\frac{a^2+b^2}{a^2-b^2} \cdot \cos 2\alpha.$$

SECT. 2. *Stability and Instability of Equilibrium.*

(1) Three weights are suspended from the angles of an isosceles triangle, the plane of which is vertical, and which is supported by a horizontal axis passing through its centre of gravity, about which it is able to revolve: to determine its positions of equilibrium, when the two weights suspended from the extremities of the base of the triangle are equal to each other, and are each of them greater than the third; and to determine the character of the equilibrium.

Let A (fig. 103) be the vertex, and AB the axis of the triangle, G being the centre of gravity. Let $AB = 3a$. Let P be the smaller weight, and Q either of the larger ones. The two weights Q may be collected at B. Let θ be the inclination of AB to the vertical. The moment of $2Q$ about G is $2Q\, a \sin\theta$, and that of P, in an opposite direction, $P \,.\, 2a \sin\theta$.

The resultant of these two moments is

$$2a\,(Q-P)\sin\theta,$$

estimated in the direction of the arrows. This moment, from $\theta = 0$ to $\theta = \pi$, always acts in the same direction, provided that θ be not actually 0 or π, in which cases the moment vanishes. Hence we see that, for equilibrium,

$$\theta = 0 \text{ or } \theta = \pi,$$

the former corresponding to stable and the latter to unstable equilibrium.

(2) AB (fig. 104) is a beam moveable about a hinge A; C is a small pully in the vertical line through A, AC being equal to AG, where G is the centre of gravity of AB; a fine string

is attached to G, which passes over C and supports a weight P; to find the stable and unstable positions of equilibrium of the beam.

Let $GA = a = CA$, $l =$ the length of the string GCP, $W =$ the weight of the beam AB, $\angle GCA = \theta$; $x =$ the vertical distance of the centre of gravity of P and the beam below the horizontal plane through C.

Now, from the geometry,

$$CP = l - 2a\cos\theta;$$

and the distance of G below the horizontal plane through C is

$$a + a\cos 2\theta;$$

hence, by the property of the centre of gravity of bodies,

$$(P + W)\,x = P(l - 2a\cos\theta) + Wa\,(1 + \cos 2\theta).$$

Now, when there is equilibrium, $\dfrac{dx}{d\theta} = 0$ and therefore $\dfrac{du}{d\theta} = 0$, where

$$u = W\cos 2\theta - 2P\cos\theta;$$

therefore

$$\frac{du}{d\theta} = -2W\sin 2\theta + 2P\sin\theta = 0,$$

and therefore $\quad \sin\theta\,(2W\cos\theta - P) = 0;$

hence for equilibrium it is necessary that $\sin\theta = 0$, and therefore $\theta = 0$, or $\cos\theta = \dfrac{P}{2W}$.

Differentiating u a second time, we get

$$\frac{d^2u}{d\theta^2} = -4W\cos 2\theta + 2P\cos\theta;$$

if $\theta = 0$, we have $\quad \dfrac{d^2u}{d\theta^2} = -4W + 2P;$

hence $\dfrac{d^2u}{d\theta^2}$ and therefore $\dfrac{d^2x}{d\theta^2}$ will be positive or negative, and therefore x a minimum or a maximum according as P is

greater or less than $2W$; hence, if P be greater than $2W$, $\theta = 0$ gives a position of unstable equilibrium, and, if P be less than $2W$, one of stability.

Again, if $\cos\theta = \dfrac{P}{2W}$, we shall have

$$\frac{d^2u}{d\theta^2} = -4W(2\cos^2\theta - 1) + 2P\cos\theta = \frac{4W^2 - P^2}{W};$$

if then $2W$ be greater than P, $\dfrac{d^2u}{d\theta^2}$ and therefore $\dfrac{d^2x}{d\theta^2}$ is positive, and therefore the altitude of the centre of gravity of P and the beam is a maximum, and therefore the position will be one of unstable equilibrium; if $2W$ be less than P, $\cos\theta$ will be impossible, or the only position of equilibrium will be the unstable one given by $\theta = 0$.

(3) A uniform beam PQ (fig. 105) is placed upon two smooth inclined planes AB, AC; to find whether its position of equilibrium is one of stability or of instability.

Let G be the centre of gravity of the beam; from P and G draw PM, GH, at right angles to the horizontal plane bAc through A. Let $\angle BAb = \alpha$, $\angle CAc = \beta$, $PG = a = QG$, $\theta =$ the angle of inclination of PQ to the horizon, $GH = z$. Then, by the geometry,

$$z = a\sin\theta + PM = a\sin\theta + AP\sin\alpha$$

$$= a\sin\theta + \sin\alpha \,.\, 2a\frac{\sin(\beta - \theta)}{\sin(\alpha + \beta)}$$

$$= \frac{a}{\sin(\alpha+\beta)}\{\sin(\beta - \alpha)\sin\theta + 2\sin\alpha\sin\beta\cos\theta\};$$

then, if z be a maximum or minimum,

$$u = \sin(\beta - \alpha)\sin\theta + 2\sin\beta\sin\alpha\cos\theta$$

will be a maximum or minimum; hence

$$\frac{du}{d\theta} = \sin(\beta - \alpha)\cos\theta - 2\sin\beta\sin\alpha\sin\theta = 0;$$

and therefore, for equilibrium,

$$\tan\theta = \frac{\sin(\beta - \alpha)}{2\sin\beta\sin\alpha},$$

a positive quantity, if, as we will suppose, β be greater than α.

Differentiating u a second time, we have

$$\frac{d^2u}{d\theta^2} = -\sin(\beta - \alpha)\sin\theta - 2\sin\beta\sin\alpha\cos\theta;$$

from this it appears that, since θ is clearly less than $\frac{\pi}{2}$, $\frac{d^2u}{d\theta^2}$ will be negative, or that, in the position of equilibrium the centre of gravity is at its maximum altitude; hence the equilibrium will be unstable.

(4) A square board hangs in a vertical plane by a string, which, passing over a smooth nail, is fastened at its ends to two points symmetrically situated in one edge of the board. To investigate the positions and circumstances of equilibrium.

Let G (fig. 106) be the centre of gravity of the board, KCL the string passing over the nail C and attached to the board at the points K, L; draw GH at right angles to KL, and let ACB be an indefinite horizontal line.

Let l = the length of the string, $KL = a$, c = the length of a side of the square, θ = the inclination of HG to the vertical, $\frac{1}{2}u$ = the distance of G below AB.

Since $CK + CL = l$, the locus of C, relatively to KL, is an ellipse of which K, L, are the foci. Now conceive θ to be for the present invariable: then it is evident that H and therefore G will be the lower, the lower the highest point of the ellipse: the highest point must therefore coincide with C, and AB must accordingly be a tangent to the ellipse.

The distance of H from the tangent of the ellipse, the axis major of the ellipse being l and eccentricity $\frac{a}{l}$, is equal to

$$\tfrac{1}{2}(l^2 - a^2\cos^2\theta)^{\frac{1}{2}}:$$

hence
$$u = c\cos\theta + (l^2 - a^2\cos^2\theta)^{\frac{1}{2}}.$$

Differentiating we shall get

$$\frac{du}{d\theta} = \sin\theta\left\{\frac{a^2\cos\theta}{(l^2 - a^2\cos^2\theta)^{\frac{1}{2}}} - c\right\},$$

$$\frac{d^2u}{d\theta^2} = \cos\theta\left\{\frac{a^2\cos\theta}{(l^2 - a^2\cos^2\theta)^{\frac{1}{2}}} - c\right\} - \frac{l^2a^2\sin^2\theta}{(l^2 - a^2\cos^2\theta)^{\frac{3}{2}}}.$$

Putting $\frac{du}{d\theta} = 0$, we shall obtain

$$\theta = 0 \text{ or } \cos\theta = \frac{cl}{a\,(a^2+c^2)^{\frac{1}{2}}}.$$

If
$$\theta = 0,\ \frac{d^2u}{d\theta^2} = \frac{a^2}{(l^2-a^2)^{\frac{1}{2}}} - c:$$

if $\cos\theta = \frac{cl}{a\,(a^2+c^2)^{\frac{1}{2}}}$, $\frac{d^2u}{d\theta^2} = -\frac{l^2c^2\sin^2\theta}{a^4\cos^2\theta}$ = a negative quantity.

Thus we see that, if l be less than $\frac{a}{c}(a^2+c^2)^{\frac{1}{2}}$, there will be three positions of equilibrium, and, if it be greater, only one. In the former case

$$\frac{a^2}{(l^2-a^2)^{\frac{1}{2}}} - c = \text{a positive quantity},$$

and therefore $\theta = 0$ corresponds to a position of unstable, and

$$\cos\theta = \frac{cl}{a\,(a^2+c^2)^{\frac{1}{2}}}$$

to two positions of stable equilibrium.

In the latter case

$$\frac{a^2}{(l^2-a^2)^{\frac{1}{2}}} - c = \text{a negative quantity},$$

and therefore $\theta = 0$ corresponds to a position of stable equilibrium.

(5) The extremities of a uniform slender rod, acted on by gravity, are in contact with two planes (one horizontal and the other vertical), at a point in the intersection of which is a centre of attractive force, varying inversely as the square of the distance, which, at the centre of gravity of the rod, is equal to half the force of gravity; to find the position of equilibrium of the rod, and to ascertain whether it is stable or unstable.

Let AB (fig. 107) be the rod, G its centre of gravity, P any point in it; join OP, O being the centre of attraction; draw

PM at right angles to the horizontal line OA. Let $AG = a = BG$, $AP = s$, $OP = r$, $PM = y$, $\angle\, OAB = \theta$; let the mass of a unit of the rod's length be taken as the unit of mass.

Then the attraction on an element δs of the rod at P will be equal to $g\delta s$ vertically downwards, and to $\frac{1}{2}g\frac{a^2}{r^2}\delta s$ towards the centre O. Hence, adopting the notation which was employed above in the enunciation of Maupertuis' Principle,

$$\Pi = \delta^{-1}d^{-1}\left\{g\delta s\, dy + \tfrac{1}{2}g\frac{a^2}{r^2}\delta s\, dr\right\},$$

and therefore $\quad d\Pi = g\delta^{-1}(\delta s\, dy) - \frac{1}{2}a^2 g\, d\delta^{-1}\frac{\delta s}{r}.$

Now, by the geometry,

$$y = s\sin\theta, \quad dy = s\cos\theta\, d\theta;$$

hence $\quad \delta^{-1}(\delta s\, dy) = \delta^{-1}(s\delta s\cos\theta\, d\theta) = \delta^{-1}(s\delta s)\cos\theta\, d\theta,$

and consequently, the limits of the integration being obviously 0, $2a$,

$$\delta^{-1}(\delta s\, dy) = 2a^2\cos\theta\, d\theta.$$

Again, by the geometry, we see that

$$r^2 = s^2 - 4as\cos^2\theta + 4a^2\cos^2\theta;$$

hence we have

$$\delta^{-1}\frac{\delta s}{r} = \delta^{-1}\frac{\delta s}{(s^2 - 4as\cos^2\theta + 4a^2\cos^2\theta)^{\frac{1}{2}}}$$

$$= C + \log\{s - 2a\cos^2\theta + (s^2 - 4as\cos^2\theta + 4a^2\cos^2\theta)^{\frac{1}{2}}\},$$

and therefore, between the limits $s = 0$, $s = 2a$,

$$\delta^{-1}\frac{\delta s}{r} = \log\frac{\sin\theta\,(1+\sin\theta)}{\cos\theta\,(1-\cos\theta)} = \log\frac{\tan\frac{1}{4}(\pi+2\theta)}{\tan\frac{1}{2}\theta}.$$

Hence $\quad d\delta^{-1}\frac{\delta s}{r} = \frac{\frac{1}{2}\sec^2\frac{1}{4}(\pi+2\theta)}{\tan\frac{1}{4}(\pi+2\theta)} - \frac{\frac{1}{2}\sec^2\frac{1}{2}\theta}{\tan\frac{1}{2}\theta}$

$$= \frac{1}{\cos\theta} - \frac{1}{\sin\theta}.$$

Putting for $\delta^{-1}(\delta s\, dy)$ and $d\delta^{-1}\dfrac{\delta s}{r}$ their values in the expression for $d\Pi$, we get

$$\frac{d\Pi}{d\theta} = \tfrac{1}{2}a^2 g\left(-\frac{1}{\cos\theta} + \frac{1}{\sin\theta} + 4\cos\theta\right).$$

Now there will be equilibrium if Π have a maximum or a minimum value, and therefore if

$$\sin\theta - \cos\theta - 4\sin\theta\cos^2\theta = 0;$$

multiplying this equation by $\cos\theta + \sin\theta$, a quantity which cannot be equal to zero from $\theta = 0$ to $\theta = \frac{1}{2}\pi$, we get

$$-\cos 2\theta - (1 + \cos 2\theta)(1 + \sin 2\theta - \cos 2\theta) = 0,$$

$$\cos 2\theta + \sin 2\theta + \sin 2\theta\,(\cos 2\theta + \sin 2\theta) = 0,$$

$$(1 + \sin 2\theta)(\cos 2\theta + \sin 2\theta) = 0;$$

but it is evident that $1 + \sin 2\theta$ cannot become zero for any value of θ from 0 to $\frac{1}{2}\pi$; hence

$$\cos 2\theta + \sin 2\theta = 0, \quad \tan 2\theta = -1, \quad \theta = \tfrac{3}{8}\pi,$$

which determines the position of equilibrium.

Again, differentiating the expression for $\dfrac{d\Pi}{d\theta}$, we have

$$\frac{d^2\Pi}{d\theta^2} = -\tfrac{1}{2}a^2 g\left(\frac{\sin\theta}{\cos^2\theta} + \frac{\cos\theta}{\sin^2\theta} + 4\sin\theta\right),$$

which is evidently a negative quantity when $\theta = \frac{3}{8}\pi$; hence, for this value of θ, Π receives a maximum value, and therefore the equilibrium is one of instability.

(6) A particle is placed in a position of equilibrium between two centres of attractive force, varying according to any power of the distance; to determine for what laws of force the equilibrium is stable and for what unstable.

The equilibrium will be stable or unstable according as the forces attract in direct or inverse powers respectively.

(7) Two heavy particles, connected together by a thread PAQ (fig. 108) passing over the convex side of a circle situated

in a vertical plane, balance each other when placed at P and Q; to determine the position of P, Q, and to ascertain whether the equilibrium is stable or unstable, the weight of the thread being neglected.

Let O be the centre of the circle, OA a vertical radius; let $\angle POQ = \alpha$, $\angle POA = \theta$, $\angle QOA = \phi$; and let m, n, denote the masses of the particles. Then we shall have for the equilibrium, which will be unstable, the equation

$$\tan \tfrac{1}{2}(\phi - \theta) = \frac{m-n}{m+n} \tan \tfrac{1}{2}\alpha.$$

(8) A uniform rod passes through a hole in a spherical shell, and rests with one end against the internal surface, the length of the rod being equal to twice that of the diameter; having given the inclination of the rod to the vertical when it is in a position of stable equilibrium, to determine its inclinations to the vertical when in its positions of unstable equilibrium.

If α denote its inclination to the vertical when in its position of stable equilibrium, then its inclinations for its two positions of unstable equilibrium will be

$$\tfrac{1}{3}(\pi + \alpha) \text{ and } \tfrac{1}{3}(\pi - \alpha).$$

(9) A board in the form of an isosceles triangle PQR (fig. 109), of which PQ is the base, is placed upon two inclined planes AL, AM, at right angles to each other, the plane of the triangle being vertical and perpendicular to the intersection of the two inclined planes: to find the position of equilibrium and to determine whether it is stable or unstable.

If $PQ = 2a$, $h =$ the altitude of the triangle, $\alpha =$ the inclination of AP and $\theta =$ that of PQ to the horizon; then the equation

$$\tan \theta = \frac{a \cos 2\alpha}{a \sin 2\alpha + \frac{1}{3}h}$$

will define a position of unstable equilibrium.

(10) To find the position and nature of the equilibrium of a cube which rests between two smooth inclined planes, two edges

of the lowest face of the cube being parallel to the intersection of the planes.

If α, α', denote the inclinations of the two planes, and θ the inclination of the base of the cube to the horizon, the position of equilibrium, which is unstable, is given by the equation

$$\tan\theta = \frac{\sin(\alpha' - \alpha)}{\sin(\alpha + \alpha') + 2\sin\alpha\sin\alpha'}.$$

(11) A very small bar of matter is moveable about one extremity which is fixed half way between two centres of force attracting inversely as the square of the distance: to find the positions of the equilibrium of the bar and to determine their nature.

Let A, B, be the two centres of force, C the middle point between them, CL the position of the bar at rest. Let $AB = 2a$, $CL = l$, $\angle BCL = \phi$, and let μ, μ', denote the absolute forces of A, B, respectively.

Of the two quantities μ, μ', let μ be not the smaller: then, if

$$\frac{\mu}{\mu'} > \frac{a + 2l}{a - 2l} \quad\text{.......................} \quad (1),$$

there will be only two positions of equilibrium, defined by $\phi = 0$, $\phi = \pi$, the former unstable, the latter stable.

If the inequality (1) be not satisfied, $\phi = 0$, $\phi = \pi$, correspond to two positions of stable equilibrium; two unstable positions being given by the equation

$$\cos\phi = \frac{\mu - \mu'}{\mu + \mu'} \cdot \frac{a}{2l}.$$

(12) A heavy uniform rod AB hangs vertically downwards from a smooth hinge at A: each particle of the rod is attracted, according to the law of the first power of the distance, towards a centre of force at a point C vertically above A, AC being equal to AB: to ascertain the condition for stability or instability.

Let μ denote the absolute force, a the length AC or AB; then, if $\mu a < g$, the equilibrium is stable, and, if $\mu a > g$, unstable.

(13) A uniform rod is constrained to slide with its extremities on a conic section, the major axis of which is vertical and of which the latus-rectum is less than the length of the rod: to find the position of stable equilibrium.

The equilibrium will be stable when the rod passes through the focus.

H. G.; now Bishop of Carlisle. *Quarterly Journal of Pure and Applied Mathematics*, Vol. II. p. 66.

CHAPTER VII.

ATTRACTIONS.

(1) To find the attraction of the solid, generated by the revolution of the curve $r^2 = a^2 \cos\theta$ round its axis, on a particle placed at the origin, the particles attracting inversely as the squares of the distances.

The required attraction, μ denoting the absolute force, is equal to

$$\int_0^{\frac{1}{2}\pi}\int_0^r r d\theta dr \,.\, 2\pi r \sin\theta \,.\, \frac{\mu}{r^2}\cos\theta$$

$$= 2\pi\mu \int_0^{\frac{1}{2}\pi}\int_0^r \sin\theta\cos\theta \, d\theta \, dr$$

$$= 2\pi\mu a \int_0^{\frac{1}{2}\pi} (\cos\theta)^{\frac{3}{2}} \sin\theta \, d\theta$$

$$= -2\pi\mu a \, {}_1^0\left\{\frac{2}{5}(\cos\theta)^{\frac{5}{2}}\right\}$$

$$= \frac{4}{5}\pi\mu a.$$

(2) To determine that point in the axis of a hemispherical body, the particles of which attract inversely as the square of the distance, where a corpuscle must be placed so as to remain in equilibrium by the equal and contrary action of the matter of the hemisphere surrounding it.

Let CA (fig. 110) be the axis of the hemisphere, DCD' a diameter of its base, and O the required position of the corpuscle; DAD' the intersection of the plane through CA, DCD', with the surface of the hemisphere; draw BOB' at right angles to CA, join OD; take any points P, p, in the arcs AB, BD, join PO, po, and draw PM, pm, at right angles to CA. Let $CA = a = CD$, $CO = c$, $OB = b$, $OD = b'$, $OP = r$, $OM = x$,

$PM = y$, $Op = r'$, $Om = x'$, $pm = y'$; $\mu =$ the absolute attraction of a unit of mass of the hemisphere, and $\rho =$ its density; $A =$ the attraction of the portion BAB' of the hemisphere on the corpuscle, and B of the portion $BDB'D'$.

The attraction of a thin slice of the hemisphere at right angles to its axis at the point M, and having a thickness dx, will be

$$2\pi\mu\rho dx \left(1 - \frac{x}{r}\right),$$

as may be seen in elementary treatises on attraction[1]; hence

$$A = 2\pi\mu\rho \int_0^{a-c} dx \left(1 - \frac{x}{r}\right),$$

$$\frac{A}{2\pi\mu\rho} = a - c - \int_0^{a-c} \frac{x\,dx}{r} \quad\text{..........................}\quad (1);$$

similarly we have

$$B = 2\pi\mu\rho \int_0^c dx' \left(1 - \frac{x'}{r'}\right),$$

$$\frac{B}{2\pi\mu\rho} = c - \int_0^c \frac{x'\,dx'}{r'} \quad\text{..................................}\quad (2).$$

Now from the geometry we see that

$$r^2 = x^2 + y^2 = x^2 + a^2 - (x + c)^2 = a^2 - c^2 - 2cx = b^2 - 2cx;$$

hence $\qquad 2cx = b^2 - r^2, \qquad c\,dx = -r\,dr,$

and therefore $\qquad \dfrac{x\,dx}{r} = -\dfrac{b^2 - r^2}{2c^2}\,dr;$

hence from (1), it being observed that r is equal to $a - c$, b, when x is equal to $a - c$, 0, we have

$$\frac{A}{2\pi\mu\rho} = a - c + \frac{1}{2c^2}\int_b^{a-c} (b^2 - r^2)\,dr \quad\text{............}\quad (3).$$

Again, from the geometry,

$$r'^2 = x'^2 + y'^2 = x'^2 + a^2 - (c - x')^2 = a^2 - c^2 + 2cx' = b^2 + 2cx';$$

hence $\qquad 2cx' = r'^2 - b^2, \qquad c\,dx' = r'\,dr',$

and therefore $\qquad \dfrac{x'\,dx'}{r'} = -\dfrac{b^2 - r'^2}{2c^2}\,dr';$

[1] Todhunter's *Analytical Statics*, p. 235.

hence from (2), since r' is equal to b', b, when x' is equal to c, 0,

$$\frac{B}{2\pi\mu\rho} = c + \frac{1}{2c^2}\int_b^{b'} (b^2 - r'^2)\, dr';$$

but it is evident that

$$\int_b^{b'} (b^2 - r'^2)\, dr' = \int_b^{b'} (b^2 - r^2)\, dr;$$

hence $$\frac{B}{2\pi\mu\rho} = c + \frac{1}{2c^2}\int_b^{b'} (b^2 - r^2)\, dr \quad\ldots\ldots\ldots\ldots (4).$$

But, since the corpuscle is in equilibrium, we must have $A = B$, and therefore, by (3) and (4),

$$a - c + \frac{1}{2c^2}\int_b^{a-c} (b^2 - r^2)\, dr = c + \frac{1}{2c^2}\int_b^{b'} (b^2 - r^2)\, dr;$$

hence $$a - 2c = \frac{1}{2c^2}\int_{a-c}^{b'} (b^2 - r^2)\, dr:$$

performing the integration, and putting for b' its value $(a^2 + c^2)^{\frac{1}{2}}$, we shall get, after certain obvious simplifications,

$$a^3 - 4c^3 = (a^2 - 2c^2)\,(a^2 + c^2)^{\frac{1}{2}};$$

squaring both sides and simplifying,

$$12c^4 - 8a^3c + 3a^4 = 0,$$

an equation from which the value of c is to be determined; as an approximation $c = \frac{3}{5}a$.

Diarian Repository, p. 629.

(3) Two infinite lines in space, inclined to each other at a given angle, attract each other with forces varying inversely as the square of the distance: to find the whole attraction in the direction of the shortest line between them, the mutual attraction of two units of length, collected at centres and separated by the unit of distance, being considered equal to unity.

Let EE', FF', (fig. 111) be the two straight lines; AB the shortest distance between them. Take P, Q, any two points in the lines: join PQ. Through B draw a plane at right angles to AB, and let QK be the projection of QP on this plane.

Let $AB = c = PK$, $PQ = s$, $AP = r = BK$, $BQ = r'$, $\angle QPK = \phi$. Let θ = the angle between the two lines EE', FF'.

The mutual attraction of P and Q is equal to $\dfrac{dr\,.\,dr'}{s^2}$: and its resolved part parallel to AB is equal to

$$\frac{dr\,.\,dr'\,.\cos\phi}{s^2} = \frac{dr\,.\,dr'\,.\,s\cos\phi}{s^3} = \frac{dr\,.\,dr'\,.\,c}{s^3}.$$

But $$s^2 = QK^2 + PK^2 = r^2 + r'^2 - 2rr'\cos\theta + c^2.$$

Hence the whole mutual attraction parallel to AB is equal to

$$c\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{dr\,.\,dr'}{(c^2 + r^2 + r'^2 - 2rr'\cos\theta)^{\frac{3}{2}}}$$

$$= c\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{dr\,dr'}{\{(r' - r\cos\theta)^2 + c^2 + r^2\sin^2\theta\}^{\frac{3}{2}}}$$

$$= c\int_{-\infty}^{+\infty}\Big[\frac{(r' - r\cos\theta)\,dr}{(c^2 + r^2\sin^2\theta)\,\{(r' - r\cos\theta)^2 + c^2 + r^2\sin^2\theta\}^{\frac{1}{2}}}\Big]_{-\infty}^{+\infty}$$

$$= 2c\int_{-\infty}^{+\infty}\frac{dr}{c^2 + r^2\sin^2\theta} = \frac{2}{\sin\theta}\Big\{\tan^{-1}\frac{r\sin\theta}{c}\Big\}_{-\infty}^{+\infty} = \frac{2\pi}{\sin\theta}.$$

(4) A slender ring DEF, (fig. 112), is attached to another slender ring ABC by means of a string AD, the length of which is equal to the radius of ABC; supposing DEF to lie entirely within ABC, to determine the tension of the string, when DEF is at rest: the force of attraction of ABC varying inversely as the cube of the distance.

It is plain that, when the smaller ring is at rest, AD will coincide with a radius of the larger.

Let P be any point in the smaller ring, R in the larger. Join PR, DR, DP, and produce DP to R. Let a = the radius of the larger ring, $DE = a'$, $DP = e$, $PR = \rho$, $\angle PDC = \theta$, $\angle RDQ = \phi$, $\angle RPQ = \psi$; let ds, ds', be elements of the two rings at R, P, respectively, and μ, μ', the masses of units of length of the two rings; let T = the required tension.

Then $$T = \mu\mu' \int\left\{\cos\theta\, ds' \int \frac{ds}{\rho^3}\cos\psi\right\}.$$

Now $$ds = a\,d\phi, \quad ds' = \tfrac{1}{2}a'd\,(\pi - 2\theta) = -a'd\theta.$$

Hence $$T = \mu\mu' aa' \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\left\{\cos\theta\, d\theta \int_0^{2\pi}\frac{d\phi}{\rho^3}\cos\psi\right\}.$$

$$\int_0^{2\pi}\frac{d\phi}{\rho^3}\cos\psi = \int_0^{2\pi}\frac{a\cos\phi - c}{(a^2+c^2-2ac\cos\phi)^2}\,d\phi = \frac{1}{2}\frac{d}{dc}\int_0^{2\pi}\frac{d\phi}{a^2+c^2-2ac\cos\phi}$$

$$= \frac{1}{2}\frac{d}{dc}\int_0^{2\pi}\frac{\sec^2\frac{\phi}{2}\,d\phi}{(a^2+c^2)\left(1+\tan^2\frac{\phi}{2}\right) - 2ac\left(1-\tan^2\frac{\phi}{2}\right)}$$

$$= \frac{d}{dc}\int_0^{2\pi}\frac{d\tan\frac{\phi}{2}}{(a-c)^2 + (a+c)^2\tan^2\frac{\phi}{2}}$$

$$= \frac{d}{dc}\Big[\ \Big]_0^{2\pi}\left\{\frac{1}{a^2-c^2}\cdot\tan^{-1}\left(\frac{a+c}{a-c}\tan\frac{\phi}{2}\right)\right\}$$

$$= \frac{d}{dc}\left(\frac{\pi}{a^2-c^2}\right) = \frac{2\pi c}{(a^2-c^2)^2}.$$

$$\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\cos\theta\, d\theta\cdot\frac{2\pi c}{(a^2-c^2)^2} = 2\pi a'\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\frac{\cos^2\theta\, d\theta}{(a^2-a'^2\cos^2\theta)^2}$$

$$= 2\pi a'\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\frac{\sec^2\theta\, d\theta}{(a^2-a'^2+a^2\tan^2\theta)^2}$$

$$= \frac{2\pi a'}{a}\int_{-\infty}^{+\infty}\frac{dx}{(a^2-a'^2+x^2)^2} = \frac{2\pi^2 a'}{a\,(a^2-a'^2)^{\frac{3}{2}}}.$$

Hence $$T = \frac{2\pi^2\mu\mu' a'^2}{(a^2-a'^2)^{\frac{3}{2}}}.$$

(5) To determine how much of the Earth's surface, considered spherical, a person can see, who is raised to such a height as to lose $\left(\frac{1}{n}\right)^{\text{th}}$ part of his weight.

If r = the radius of the Earth, the visible area is equal to

$$2\pi r^2\left\{1-\left(\frac{n-1}{n}\right)\right\}.$$

(6) A body of uniform density is bounded by the surface generated by the revolution of a loop of the curve $r^2 = a^2 \cos 2\theta$ about its axis: to find the resultant attraction of the body on a particle at the node of the curve, supposing the law of attraction to be that of the inverse square.

The required attraction is equal to $\frac{1}{3}\pi\mu a$, where μ is the attraction of a unit of mass at a unit of distance.

(7) CAC' is a thin lamina, bounded by an arc of a lemniscate $r^2 = a^2 \cos 2\theta$, the node of which is O and vertex A, and CC', an arc of a circle, of which O is the centre and radius $a \sin \epsilon$. To find the law of the variation of the resultant attraction of the lamina upon a molecule at O, when ϵ varies; the particles of the lamina being supposed to attract according to the law of nature.

The resultant attraction varies as

$$\log\left(\cot\frac{\epsilon}{2}\right) - \cos\epsilon.$$

(8) The sides of an isosceles triangle are formed of slender uniform prisms, attracting with forces which vary inversely as the square of the distance; to determine the vertical angle in order that a particle may remain at rest in a point which divides the perpendicular, drawn from the vertex to the base, in a given ratio.

If a be the distance of the particle from the vertex, and b from the base, then

$$\text{the vertical angle} = 2\sin^{-1}\left(\frac{b}{a}\right).$$

(9) The resultant attraction of a uniform rod upon a particle passes through a given point equidistant from the ends of the

rod, the law of attraction being that of the inverse square: to find the locus of the particle.

The required locus is a circle of which the rod is a chord, the diameter of the circle being equal to $\frac{a^2}{b}$, where a is the distance of the given point from an end and b from the middle of the rod.

(10) Two straight lines AB, AC, at right angles to one another, attract a particle P placed at the point where the perpendicular AP meets BC; to find the direction and magnitude of the force necessary to keep the particle at rest, the law of attraction being that of the inverse square.

Let $AB = a$, $AC = b$, $BC = c$, $\mu =$ the absolute force of a unit of length of the attracting lines condensed into a point; then the direction of the required force will make an angle of 45° with AB, and its magnitude will be equal to

$$\frac{\sqrt{(2)}\,\mu c^3}{a^2 b^2}.$$

(11) A particle is attached, by means of a fine string, to the centre of the rim of a thin hemispherical shell of attractive matter; to determine the tension of the string, supposing its length to be less than the radius of the shell, the force of attraction varying inversely as the square of the distance.

If $r =$ the radius of the shell, $c =$ the length of the string, $\mu =$ the attraction of a unit of the shell's mass condensed at a unit of distance, $\tau =$ the thickness of the shell, the tension will be equal to

$$\frac{2\pi\mu\tau r^2}{c^2}\left\{1 - \frac{r}{(c^2 + r^2)^{\frac{1}{2}}}\right\}.$$

(12) A molecule is placed at a point within a triangle ABC, formed of three uniform rods of equal thickness, which attract according to the law of the inverse square, the densities of the rods BC, CA, AB, being λ, μ, ν, respectively: to find the conditions for the equilibrium of the particle.

If p, q, r, be the perpendicular distances of the molecule from BC, CA, AB, respectively, then

$$\frac{p}{\lambda}=\frac{q}{\mu}=\frac{r}{\nu}.$$

If $\lambda=\mu=\nu$, then $p=q=r$, or the molecule will rest at the centre of the inscribed circle, a theorem proved by Ferdinand Joachimsthal, in the *Cambridge and Dublin Mathematical Journal*, Vol. III. p. 93.

(13) Two equal straight rods, the particles of which attract each other with a force varying inversely as the square of the distance, are parallel to each other and perpendicular to the lines joining their ends, and are held asunder by strings attached to their middle points: to determine the tension of the strings when the rods are at a given distance from each other.

If $a=$ the distance between the rods and $b=$ the length of either, the required tension is equal to

$$\frac{2\mu}{a}\{(a^2+b^2)^{\frac{1}{2}}-a\}.$$

(14) Each particle of two rods of infinite length, which coincide in direction with two conjugate diameters of an elliptic wire, attracts with a force varying inversely as the square of the distance: to find the position of equilibrium of a bead moveable along the wire.

Let a, b, be the semi-axes of the ellipse, ω the acute angle between the two conjugate diameters: then the particle will rest at the intersections of the ellipse with a concentric circle the square of the radius of which is equal to

$$\frac{1}{2}(a^2+b^2-ab\cot\omega).$$

(15) Each particle of two infinite rods, at right angles to one another, attracts with a force varying inversely as the n^{th} power of the distance: to find the form of the rigid curve on which a particle, subject to the attraction of the rods, will rest in all positions.

The rods being taken as axes of co-ordinates, the equation to the curve is

$$x^{2-n} + y^{2-n} = c^{2-n};$$

unless $n = 2$, in which case it is $xy = c^2$.

(16) To find the resultant attraction of a homogeneous globe on an external particle, the law of attraction being that of the inverse cube.

If a be the radius of the globe, c the distance of the attracted particle from the centre, and μ the attraction of a unit of mass at a unit of distance, the required attraction is equal to

$$\tfrac{1}{2}\pi\mu \left\{\frac{c^2 + a^2}{c^2} \log\left(\frac{c + a}{c - a}\right) - \frac{2a}{c}\right\}.$$

(17) A quantity of homogeneous matter of uniform density is in the form of a portion of a paraboloid of revolution bounded by a plane perpendicular to the axis: to find its attraction on a particle of unit mass at the vertex.

If ρ be the density, c the length of the axis of the frustum, and l the length of the latus rectum, the attraction is equal to

$$\pi\rho l \log(\omega \,.\, e^{\omega - 1}),$$

where $\omega = \cot\phi$, ϕ being defined by the equation

$$\frac{l}{2c + l} = \sin 2\phi.$$

(18) A portion of a spherical surface, intercepted between two parallel planes, attracts a particle placed in a normal to the two planes through the centre of the sphere: to find the resultant attraction on the particle, the law of attraction being that of the inverse square.

Let r', r'', be the distances of the particle from the two boundaries of the portion of the surface, c its distance from the centre of the sphere, a the radius of the sphere: then the required attraction is equal to

$$\frac{\pi\mu a}{c^2}(r' - r'')\left(1 + \frac{c^2 - a^2}{r'r''}\right),$$

where μ is the attraction of a unit of area of the spherical surface condensed at a unit of distance from the particle.

(19) A portion of a thin spherical shell, the projections of which upon three planes through the centre at right angles to each other are given, attracts a particle at the centre: supposing the law of attraction to be that of any function of the distance, to find the direction of the resultant attraction on the particle.

If A, B, C, be the given projections and the three planes be co-ordinate planes, the equations to the required direction are

$$\frac{x}{A}=\frac{y}{B}=\frac{z}{C}.$$

Ferrers and Jackson: *Solutions of the Cambridge Problems*, 1848 to 1851, p. 373.

(20) A brittle rod AB, attached to smooth hinges at A and B, is attracted towards a centre of force C according to the law of nature. Supposing the absolute force to be indefinitely augmented, to determine where the rod will eventually snap.

If E be the point of snapping, then, α, β, denoting the angles BAC, ABC, respectively,

$$\cos \angle AEC=\frac{\sin\dfrac{\alpha-\beta}{2}}{\sin\dfrac{\alpha+\beta}{2}}.$$

Mackenzie and Walton: *Solutions of the Cambridge Problems for* 1854.

CHAPTER VIII.

MISCELLANEOUS PROBLEMS.

(1) STRINGS are fastened to any number of points A, B, C, ..., in space, and are pulled towards a point P with forces proportional to PA, PB, PC, ...: shew that, wherever the point P be situated, the resultant of these forces will always pass through a fixed point.

Let a, b, c, be the co-ordinates of P referred to three rectangular axes: then, x, y, z, being the co-ordinates of any one of the points A, B, C,......, the components of the resultant will be equal to

$$\mu\,(na - \Sigma x), \qquad \mu\,(nb - \Sigma y), \qquad \mu\,(nc - \Sigma z),$$

which will therefore be proportional to the direction-cosines of the resultant. The equations to the resultant will therefore be

$$\frac{x' - a}{na - \Sigma x} = \frac{y' - b}{nb - \Sigma y} = \frac{z' - c}{nc - \Sigma z}:$$

multiplying each of these fractions by n and adding unity to each we get

$$\frac{nx' - \Sigma x}{na - \Sigma x} = \frac{ny' - \Sigma y}{nb - \Sigma y} = \frac{nz' - \Sigma x}{nc - \Sigma z}.$$

Hence we see that the resultant always passes through a point of which the co-ordinates are

$$\frac{1}{n}\Sigma x, \qquad \frac{1}{n}\Sigma y, \qquad \frac{1}{n}\Sigma z.$$

(2) Find the amount of *work done* in drawing up a common Venetian blind. How must the same problem be solved for a curtain?

Let W = the weight of each bar of the blind; a = the distance between two consecutive bars; n = their number. Then the work done will be equal to

$$W(a + 2a + 3a + \ldots\ldots + na)$$
$$= \tfrac{1}{2} n(n+1) Wa.$$

Let P = the sum of the weights and l = the height of the window: then $P = nW$ and $l = na$, and the work done is equal to

$$\frac{1}{2}\left(1 + \frac{1}{n}\right) Pl.$$

Let $n = \infty$: then the Venetian blind is mechanically the same as a curtain, the number of its bars being infinite and the weight of each indefinitely small. Thus, P being the weight and l the length of the curtain, the work done is equal to

$$\tfrac{1}{2} Pl.$$

The work done in raising the curtain may also be estimated by integration.

The weight of a length dx of the curtain is $\frac{P}{l}\,dx$: hence the work done

$$= \int_0^l \frac{P}{l}\,dx \,.\, x = \tfrac{1}{2} Pl.$$

(3) The frustum of a paraboloid of revolution, the density of its circular sections varying as their areas, stands upon its vertex on a horizontal plane: to find the length of its axis when the equilibrium is indifferent.

If the vertex of a solid of revolution, of which the axis extends vertically upwards, be placed upon the summit of another solid of revolution the axis of which extends vertically downwards, then, as is proved in most works on Statics, the equilibrium will be stable, unstable, or indifferent, accordingly as the altitude of the centre of gravity of the upper body above the point of contact is less than, greater than, or equal to

$$\frac{rr'}{r + r'},$$

r, r', being the radii of curvature of the two surfaces at the point of contact.

If r', the radius of curvature of the lower surface, be equal to ∞, the lower surface becomes a plane, and the expression

$$\frac{rr'}{r+r'}, \text{ becomes } r.$$

In the present question, as we may easily ascertain, the altitude of the centre of gravity is equal to $\frac{3}{4}c$, c being the length of the axis. Also the radius of curvature at the vertex is equal to $\frac{1}{2}l$, l being the latus rectum. Hence

$$\tfrac{3}{4}c = \tfrac{1}{2}l, \qquad c = \tfrac{2}{3}l.$$

(4) A beam can turn in every direction about one end, which is fixed. The other rests on the upper surface of a rough plane, (the coefficient of friction being μ), which is inclined to the horizon at an angle α. If β be the angle the beam makes with the plane, prove that the beam will rest in any position if $\tan\alpha$ be not greater than

$$\frac{\mu}{(1+\mu^2\tan^2\beta)^{\frac{1}{2}}}.$$

Let O, (fig. 113), be the fixed end; OC a perpendicular upon the rough plane; CB a section of the rough plane by a vertical plane through OC; OER a horizontal line cutting BC in E; the circular quadrantal arc APB the locus of the free end of the beam; P the place of the end of the beam for a limiting position of equilibrium; PQ a line parallel to AC, and QR at right angles to OER.

Let l = the length of the beam, $\angle PCQ = \theta$, R = the normal reaction of the rough plane at P; then, the horizontal component of R being $R\sin\alpha$, parallel to EO, and the horizontal components of μR being $\mu R\cos\theta$ along PQ and $\mu R\sin\theta\cos\alpha$ parallel to EO, we have for the equilibrium of the beam, taking moments about a vertical line through O,

$$R\sin\alpha\,.\,PQ = \mu R\cos\theta\,.\,OR + \mu R\sin\theta\cos\alpha\,.\,PQ,$$

$$\sin\alpha\,.\,l\cos\beta\sin\theta$$

$$= \mu\cos\theta\left\{l\frac{\sin\beta}{\sin\alpha} + (l\cos\beta\cos\theta - l\sin\beta\cot\alpha)\cos\alpha\right\}$$

$$+\mu\sin\theta\cos\alpha\,.\,l\cos\beta\sin\theta,$$

$$\sin\alpha\cos\beta\sin\theta = \mu\sin\beta\cos\theta\sin\alpha + \mu\cos\alpha\cos\beta,$$

$$\tan\alpha\tan\theta - \mu\tan\alpha\tan\beta = \mu(1+\tan^2\theta)^{\frac{1}{2}},$$

$$(\tan^2\alpha - \mu^2)\tan^2\theta - 2\mu\tan^2\alpha\tan\beta\tan\theta + \mu^2(\tan^2\alpha\tan^2\beta - 1) = 0.$$

That $\tan\theta$ may be impossible, or that there may be no *limiting* equilibrium, we must have

$$\tan^4\alpha\tan^2\beta < (\tan^2\alpha - \mu^2)(\tan^2\alpha\tan^2\beta - 1),$$

or
$$\tan^2\alpha(1+\mu^2\tan^2\beta) < \mu^2.$$

A different solution of this problem may be seen in the *Solutions of the Senate-House Problems* for 1844, by O'Brien and Ellis.

(5) Each element of an octant of an ellipsoidal surface, bounded by the three principal planes, is acted on, along the normal, by a force proportional to the area of the element: to investigate the resultant effect.

Let X, Y, Z, denote the components of the resultant force, if there be one, and L, M, N, those of the resultant couple.

Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ represent the ellipsoidal surface and dS an element of it: then, μ being a constant,

$$X = \mu\iint \frac{\frac{x}{a^2}\,.\,dS}{\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^{\frac{1}{2}}} = \mu\iint dydz = \tfrac{1}{4}\pi\mu bc.$$

Similarly $\qquad Y = \frac{1}{4}\pi\mu ca, \qquad Z = \frac{1}{4}\pi\mu ab.$

Again
$$L = \mu\iint dS\,.\,\frac{y\,.\,\frac{z}{c^2} - z\,.\,\frac{y}{b^2}}{\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^{\frac{1}{2}}}$$

$$= \mu\iint(y\,dxdy - z\,dxdz)$$

$$= \tfrac{1}{2}\mu\int(y^2 - z^2)dx$$

$$= \frac{\mu(b^2-c^2)}{2a^2}\int(a^2 - x^2)dx = \tfrac{1}{3}\mu a(b^2 - c^2).$$

Similarly $\quad M = \frac{1}{3}\mu b\,(c^2 - a^2), \qquad N = \frac{1}{3}\mu c\,(a^2 - b^2).$

Thus $\qquad LX + MY + NZ = 0,$

and therefore there exists a single resultant.

If x, y, z, be the co-ordinates of any point in the resultant,

$$Zy - Yz = L, \quad Xz - Zx = M, \quad Yx - Xy = N,$$

and therefore the equations to the resultant are

$$\frac{by - cz}{b^2 - c^2} = \frac{cz - ax}{c^2 - a^2} = \frac{ax - by}{a^2 - b^2} = \frac{4}{3\pi}.$$

(6) A system consists of n equal particles which have no initial velocities: prove that it will remain at rest, if their co-ordinates can only vary subject to the condition

$$n\Sigma\,(x^2 + y^2 + z^2) - (\Sigma x)^2 - (\Sigma y)^2 - (\Sigma z)^2 = \text{a constant}:$$

the particles attracting one another with forces which vary as the distance.

The attraction between any two of the particles P_1, P_2, at a distance r from each other is proportional to r. Conceive the system to experience any indefinitely small displacement consistently with its geometrical connections, and let α denote the component of P_1's motion estimated along P_1P_2, and β that of P_2's motion estimated along P_2P_1. Then $(\alpha + \beta)\,r$ will denote the sum of the *moments* of the two forces; but $\alpha + \beta = -dr$: hence, by the principle of Virtual Velocities, equating to zero the sum of the *moments* of the whole system, we have

$$0 = \Sigma\,(r\,dr),$$

whence
$$\begin{aligned} C &= 2\int\Sigma\,(rdr) \\ &= \Sigma\,(r^2) \\ &= \Sigma\,\{(x' - x)^2 + (y' - y)^2 + (z' - z)^2\} \\ &= (n-1)\,\Sigma(x^2 + y^2 + z^2) - 2\Sigma\,(xx' + yy' + zz') \\ &= n\Sigma\,(x^2 + y^2 + z^2) - (\Sigma x)^2 - (\Sigma y)^2 - (\Sigma z)^2. \end{aligned}$$

(7) If an elliptic board be placed, so that its plane is vertical, on two pegs which are in a horizontal line, there will be

equilibrium if these pegs be at the extremities of a pair of conjugate diameters. What are the limits which the distance between the pegs must not exceed or fall short of, in order that this position of equilibrium may be possible? Shew that the position is one of unstable equilibrium.

Let P', P'', (fig. 114), be the two pegs: C the centre of the ellipse, CA the semi-axis major: draw $P'M'$, $P''M''$, at right angles to CA, and CQ at right angles to $P'P''$.

Let $P'P''=c$, $CQ=u$, $CM'=x'$, $P'M'=y'$, $CM''=x''$, $PM''=y''$.

Then, equating CQ to the difference of the projections of CM', $P'M'$, upon its direction, we get

$$u=\frac{x'+x''}{c}\,y'-\frac{y'-y''}{c}\,x'=\frac{x''y'+x'y''}{c}.$$

Put $x'=a\cos\phi'$, $y'=b\sin\phi'$, $x''=a\cos\phi''$, $y''=b\sin\phi''$, a and b being the semi-axes of the ellipse: then, if $\phi'+\phi''=\psi$,

$$cu=ab\sin(\phi'+\phi'')=ab\sin\psi.$$

That u may be a maximum or minimum,

$$c\frac{du}{d\psi}=ab\cos\psi=0,$$

whence $\psi=\frac{1}{2}\pi$, which shews, by a known property of the ellipse, that P', P'', are extremities of conjugate diameters of the ellipse.

Again,

$$c\frac{d^2u}{d\psi^2}=-ab\sin\psi=-ab:$$

hence u is a maximum, or the equilibrium is unstable.

Moreover,
$$\begin{aligned} c^2 &= (x'+x'')^2+(y'-y'')^2 \\ &= a^2(\cos\phi'+\cos\phi'')^2+b^2(\sin\phi'-\sin\phi'')^2 \\ &= a^2(\cos\phi'+\sin\phi')^2+b^2(\sin\phi'-\cos\phi')^2 \\ &= a^2+b^2+(a^2-b^2)\sin 2\phi'. \end{aligned}$$

Hence we see that the greatest and least limits of c are $a\sqrt{2}$ and $b\sqrt{2}$.

(8) A flexible thread rests upon a smooth surface, under the action of any forces: to investigate its form.

Let R = the reaction of the surface at any point (x, y, z) of the thread, t being the tension at that point. Then, for the equilibrium of the thread, λ denoting a certain coefficient, and $m\delta s$ the mass of an element δs of the thread,

$$\frac{d}{ds}\left(t\frac{dx}{ds}\right) + mX + \lambda R\frac{du}{dx} = 0 \quad\text{..................(1)},$$

$$\frac{d}{ds}\left(t\frac{dy}{ds}\right) + mY + \lambda R\frac{du}{dy} = 0 \quad\text{..................(2)},$$

$$\frac{d}{ds}\left(\lambda\frac{dz}{ds}\right) + mZ + \lambda R\frac{du}{dz} = 0 \quad\text{..................(3)}.$$

Also, if $u = 0$ be the equation to the surface,

$$\frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dz}dz = 0 \quad\text{..................(4)}.$$

From (1), (2), (3), eliminating $\frac{dt}{ds}$ and λR by cross multiplication, we get

$$0 = \left(mX + t\frac{d^2x}{ds^2}\right)\left(\frac{dy}{ds}\frac{du}{dz} - \frac{dz}{ds}\frac{du}{dy}\right) + \left(mY + t\frac{d^2y}{ds^2}\right)\left(\frac{dz}{ds}\frac{du}{dx} - \frac{dx}{ds}\frac{du}{dz}\right)$$
$$+ \left(mZ + t\frac{d^2z}{ds^2}\right)\left(\frac{dx}{ds}\frac{du}{dy} - \frac{dy}{ds}\frac{du}{dx}\right) \quad\text{.............. (5)}.$$

But, from (1), (2), (3), (4),

$$0 = dx\frac{d}{ds}\left(t\frac{dx}{ds}\right) + dy\frac{d}{ds}\left(t\frac{dy}{ds}\right) + dz\frac{d}{ds}\left(t\frac{dz}{ds}\right)$$
$$+ m(Xdx + Ydy + Zdz),$$

and therefore

$$0 = dt + m(Xdx + Ydy + Zdz) \quad\text{................ (6)}.$$

The equations (5), (6), together with the equation to the surface, determine the form of the thread.

(9) From a square $ABCD$ a triangle AEF is cut, AE being the fourth part of AD, and AF three-fourths of AB; to find the centre of gravity of the remaining figure $BCDEF$.

If a denote a side of the square, the distances of the required centre of gravity from AB, AD, respectively, are

$$\frac{63}{116}a, \qquad \frac{61}{116}a.$$

(10) The points D, E, F, divide the sides BC, CA, AB, of a triangle proportionally, that is, so that

$$BD : CE : AF :: DC : EA : FB;$$

shew that the centre of gravity of the triangle DEF coincides with that of the triangle ABC.

(11) The diagonals of a trapezium intersect at right angles at a fixed point, and have always the same directions, the magnitudes of the diagonals and of one side being given: prove that the locus of the centre of gravity of the trapezium is a circle of which the radius is $\frac{2}{3}c$.

(12) A triangular plate hangs by three parallel threads attached at the corners, and supports a heavy particle. Prove that, if the threads are of equal strength, a heavier particle may be supported at the centre of gravity than at any other point of the disk.

(13) Two forces in the ratio of $1 + n$ to 1, where n is small, act upon a point in directions including an angle α; shew that the sine of the angle which the direction of the resultant makes with that of the larger force is nearly equal to

$$(1 - \tfrac{1}{2}n) \sin \frac{\alpha}{2}.$$

(14) If three forces, represented in magnitude and direction by lines OA, OB, OC, act at a point O, not necessarily in the plane of ABC, prove that their resultant will be represented in magnitude and direction by $3\,OG$, G being the centre of gravity of the triangle ABC.

(15) Forces represented by $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$, act at the angular points of a triangle ABC, right-angled at C, in the directions of the

sides taken in order: prove that the resultant is represented by

$$\left(\frac{c^2}{a^2b^2} - \frac{3}{c^2}\right)^{\frac{1}{2}},$$

that it is inclined to AC at an angle $\cos^{-1}\dfrac{a^3}{(a^6+b^6)^{\frac{1}{2}}}$, and that it cuts BC produced at a distance $\dfrac{b^2}{a}$ from C.

(16) The vertex of a triangle is at a fixed point in the circumference of a circle, of which the base is a chord: prove that, if the angle at the vertex be of given magnitude, the locus of the centre of gravity of the triangular area is another circle.

(17) If forces P, Q, R, acting at the centre O of a circular lamina along the radii OA, OB, OC, be equivalent to forces P', Q', R', acting along the sides BC, CA, AB, of the inscribed triangle ABC, prove that

$$\frac{P.P'}{BC} + \frac{Q.Q'}{CA} + \frac{R.R'}{AB} = 0.$$

(18) A series of lighted candles, of various compositions and altitudes, stand in a vertical plane on a table: prove that generally the centre of gravity of the candles describes a series of arcs of successive hyperbolas.

Prove also that, at any instant of time, the corresponding hyperbola has a vertical asymptote which passes through the original centre of gravity of the pieces, which have been consumed, of the candles still burning.

(19) Shew that a system of forces, acting in one plane and represented by the sides of a polygon, is equivalent to a couple the moment of which is represented by twice the area of the polygon.

(20) Three forces, acting at the angles of a triangle, are in equilibrium, one bisecting the angle at which it acts, and the other two making equal angles with the side opposite to that angle; shew that the forces are proportional to the sides opposite to their points of application.

(21) Two spheres, attached to the two ends of a fine string, which hangs over a fixed point, rest in contact: prove that their weights are inversely proportional to the distances of their centres from the point of suspension.

(22) A pack of cards is laid on a table; each projects in the direction of the length of the pack beyond the one below it: if each projects as far as possible, prove that the distances between the extremities of the successive cards will form an harmonic progression.

(23) Four unequal forces P, Q, R, S, act upon a rigid body along the sides OA, AB, BC, CO, of a square $OABCO$: prove that, OA and OC being taken as the axes of co-ordinates, there will be a single resultant force the equation to which, if a be a side of the square, is

$$(Q-S)\,x+(R-P)\,y=a\,(Q+R),$$

and of which the magnitude is

$$\{(P-R)^2+(Q-S)^2\}^{\frac{1}{2}}.$$

(24) A quadrilateral is formed by four rigid rods jointed at the ends; shew that two of its sides must be parallel in order that it may preserve its form when the middle points of either pair of opposite sides are joined together by a string in a state of tension.

(25) A rectangular column is formed by placing a number of smooth cubical blocks one above another, the base of the column resting upon a horizontal plane: all the blocks above the lowest are then twisted in the same direction about an edge of the column, first the highest, then the two highest, and so on, in each case as far as is consistent with equilibrium. Prove that the sum of the sines of the inclinations of a diagonal of the base of any block to the like diagonals of the bases of all the blocks above it is equal to the sum of the cosines.

(26) A triangular disk, the sides of which are a, b, c, is suspended from a fixed point by threads attached to its corners,

α, β, γ, being the lengths of the threads attached to the corners opposite to a, b, c, respectively. If the plane of the disk be horizontal, prove that

$$a^2 + 3\alpha^2 = b^2 + 3\beta^2 = c^2 + 3\gamma^2.$$

(27) Three equal heavy rods, in the position of the three edges of an inverted triangular pyramid, are in equilibrium under the following circumstances: their upper extremities are connected by fine strings of equal lengths, and their lower extremities are attached to a hinge about which the rods may move freely in all directions. Shew that the increase of the tension of the strings, corresponding to a given small increase of their lengths, varies inversely as $\sin^3\theta$, where θ is the inclination of each of the rods to the horizon.

(28) Assuming friction to consist of the sum of two parts, the one proportional to the pressure, and the other to the surface in contact, shew that, when a parallelepiped, the edges of which are a, b, c, is supported with one edge parallel to the horizon on a given inclined plane by the least possible force acting in a given direction through its centre of gravity at right angles to this edge, we shall have

$$\frac{q-r}{a} + \frac{r-p}{b} + \frac{p-q}{c} = 0;$$

p, q, r, being the values of the force in question, when the parallelepiped rests on the faces bc, ca, ab, respectively.

(29) Eight central forces, the centres of which are at the corners of a cube, attract, according to the same law and with the same absolute intensity, a particle placed very near the centre of the cube: shew that their resultant action passes through the centre of the cube, unless the law of force be that of the inverse square of the distance.

(30) Prove that a cone, the density of the circular sections of which varies as their distances from its vertex, will balance on the middle point of its axis, if a weight equal to three-fifths of its own weight be suspended at the vertex.

(31) A solid is generated by the revolution of a semicircle round its chord through an angle of 60°; to determine the moment of the couple which will keep the axis of revolution of the solid, placed upon a smooth horizontal plane, in a vertical position.

If $a =$ the radius of the circle, and $\rho =$ the density of the material, the required moment is equal to $\frac{1}{8}g\rho\pi a^4$.

(32) Prove that, if, at each point of space, a force f act which is any function of the distance of the point from a given point A, and θ be the angle at which the tangent at a point P of an arbitrary curve, connecting any two points P_1, P_2, in space, is inclined to the direction of the force at P, then $\int f \cos\theta ds$ from P_1 to P_2 depends only on the distances AP_1, AP_2.

(33) If the frustum of a cone be bisected by a plane through its axis, prove that either half will just stand upon the smaller end on a horizontal plane, if

$$\frac{h+h'}{h} = \pi\frac{h^2+hh'+h'^2}{h^2+h'^2},$$

where h, h', are the heights of the smaller and larger cones the difference of which constitutes the frustum.

(34) Shew that the centre of gravity of any arc of the curve $r^3 = a^3 \sec 3\theta$ is in the straight line which joins the pole with the intersection of the tangents at the extremities of the arc.

(35) A small weightless ring, fixed at the lower end of a straight uniform rod, is moveable along a fixed vertical bar: the rod is moved so that its middle point shall always lie in a given horizontal line passing through the bar: shew that the rod would rest in every position on the curve, if rigid, which it always touches; and prove that the pressure of the rod on the curve varies inversely as the cube root of the distance of the point of contact from the bar.

(36) If every element of a uniform wire, in the form of a closed plane curve, be acted on tangentially, in the same

rotatory direction, by a force varying inversely as the square of the distance of the element from a given point in the area enclosed by the wire, prove that, the given point being chosen as the origin of moments, the resultant couple is independent of the length and form of the wire.

(37) On a thin uniform lamina in the form of a cardioid, the equation to which is $r = a(1 - \cos\theta)$, are traced an infinite number of similar and similarly situated cardioids about the same pole: on each element of the lamina an accelerating force, varying inversely as the square of the distance of the element from the pole, acts, in the same rotatory direction about the pole, tangentially to the cardioid on which the element lies: to find the magnitude of the resultant moment about the pole.

The required moment, μ being the absolute accelerating force, is equal to $\frac{16}{3}\mu a$.

(38) The lower end A of a thin uniform rod is attached to a smooth hinge, its upper end B resting against a smooth vertical plane: prove that the tendency to break at any point P of the rod varies as $AP.BP$.

(39) A hollow homogeneous cylinder, of given material, which is perfectly brittle and incompressible, is partially inserted into a fixed horizontal tube just wide enough to admit it: prove that the greatest length which the free portion of the cylinder can have, without snapping off, varies as the square root of the radius of its external surface.

(40) A uniform beam, on which at a given point is placed a given weight, is supported horizontally at its extremities: to find the tendency of the beam to break at any point, supposing the weight of the beam so small that it may be neglected.

Let l be the length of the beam, W the given weight, a the distance of the weight from one end: then the tendency to

break at a distance x from this end will be represented by the expression

$$\frac{W}{l} \cdot \frac{x(l-a)\,0^x + a(l-x)\,0^a}{0^a + 0^x}.$$

Archibald Smith; *Cambridge Mathematical Journal,* Vol. I. p. 276.

(41) The lower radius of any fixed sector of a circle is horizontal, the plane of the sector being vertical: shew that a heavy uniform chain, laid over the arc and upper radius, so as just to coincide with them in every part, will remain in equilibrium.

(42) The ends of a uniform chain are fastened to two fixed points A and B in a horizontal line: a link C of the chain rests upon a rigid rectilinear wire which joins A and B, so that the chain forms two festoons AC and CB. Prove that, if there be no friction between the wire and chain, the smaller of these festoons is equal and similar to a portion of the larger.

(43) The ends of a uniform chain of length $2l$ are attached to two fixed points in a horizontal line; if $2a$ be the distance between the points of support, t, c, the respective lengths of the chain the weights of which are equal to the tensions at either point of support and at the lowest point of the chain, prove that, when l has such a value that t is a minimum,

$$ct = al.$$

(44) To determine the form in which a chain will hang, suspended at two points, when the density at any point varies as the tension at that point; the thickness of the chain being uniform.

The axis of x being horizontal and that of y vertical, and the origin being at the lowest point, the equation to the curve will be

$$e^{\frac{y}{c}} = \sec\frac{x}{c},$$

c being a constant.

(45) A flexible chain, the ends of which are united, hangs over two pegs, which are fixed in a horizontal line, in the form of two festoons; if P, P', be the tensions at the vertices of the festoons, and α, α', the inclinations of the festoons to the horizon at either peg, prove that the weight of half the chain is equal to

$$P \tan \alpha + P' \tan \alpha'.$$

Prove also that the weight of a piece of the chain, equal in length to the distance between the vertices of the festoons, is equal to $P \sim P'$.

(46) Of an endless uniform chain one part lies within a fixed horizontal tube and the other hangs in a festoon below: if the weight of the part within the tube be equal to the tension at the lowest point of the festoon, prove that the tension of the string within the tube is to the weight of half the festoon as $e+1$ to $e-1$.

(47) A uniform flexible string rests on the surface of a sphere, to the highest point of which one end of the string is attached: supposing the length of the string to be equal to that of a quadrant of a great circle of the sphere, prove that the whole pressure exerted on the sphere by the third part of the string, reckoned from the highest point, is equal to the tension at the highest end of the string.

(48) A fine thread just encloses, without tension, the circumference of an ellipse: supposing a centre of force, attracting inversely as the square of the distance, to be placed at one of the foci, prove that the sum of the tensions of the thread at the ends of any focal chord is invariable, and that the normal pressure on the ellipse at any point varies inversely as the cube of the conjugate diameter.

(49) If G be the principal moment of a system of forces with respect to any origin O, and K the least principal moment, prove that the locus of an origin, the axis with respect to which is perpendicular to that of G, is a plane, the normal to which through O intersects the central axis at right angles and is divided by it in the ratio of K^2 to $G^2 - K^2$.

(50) A uniform wire in the form of a lemniscate attracts a particle at the node, the law of force being that of the direct distance: prove that the resultant attraction of any arc will be the same in all respects as that of an arc, subtending the same angle at the node, of a circular wire, of the same material, which touches the lemniscate at the vertices.

(51) A uniform homogeneous wire, of which A is the middle point and P an end, is bent into the form of an arc of a loop of the lemniscate of which A becomes the vertex: prove that the resultant attraction on the wire, arising from a centre of force at the node O, attracting according to the law of the inverse square, varies as

$$\left(\frac{1}{OP^2} - \frac{1}{OA^2}\right)^{\frac{1}{2}}.$$

(52) A cone rests with its base upon the vertex of a given paraboloid: prove that, for stability of equilibrium, it is necessary that the height of the cone be less than twice the latus rectum of the paraboloid.

(53) If a cone of the same substance and of equal base with a hemisphere be fixed to the latter, so that their bases coincide, to find the greatest height of the cone in order that the equilibrium may be stable, when the hemisphere rests symmetrically on a horizontal plane.

The height of the cone must be less than $r\sqrt{3}$, r being the radius of the hemisphere.

(54) If a solid cylinder be cut by a plane which touches the circumference of its base at a point A and meets the axis at an angle of 45°, prove that the piece of the cylinder included between the cutting plane and the base will rest in indifferent equilibrium, if placed with its circular end on the vertex of a paraboloid the latus rectum of which is $\frac{3}{8}$ths the diameter of the base, the point of contact being also at this same distance from A.

(55) If five given lines have a common transversal, then, taking the remaining transversal of each four of the given lines, shew by statical considerations that the five transversals have a common transversal.

Cayley; *Messenger of Mathematics*, Vol. IV. p. 219.

(56) Shew that the attraction of an indefinitely thin double-convex lens on a point at the centre of one of its faces is equal to that of the infinite plate included between the tangent plane at the point and the parallel tangent plane of the other face of the lens.

Cayley; *Messenger of Mathematics*, Vol. V. p. 194.

DYNAMICS.

CHAPTER I.

IMPACT AND COLLISION. SMOOTH SPHERICAL BODIES.

CONCEIVE two spherical bodies, which are composed of the same material, to be moving in the same straight line, namely, in the line joining their centres, and at any time during their motion to impinge against each other. Let m, m', denote the masses of the two bodies; and let u, u', represent their velocities before and v, v', their velocities after collision; the symbols which represent the velocities being positive when motion takes place in one direction and negative when it takes place in the other; then, whatever be the magnitudes of u, u', or of m, m',

$$u - u' : v' - v :: 1 : e,$$

or

$$v' - v = e\,(u - u') \text{..........................(A)},$$

where e is a numerical quantity not greater than unity, which is invariable while the material of the bodies remains the same, but which changes generally with a change in their substance. The bodies are said to be inelastic if e be equal to zero; imperfectly elastic if it be equal to any fraction between zero and unity; and perfectly elastic if it be equal to unity.

The theory of collision furnishes us likewise with the following general relation,

$$m\,(u - v) = m'\,(v' - u') \text{......................(B)}.$$

The equations (A) and (B) are usually more convenient when written in the forms

$$eu + v = eu' + v',$$

$$mu + m'u' = mv + m'v'.$$

The signification of the equation (A) is, that the relative velocity of the two bodies after collision bears a constant ratio to the relative velocity before collision, so long as the material of the bodies remains unchanged; and the equation (B) implies, that the momentum which one body gains by the collision in the positive direction of motion, is equal to that which the other loses. These are the two fundamental principles in the theory of collision.

Suppose that u' is equal to zero, and that m is inconsiderable in comparison with m'; then clearly, by (A) and (B),

$$v' - v = eu \text{ and } v' = u' = 0,$$

and therefore $$v = -eu \qquad\qquad (C),$$

or the small body is reflected backwards with a velocity which is to the velocity of impact as e to 1; while the large body experiences no appreciable motion from the collision. This is evidently the case of bodies impinging and rebounding upon the surface of the earth, or upon other bodies firmly attached to it, the earth being regarded as stationary.

In the year 1639, J. Marc Marci de Crownland[1], a Hungarian physician, published at Prague a work entitled *De Proportione Motus, seu Regula Sphymica,* in which he has treated of the collision of perfectly elastic and perfectly inelastic bodies. He occupies himself principally with the consideration of perfectly elastic bodies, and lays down precisely the same rules for their collision which are now commonly adopted. This work, the earliest in which the theory of collision had been correctly propounded, having fallen into general oblivion in the scientific world, the subject was again correctly investigated by the independent efforts of Wallis, Wren, and Huyghens, who apparently had not the slightest knowledge even of the existence of the work by Marci. The laws of the collision of perfectly inelastic bodies were laid down by Wallis, *Phil. Trans.* 1668, p. 864, and of perfectly elastic bodies by Wren, *Phil. Trans.* 1668, p. 867, and Huyghens, *Phil. Trans.* 1669, p. 925, and *Journal des Sçavans* of March 18, 1669. Wren and Lawrence Rook

[1] Montucla; *Histoire des Mathématiques*, Tom. II. p. 406.

had, several years earlier than this, exhibited various experiments before the Royal Society, in illustration of the principles of collision. The conclusions of Wallis, Wren, and Huyghens, which had been presented to the Royal Society in a very brief shape, were afterwards given more at large by Wallis, *Mechanica, Pars Tertia,* 1671; Keill, *Introductio ad Veram Physicam,* Lect. 12, 13, 14; and Mariotte, *Traité de Percussion.* There are some ingenious experiments by Smeaton on the theory of collision in the *Phil. Trans.*, April 18, 1782. The principles of the collision of imperfectly elastic bodies were first propounded by Newton, *Principia,* Lib. I., Scholium to the Laws of Motion, who inferred experimentally the truth of the equation (A) for any value whatever of e between zero and unity; preceding philosophers having directed their attention to those cases alone in which e is supposed to be either zero or unity. The physical value of Newton's generalization is the more striking when it is considered that natural bodies are never actually endowed with perfect elasticity. For the mathematical formulæ in the theory of the collision of imperfectly elastic spheres, the reader is referred to Maclaurin, *Choc des Corps, Prix de l'Academie,* Tom. I. and to Bossut, *Cours de Mathématique,* Tom. III. The results of a series of experiments on the elasticity of bodies, by Mr Hodgkinson, are to be found in Vol. III. p. 534, of the *Reports of the British Association for the Advancement of Science,* where he has shewn that the quantity e, in the equation (A), is not, as we stated, and as we shall suppose for the sake of mathematical simplicity, entirely independent of the velocities of the impinging bodies, as Newton had concluded, but that it decreases as the relative velocity increases, assuming however a nearly constant value when the relative velocity of collision becomes considerable.

(1) Two inelastic bodies are moving in opposite directions with given velocities; to find their velocities after collision.

Let m, m', denote the masses of the bodies; a, a', their velocities before collision. Then, putting in the formulæ (A) and (B),

$$u = a, \quad u' = -a', \quad e = 0,$$

we have $$v' - v = 0, \quad m(a - v) = m'(v' + a'),$$

and therefore $$m(a-v)=m'(a'+v),$$

$$v'=v=\frac{ma-m'a'}{m+m'}.$$

If then ma be greater than $m'a'$, the bodies will, after collision, move along, in the direction in which m originally moved, with a common velocity $(ma-m'a') : (m+m')$; and, if ma be less than $m'a'$, they will move in the opposite direction with a common velocity $(m'a'-ma) : (m'+m)$. If ma be equal to $m'a'$, the collision will reduce both the bodies to rest.

Wallis; *Mechan. Pars Tertia, de Percussione*, Prop. IV.

(2) Two perfectly elastic bodies are moving in opposite directions with given velocities; to find their velocities after collision.

The notation being the same as in the preceding example, we have, e being in this case equal to unity,

$$v'-v=a+a', \quad m(a-v)=m'(a'+v').$$

Eliminating v' from these two equations, we have

$$v=\frac{ma-m'a}{m+m'}-\frac{2m'a'}{m+m'}.$$

Eliminating v, we have

$$v'=\frac{ma'-m'a'}{m+m'}+\frac{2ma}{m+m'}.$$

Wallis; *Ib. de Elatere et Resilitione*, Prop. X.

(3) Three bodies m, m', m'', are placed in a row. If the body m be projected with a given velocity towards m', to find the magnitude of m' in order that the velocity communicated to m'' by its intervention may be the greatest possible.

Let a be the velocity with which m is projected; a' the velocity which m' acquires on being struck by m, and a'' that which m'' receives on being struck by m'. Then

$$a'=\frac{2ma}{m+m'}, \qquad a''=\frac{2m'a'}{m'+m''},$$

and therefore $$a''=\frac{4mm'a}{(m+m')(m'+m'')}.$$

Since a'' is to be a maximum, the expression

$$\left(\frac{m}{m'}+1\right)\left(m'+m''\right)$$

must be a minimum: hence, differentiating with respect to the variable m',

$$\frac{m}{m'}+1-\frac{m}{m'^2}(m'+m'')=0,$$

$$m'^2-mm''=0, \quad m'=(mm'')^{\frac{1}{2}}.$$

Huyghens; *Phil. Trans.* 1669, p. 928.
Wolff; *Elementa Matheseos Universæ,* Tom. II. p. 158.

(4) A perfectly elastic sphere impinges with a given velocity and in a given direction against a smooth plane; to determine the velocity and direction of reflection.

Let u, v, denote the velocities of incidence and of reflection, and α, β, the angles which the directions of the motion before and after impact make with a normal to the plane.

The resolved parts of the velocities, parallel to the plane, are $u\sin\alpha$ and $v\sin\beta$, and, at right angles to it, $u\cos\alpha$ and $v\cos\beta$. But, the plane being perfectly smooth, the resolved parts of the velocities parallel to the plane will be equal to each other, and therefore

$$v\sin\beta=u\sin\alpha;$$

while the other resolved parts of the velocities will bear to each other the same relation as if the impact had been direct, and therefore, by (C),

$$v\cos\beta=u\cos\alpha.$$

From these two equations it is evident that

$$\tan\beta=\tan\alpha, \quad \beta=\alpha, \quad \text{and} \quad v=u,$$

or the angle of reflection is equal to that of incidence, and the velocity of reflection to the velocity of incidence.

Wallis; *Mechan. Pars Tertia, De Elatere,* &c. Prop. II.

(5) Two smooth spheres, moving with given velocities and in given directions, impinge against each other; the spheres

being supposed to be perfectly elastic, to determine their velocities and the directions of their motions after collision.

Let AB, $A'B'$ (fig. 115), be the directions of the motions of the two bodies before collision; and O, O', the positions of their centres at the instant of contact; produce $O'O$ indefinitely to a point C. Let a, a', denote their velocities before collision, and α, α', the angles AOC, $A'O'C$.

Then the resolved parts of the velocities of the spheres O, O', in the direction COO' will be $a \cos \alpha$, $a' \cos \alpha'$, and, at right angles to OO' in the planes AOC, $A'O'C$, respectively, $a \sin \alpha$, $a' \sin \alpha'$. These latter resolved velocities will not be affected by the collision. The former will be affected exactly as if the sphere O moving along COO' with a velocity $a \cos \alpha$ were to impinge directly upon the sphere O' moving with a smaller velocity $a' \cos \alpha'$ estimated in the same direction. Hence, if v, v', denote the resolved parts of the velocities after collision parallel to the line COO', we have, as may be readily ascertained by the principles of this chapter,

$$v = \frac{ma \cos \alpha - m'a \cos \alpha}{m + m'} + \frac{2m'a' \cos \alpha'}{m + m'},$$

$$v' = \frac{m'a' \cos \alpha' - ma' \cos \alpha'}{m + m'} + \frac{2ma \cos \alpha}{m + m'}.$$

Let V, V', denote the velocities of the spheres O, O', after collision, and ϕ, ϕ', the angles which the directions of their motions make with OO'; then

$$V^2 = v^2 + a^2 \sin^2 \alpha, \quad V'^2 = v'^2 + a'^2 \sin^2 \alpha',$$

$$\tan \phi = \frac{a \sin \alpha}{v}, \quad \tan \phi' = \frac{a' \sin \alpha'}{v'},$$

their motions still taking place in the planes AOC, $A'O'C$.

Keill; *Introductio ad Veram Physicam*, Lect. 14.

(6) Two imperfectly elastic bodies are moving in the same direction along the same straight line with given velocities; the one overtakes the other and collision ensues; to find the velocities of the two bodies after collision.

If m, m', be the masses of the two bodies, e their common

elasticity; a, a', their velocities before, and v, v', their velocities after collision,

$$v = \frac{ma + m'a'}{m + m'} - \frac{em'(a - a')}{m + m'},$$

$$v' = \frac{ma + m'a'}{m + m'} + \frac{em(a - a')}{m + m'}.$$

Maclaurin; *Choc des Corps*, p. 30, *Prix de l'Academie*, Tom. I.

(7) To find with what velocity a ball must impinge upon another equal ball moving with a given velocity, in order that the impinging ball may be reduced to rest by the collision, the common elasticity of the balls being known.

If e be the common elasticity, and a the velocity of the ball which is struck, the impinging ball must impinge with an opposite velocity equal to

$$\frac{1 + e}{1 - e} a.$$

(8) To find the elasticity of two spheres A and B, and the ratio between their masses, in order that, when A impinges upon B at rest, A may be reduced to rest, and B move on with the n^{th} part of A's velocity.

If m, m', denote the masses of A, B, and e their common elasticity, then

$$e = \frac{1}{n}, \quad \frac{m'}{m} = n.$$

(9) Two perfectly elastic spheres meet directly with equal velocities; to find the relation between their magnitudes, in order that after collision one of them may remain at rest.

If m, m', denote their masses, m' corresponding to the one which remains at rest,

$$m' : m :: 3 : 1.$$

(10) To determine the velocities of two bodies A and B of given elasticity and given masses moving in the same direction, in order that after collision A may remain at rest and B may move along with an assigned velocity.

If m, m', be the masses of A, B, e their elasticity, β the velocity which B is to have after collision, and a, b, the required velocities of A, B, before collision,

$$a = \frac{1+e}{e}\,\frac{m'\beta}{m+m'}, \quad b = \frac{(em'-m)\,\beta}{e\,(m+m')}.$$

Maclaurin; *Choc des Corps*, p. 52, *Prix de l'Acad.* Tom. I.

(11) A spherical body A impinges directly with a certain velocity upon a spherical body B at rest, the common elasticity of the two bodies being given; to find the mass of a third body which, moving with the velocity which A has before the collision, shall have the same momentum which B has after the collision.

If m, m', denote the masses of A, B; e the common elasticity of A, B; and m'' the mass of the required body,

$$m'' = (1+e)\,\frac{mm'}{m+m'}.$$

(12) A, B, C, are three perfectly elastic balls in the same straight line, the masses of which are as 2, 3, 5: B and C being at rest, A impinges on B with a velocity 1, and B is thus made to impinge on C: to find the velocity of B, after the first impact, and of B and C after the second impact; and to ascertain whether A and B ever come together again.

After the first impact, B's velocity is $\frac{4}{5}$, and, after the second impact, C's velocity is $\frac{3}{5}$, and B's, in an opposite direction, is $\frac{1}{5}$.

A and B part to meet no more.

(13) A number of balls of given elasticity A_1, A_2, A_3, ... are placed in a line; A_1 is projected with a given velocity so as to impinge on A_2; A_2 then impinges on A_3, and so on; to find the masses of the balls A_2, A_3, ... in order that each of the balls A_1, A_2, A_3, ... may be at rest after impinging on the next; and to find the velocity of the n^{th} ball after its collision with the $(n-1)^{\text{th}}$.

If u_1 = the original velocity of A_1, the final velocity of the n^{th} ball is equal to $e^{n-1}u_1$; also

$$A_r = \frac{A_1}{e^{r-1}}.$$

(14) Any number of spheres of given elasticity are arranged in a straight line: to one of the extreme ones a given velocity is then communicated so as to bring it into direct collision with the adjacent sphere of the series; to determine the velocity ultimately acquired by the last sphere.

If r be the number of the spheres, e their common elasticity, $m_1, m_2, m_3, \ldots\ldots m_r$, their masses, and a the velocity with which the first is projected; then, v being the velocity acquired at last by m_r,

$$v = (1+e)^{r-1}\frac{m_1}{m_1+m_2}\cdot\frac{m_2}{m_2+m_3}\cdot\frac{m_3}{m_3+m_4}\cdots\cdots\cdots\frac{m_{r-1}}{m_{r-1}+m_r}\cdot a.$$

Maclaurin; *Choc des Corps*, p. 54, *Prix de l'Acad.* Tom. I.

(15) A number of equal spheres are placed on a smooth table in a straight line and close together; they are connected together by equal inelastic threads; a motion is given to the first in the direction of the line which they form so as to separate it from the second; to find the time which elapses before the last sphere is put in motion.

If n be the number of spheres, a the length of each of the connecting threads, and β the velocity with which the first sphere is projected, then

$$\text{the time required} = \frac{n(n-1)}{1.2}\,\frac{a}{\beta}.$$

(16) To find the sum of the *vires vivæ* of two perfectly elastic bodies after direct collision.

If a, a', be the velocities before, and v, v', after collision,

$$mv^2 + m'v'^2 = ma^2 + m'a'^2.$$

Huyghens; *De Motu Corporum ex Percuss.* Prop. XI.
John Bernoulli; *Discours sur le Mouvement*, chap. X.

(17) To find the sum of the *vires vivæ* of two imperfectly elastic bodies after direct collision.

The notation remaining the same as in the preceding example, and e denoting the elasticity,

$$mv^2 + m'v'^2 = ma^2 + m'a'^2 - \frac{(1-e^2)\,mm'\,(a-a')^2}{m+m'},$$

which shews that *vis viva* is lost by the collision.

(18) Two balls, of elasticity e, are projected along a smooth fixed tube in the form of any closed curve, lying in a horizontal plane, from any two points in the tube: supposing u, v, to be the velocities of projection, estimated in the same direction, and c to be the length of the tube, to find the whole interval of time between the 1st and $(n+1)^{\text{th}}$ collisions.

The required interval is equal to

$$\frac{c}{u-v} \cdot \frac{e^{-n}-1}{1-e}.$$

(19) Three equal balls A, B, C, of elasticity e, are placed in order on a smooth horizontal plane in a straight line: velocities are impressed upon them in the direction ABC, those of A and C being each greater than that of B: two collisions having taken place, the velocities of A and B are observed to be equal to each other: to determine the ratio of the initial relative velocity of C, B, to that of A, B.

The required ratio is equal to

$$\frac{1}{2}\,\frac{(1-e)^2}{1+e}.$$

(20) Between two spheres of given masses is placed a row of spheres; a velocity is communicated to one of the original spheres so as to bring it into direct collision with the nearest of the intermediate ones; to find the requisite magnitudes of the intermediate spheres in order that the velocity acquired by the last sphere may be the greatest possible, and, the spheres being

perfectly elastic, to determine this velocity when their number becomes indefinitely great.

The intermediate spheres must be geometrical means between the two original ones. If the spheres be perfectly elastic, then, m, m', denoting the masses of the original spheres, a the velocity communicated to m, and a' that acquired by m' when the number of the intermediate spheres becomes infinite,

$$a' = \left(\frac{m}{m'}\right)^{\frac{1}{2}} a.$$

(21) An imperfectly elastic sphere impinges upon a plane; to find the angles of incidence and of reflection, in order that the velocity before may be to the velocity after impact as $2^{\frac{1}{2}} : 1$, the elasticity being equal to $\dfrac{1}{3^{\frac{1}{2}}}$.

The angle of incidence $= \frac{1}{3}\pi$, the angle of reflection $= \frac{1}{4}\pi$.

(22) The edge of a smooth elliptical table is environed by a vertical border: supposing a perfectly elastic ball to be projected along the table from one of its foci, in a direction inclined at a given angle to the major axis, to find the inclination of its path to the same axis between the n^{th} and $(n+1)^{\text{th}}$ impacts.

If θ, θ_n, represent the given and required angles respectively, and e the eccentricity of the ellipse,

$$\tan\frac{\theta_n}{2} = \left(\frac{1-e}{1+e}\right)^n . \tan\frac{\theta}{2}.$$

(23) An inelastic sphere A, moving with a given velocity, impinges upon an inelastic sphere B at rest, the line joining the centres of the two spheres at the instant of collision making a given angle with the direction of A's motion; to determine the velocity of A after collision.

If α be the given angle; m, m', the masses of the spheres A, B; and a, v, the velocities of A before and after collision,

$$v = a\left\{\sin^2\alpha + \frac{m^2}{(m+m')^2}\cos^2\alpha\right\}^{\frac{1}{2}}.$$

(24) **A sphere A in motion is struck by an equal one B, moving with the same velocity and in a direction making an angle α with that in which A is moving, in such a manner that the line joining their centres at the instant of collision is in the direction of B's motion; to find the velocities of the spheres after collision, and to determine for what value of α that of A will be a maximum, the common elasticity of the spheres being supposed to be known.**

If a denote the velocity of each of the spheres A, B, before, and u, v, their respective velocities after collision, then, e being their common elasticity,

$$u^2 = \tfrac{1}{4} a^2 \{1 + e + (1 - e) \cos \alpha\}^2 + a^2 \sin^2 \alpha,$$

$$v^2 = \tfrac{1}{4} a^2 \{1 - e + (1 + e) \cos \alpha\}^2.$$

When u is a maximum, $\cos \alpha = \dfrac{1 - e}{3 - e}$.

(25) **Three perfectly elastic spheres A, B, C, are placed at the three angles of a plane triangle of which the angles are known; to compare the magnitudes of the spheres, when A impinging obliquely upon B is reflected so as to strike C, and thence reflected to its first position; the lines joining the centres of the spheres A, B, and A, C, at the instants of collision, being respectively perpendicular to the opposite sides of the triangle, and their diameters being inconsiderable in comparison with the sides of the triangle.**

If m, m', m'', denote the masses of the spheres A, B, C; and α, β, γ, the angles of the triangle at which A, B, C, are placed,

$$\frac{m'}{m} = \frac{\sin \beta}{\sin (\alpha - \gamma)}, \quad \frac{m''}{m} = \frac{\sin \gamma}{\sin (\beta - \alpha)}.$$

(26) **A sphere A (fig. 116), moving in the direction EAF with an assigned velocity, impinges upon a sphere B at rest, the two spheres having the same elasticity; supposing AF' to be the direction of A's motion after impact, and $KABL$ to be a**

straight line passing through the centres of the two spheres at the instant of collision, to find the value of the angles EAK and FAF' when the latter angle has its greatest value.

If n be the ratio of the mass of A to that of B, e the common elasticity of the spheres, $\angle\, EAK = \theta$, $\angle\, FAF' = \phi$; then

$$\tan\theta = \left(\frac{n-e}{n+1}\right)^{\frac{1}{2}}, \quad \tan\phi = \frac{\frac{1}{2}(1+e)}{\{(n+1)(n-e)\}^{\frac{1}{2}}}.$$

CHAPTER II.

RECTILINEAR MOTION OF A PARTICLE.

THE determination of the circumstances of the motion of a material particle, which moves in a straight line under the action of a finite accelerating or retarding force, depends upon the two following differential equations, called the equations of motion of the particle,

$$\frac{dx}{dt}=v, \quad \frac{dv}{dt}=f,$$

where t denotes the time of the motion reckoned from an assigned epoch, x the distance of the particle at the end of this time from an assigned point in the line of its motion, v the velocity, and f the accelerating or retarding force.

From these two equations we readily deduce the two following,

$$\frac{d^2x}{dt^2}=f, \quad v\frac{dv}{dx}=f.$$

These equations, which constitute the complete expression of the circumstances of rectilinear motion in the language of the differential calculus for every condition of acceleration or retardation, are due to Varignon, and were published in the *Mém. de l'Acad. des Sciences de Paris*, 1700, p. 22. It may be observed however that, long before this, geometrical investigations of rectilinear motion for variable forces had been given by Newton[1].

From the formula $vdv=fdx$ we see that dv^2 varies as fdx: an opinion however was expressed by Daniel Bernoulli[2], that there is no reason to consider this the only possible law of variation; for instance, that we might as well have $dv^n \propto fdx$, n being any

[1] *Principia*, Lib. I. sect. 7; Lib. II. sect. 1.
[2] *Comment. Petrop.* 1727, p. 186.

quantity whatever. In opposition to Bernoulli's suggestion, Euler[1] endeavoured to prove that the law of the square of the velocity is necessarily true; and D'Alembert[2] shewed the truth of this law to depend simply upon the definition of the meaning of the symbol f.

The complete solution of a problem in rectilinear motion consists in the determination of relations between every two of the quantities x, v, f, t: now the general equations of rectilinear motion furnish us with only two independent relations between these four quantities; it is evident then that the data in every problem must consist in the expression of some particular equation, $\phi(x, v, f, t) = 0$ between x, v, f, t, so that we may have, in all, three equations connecting the four variables.

The function $\phi(x, v, f, t)$ may involve two, three, or all of the quantities x, v, f, t; and, by the theory of combinations, it is evident that there will be six varieties of the first, and four of the second class; hence the general problem of rectilinear motion resolves itself into eleven distinct classes of problems. We shall however confine ourselves to the consideration of those two classes in which the given function involves either x, f, alone; or x, f, v, alone: under the former head we shall exemplify the motion of a particle in vacuum; under the latter, in a resisting medium. The other classes are devoid of any physical interest.

SECT. 1. *Motion in Vacuum.*

(1) A particle is placed at a centre of repulsive force which varies as any power of the distance; to determine its velocity after receding to any distance from the centre, and the time of the motion.

Let μ represent the absolute force, x the distance of the particle from the centre of force after a time t, and v the velocity. Then, for the motion, we have

$$v\frac{dv}{dx} = \mu x^n.$$

[1] *Mechanica*, Tom. I. p. 62 et seq.

[2] *Traité de Dynamique*.

Integrating with respect to x, and bearing in mind that $v=0$ when $x=0$, we have

$$\tfrac{1}{2}v^2 = \mu\int_0^x x^n\,dx = \frac{\mu}{n+1}\,x^{n+1},$$

and therefore
$$v^2 = \frac{2\mu}{n+1}\,x^{n+1},$$

which gives the velocity for any value of x.

Again,
$$\frac{dx}{dt} = v = \left(\frac{2\mu}{1+n}\right)^{\frac{1}{2}} x^{\frac{1}{2}(n+1)}:$$

hence, t being equal to zero when $x=0$, there is

$$t = \left(\frac{1+n}{2\mu}\right)^{\frac{1}{2}} \int_0^x x^{-\frac{1}{2}(n+1)}\,dx$$
$$= \frac{2}{1-n}\left(\frac{1+n}{2\mu}\right)^{\frac{1}{2}} x^{\frac{1}{2}(1-n)}.$$

Euler; *Mechanica*, Tom. I. p. 123.

(2) A particle, initially at rest, is attracted by a force varying inversely as the n^{th} power of the distance: to find the value of n when the velocity acquired from an infinite distance to a distance a from the centre is equal to the velocity which would be acquired from a to $\frac{1}{4}a$.

Let μ denote the absolute force, x the distance of the particle from the centre of force after a time t, and v_1, v_2, the two velocities. Then, for the motion of the particle,

$$v\frac{dv}{dx} = -\frac{\mu}{x^n}.$$

Hence, for the former motion, v being equal to zero when $x=\infty$,

$$v_1^2 = -2\mu\int_\infty^a \frac{1}{x^n}\,dx = \frac{2\mu}{(n-1)\,a^{n-1}},$$

and, for the latter motion, since $v=0$ when $x=a$,

$$v_2^2 = -2\mu\int_a^{\frac{1}{4}a} \frac{1}{x^n}\,dx = \frac{2\mu}{n-1}\left(\frac{4^{n-1}}{a^{n-1}} - \frac{1}{a^{n-1}}\right).$$

But, by the hypothesis, v_1^2 is equal to v_2^2; hence

$$\frac{2\mu}{n-1}\frac{1}{a^{n-1}}=\frac{2\mu}{n-1}\frac{1}{a^{n-1}}(4^{n-1}-1),$$

and therefore $$1=4^{n-1}-1,\ 4^{n-1}=2;$$

whence $$n=\tfrac{3}{2}.$$

(3) The corners of a square are the centres of four equal attractive forces, their intensity varying as any function of the distance; a particle is placed in one of the diagonals of the square very near to its centre; to find the time of an oscillation.

Let O be the centre of the square (fig. 117), E the position of the particle after any time t from the commencement of the motion; let $OD=a$, $OE=x$, $AE=r=CE$. Then, for the motion of the particle, taking the sum of the forces acting upon it in the line OD, we have

$$\frac{d^2x}{dt^2}=-2\phi(r)\frac{x}{r}+\phi(a-x)-\phi(a+x),$$

and therefore, neglecting powers of the small quantity x higher than the first, we get, by Taylor's theorem,

$$\frac{d^2x}{dt^2}=-2\left\{\frac{\phi(a)}{a}+\phi'(a)\right\}x,$$

or, putting the coefficient of x equal to $-k$,

$$\frac{d^2x}{dt^2}=-kx.$$

The integral of this equation is evidently

$$x=C\cos(k^{\frac{1}{2}}t+\epsilon),$$

C and ϵ being constants: let β be the initial value of x:

then $$\beta=C\cos\epsilon:$$

but $\dfrac{dx}{dt}=0$ initially, and therefore $0=C\sin\epsilon$:

hence $$x=\beta\cos(k^{\frac{1}{2}}t).$$

Now, as soon as $k^{\frac{1}{2}}t$ becomes equal to π, x becomes equal to $-\beta$, its greatest negative value. Hence, the time of a complete oscillation being T, we have

$$k^{\frac{1}{2}}T=\pi,$$

and therefore, substituting for k its value,

$$T=\frac{\pi}{\sqrt{2}}\left\{\frac{1}{a}\phi(a)+\phi'(a)\right\}^{-\frac{1}{2}}.$$

(4) A particle A attracts a particle B with a force always to that with which B attracts A in the ratio of μ' to μ; the particles being originally at rest, to find their position as well as that of their centre of gravity after any time; the intensity of each force being directly as the distance between the particles.

Let O be a fixed point in the line of the motion of the particles (fig. 118), and let $OA=x$, $OB=x'$, at any time t.

Then, for the motion, we have

$$\frac{d^2x}{dt^2}=\mu(x'-x)\dots\dots\dots\dots(1),$$

$$\frac{d^2x'}{dt^2}=-\mu'(x'-x)\dots\dots\dots\dots(2).$$

Multiplying (1) and (2) by μ' and μ respectively, and adding the results, we get

$$\mu'\frac{d^2x}{dt^2}+\mu\frac{d^2x'}{dt^2}=0.$$

Integrating, and bearing in mind that $\frac{dx}{dt}$, $\frac{dx'}{dt}$, are both equal to zero initially,

$$\mu'\frac{dx}{dt}+\mu\frac{dx'}{dt}=0;$$

integrating again,

$$\mu'x+\mu x'=\mu'a+\mu a'\dots\dots\dots\dots(3),$$

a, a', being the initial values of x, x'.

Again, subtracting (1) from (2),

$$\frac{d^2}{dt^2}(x'-x)+(\mu+\mu')(x'-x)=0:$$

the integral of this equation is

$$x'-x=C\cos\{(\mu+\mu')^{\frac{1}{2}}t+\epsilon\},$$

C and ϵ being constants.

Now, initially, $x = a$, $x' = a'$, $\frac{dx}{dt} = 0$, $\frac{dx'}{dt} = 0$;

hence $\quad a' - a = C \cos \epsilon, \qquad 0 = C \sin \epsilon$:

the integral therefore becomes

$$x' - x = (a' - a) \cos \{(\mu + \mu')^{\frac{1}{2}} t\} \quad\ldots\ldots\ldots\ldots\ldots (4).$$

From (3) and (4) we readily obtain

$$x = \frac{\mu' a + \mu a'}{\mu' + \mu} - \frac{\mu (a' - a)}{\mu' + \mu} \cos \{(\mu' + \mu)^{\frac{1}{2}} t\},$$

$$x' = \frac{\mu' a + \mu a'}{\mu' + \mu} + \frac{\mu' (a' - a)}{\mu' + \mu} \cos \{(\mu' + \mu)^{\frac{1}{2}} t\};$$

$$(m + m') \bar{x} = mx + m'x'$$

$$= \frac{m' + m}{\mu' + \mu} (\mu' a + \mu a') + \frac{\mu' m' - \mu m}{\mu' + \mu} (a' - a) \cos \{(\mu' + \mu)^{\frac{1}{2}} t\},$$

where m, m', denote the masses of A, B, and $\bar{x}$ the distance of their centre of gravity from O at any time t.

If μ', μ, be proportional to m, m', respectively, then clearly from our general result

$$(m + m') \bar{x} = \frac{m' + m}{\mu' + \mu} (\mu' a + \mu a')$$

or

$$\bar{x} = \frac{\mu' a + \mu a'}{\mu' + \mu},$$

which shews that the centre of gravity remains stationary during the whole motion.

(5) A body not affected by gravity falls down the axis of a thin cylindrical tube, infinite in length, the particles of which attract with a force which varies inversely as the square of the distance; to find the velocity acquired in falling through a given space.

Let k be the thickness of the tube, r the radius of its interior surface, x the distance of the particle P from the extremity of the tube after a time t; then the volume of a portion of the tube contained between slices at distances s and $s + ds$ from P will be $2\pi rkds$, and therefore the attraction of this elemental portion on the particle along the axis of the tube will be, the unit of attrac-

tion being chosen to be the attraction of a unit of mass at a unit of distance,

$$2\pi\rho rkds \cdot \frac{1}{s^2+r^2} \cdot \frac{s}{(s^2+r^2)^{\frac{1}{2}}},$$

where ρ denotes the density; and therefore for the motion of the particle we have

$$\frac{d^2x}{dt^2} = 2\pi\rho rk \int_{-x}^{+\infty} \frac{sds}{(s^2+r^2)^{\frac{3}{2}}}$$

$$= 2\pi\rho rk \cdot {}_{-x}^{+\infty}\left\{-\frac{1}{(s^2+r^2)^{\frac{1}{2}}}\right\} = \frac{2\pi\rho rk}{(x^2+r^2)^{\frac{1}{2}}}:$$

multiplying by $2\dfrac{dx}{dt}$ and integrating,

$$v^2 = \frac{dx^2}{dt^2} = 4\pi\rho rk \log\{x + (x^2+r^2)^{\frac{1}{2}}\} + C.$$

But $v = 0$ when $x = 0$; hence

$$0 = 4\pi\rho rk \log r + C,$$

and therefore $\quad v^2 = 4\pi\rho rk \log\dfrac{x+(x^2+r^2)^{\frac{1}{2}}}{r}.$

(6) A particle is placed at a given distance from a centre of attractive force, the intensity of which varies inversely as the cube of the distance: to determine the position of the particle at any time during its motion towards the centre of force.

Let a be the initial distance of the particle from the centre of force, and x its distance at the end of a time t: then, μ denoting the absolute force,

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^3} \quad\text{..........................(1)},$$

whence $\quad \dfrac{dx^2}{dt^2} = c + \dfrac{\mu}{x^2},$

where c is a constant: but initially $x = a$ and $\dfrac{dx}{dt} = 0$: hence

$$0 = c + \frac{\mu}{a^2},$$

and therefore
$$\frac{dx^2}{dt^2} = \mu \left(\frac{1}{x^2} - \frac{1}{a^2}\right),$$
$$\frac{ax\,dx}{\mu^{\frac{1}{2}}(a^2 - x^2)^{\frac{1}{2}}} = -dt,$$

whence, since $x = a$ when $t = 0$, we see that
$$\frac{a}{\mu^{\frac{1}{2}}}(a^2 - x^2)^{\frac{1}{2}} = t,$$

and therefore
$$x^2 = a^2 - \frac{\mu t^2}{a^2}.$$

When $t = \frac{a^2}{\mu^{\frac{1}{2}}}$, $x = 0$, so that $\frac{a^2}{\mu^{\frac{1}{2}}}$ is the time of falling to the centre of force.

(7) Two balls are moving in a straight line, one of them only being acted on by a force; if the force be constant and tend always towards the other ball, to compare the times which elapse between consecutive impacts.

Let $v =$ the relative velocity of the balls just before the first impact: then $ev =$ their relative velocity just after: hence, in a time equal to $\frac{2ev}{f}$, the balls are again in contact, f denoting the force. Similarly, the next interval is $\frac{2e^2v}{f}$, the next $\frac{2e^3v}{f}$, and so on. Thus e is the ratio of the times between consecutive impacts.

(8) From a point A in a vertical line AB falls a particle from rest; at the same instant another particle is projected upwards from B with a given velocity: to find when and where the two particles will meet; the motion being supposed to take place in vacuum, and gravity being the only force to which the particles are subject.

Let a be the length of the line AB, β the velocity of projection of the ascending particle, x the distance from A at which collision takes place, and t the time of this event from the commencement of the motion. Then

$$x = \frac{ga^2}{2\beta^2}, \quad t = \frac{a}{\beta}.$$

Kurdwanowski; *Mém. de l'Acad. des Sciences de Berlin*, 1755, p. 394.

(9) A body is projected vertically upwards with a velocity $4g$: after two seconds, gravity ceases to act for one second, and is then doubled: to find the greatest height to which the body ascends, and to determine the velocity when it returns to the point of projection.

The greatest height and the required velocity are respectively $9g$ and $6g$.

(10) A body of known elasticity falls from a given altitude above a hard horizontal plane, and rebounds continually till its whole velocity is destroyed; to find the whole space described.

If a denote the given altitude, e the elasticity, and s the required space,

$$s = \frac{1+e^2}{1-e^2}a.$$

(11) Two perfectly elastic balls, beginning at the same instant to descend from different points in the same vertical line, impinge upon a perfectly hard plane inclined to the horizon at an angle of 45°, and then move along a fixed horizontal plane with the velocities acquired; to find what distance they will move along the horizontal plane before collision takes place.

If a, a', denote the altitudes through which they fall, and s the distance required,

$$s = 2\,(aa')^{\frac{1}{2}}.$$

(12) A particle falling in a straight line towards a centre of force, the intensity of which varies as the n^{th} power of the distance, acquires a velocity β on arriving at a distance a from the centre; to find at what distance z from the centre of force it must have commenced its motion.

Let μ denote the absolute force; then z will be given by the equation

$$z^{n+1} - a^{n+1} = \frac{n+1}{2\mu}\beta^n.$$

Euler; *Mechan.* Tom. I. p. 109.

(13) A particle falls towards a centre of force, of which the intensity varies inversely as the cube of the distance; to find the whole time of descent.

Let μ denote the absolute force and a the initial distance; then

$$\text{the time of descent} = \frac{a^2}{\mu^{\frac{1}{2}}}.$$

(14) A particle descends from an infinite distance towards a centre of force which varies inversely as the square of the distance; to find the velocity at a given distance from the centre of force.

Let μ be the absolute force and a the given distance; then

$$\text{the required velocity} = \left(\frac{2\mu}{a}\right)^{\frac{1}{2}}.$$

(15) A body is projected vertically from the surface of the Earth; to find the height to which it will ascend.

If g = the force of gravity at the Earth's surface, r = the Earth's radius, and V = the velocity of the body's projection, then the height of ascent, reckoned from the Earth's centre, is equal to

$$\frac{2gr^2}{2gr - V^2}.$$

If $V > (2gr)^{\frac{1}{2}}$, the body will never descend.

(16) A body falls from a given point towards a centre of force, the attraction at any distance r being $\frac{\mu}{r^{\frac{3}{2}}}$: to find the whole time of descent.

If a be the initial distance of the particle from the centre of force, the required time $= \dfrac{3\pi a^{\frac{3}{2}}}{8\mu^{\frac{1}{2}}}$.

(17) A particle is placed at a given distance a from a centre of attractive force, the intensity of which varies inversely as the $\left(\dfrac{2n+1}{2n-1}\right)^{\text{th}}$ power of the distance, where n is a positive integer greater than unity: to find the time of falling to the centre.

If μ denote the absolute force, the required time is equal to

$$\frac{2^{n-1} \, . \, a^{\frac{2n}{2n-1}}}{\{\mu\,(2n-1)\}^{\frac{1}{2}}} \cdot \frac{1 \, . \, 2 \, . \, 3 \ldots (n-1)}{n\,(n+1)\,(n+2) \ldots (2n-2)}.$$

(18) A body moves from rest at a distance a towards a centre of force, the force varying inversely as the distance: to determine the value of β in order that the time of describing the space between βa and $\beta^n a$ may be a maximum.

The required value of β is equal to $n^{-\frac{1}{2(n-1)}}$.

(19) A particle is placed at an assigned point between two centres of force of equal intensity attracting directly as the distance; to determine the position of the particle at any time, and the period of its oscillations.

Let a denote the initial distance of the particle from the middle point of the line joining the two centres of force, x the distance after the expiration of a time t, and μ the absolute force of each centre. Then

$x = a \cos \{(2\mu)^{\frac{1}{2}} t\}$, and the period of an oscillation $= \dfrac{\pi}{(2\mu)^{\frac{1}{2}}}$.

(20) A particle acted upon by two central forces, each attracting with an intensity varying inversely as the square of the distance, is projected from an assigned point between them towards one of the centres; to find the velocity of projection in order that the particle may just arrive at the neutral point of attraction and remain at rest there.

Let μ^2, μ'^2, denote the absolute forces of the two centres; $2a$, $2a'$, the initial distances of the particle from the two centres; and V the velocity of projection. Then

$$V = \frac{\dfrac{\mu}{a} - \dfrac{\mu'}{a'}}{\left(\dfrac{1}{a} + \dfrac{1}{a'}\right)^{\frac{1}{2}}}.$$

Jullien: *Problémes de Mécanique Rationnelle*, Tom. I. p. 233.

(21) A centre of force C (fig. 119) moves along the straight line OA with a uniform velocity, attracting, with a force varying directly as the first power of the distance, a particle P which is moving in the same straight line; having given the initial position of C, and both the initial position and the initial velocity of P, to find the position of P at any time.

Let a, a', be the initial distances of C, P, from O; β the uniform velocity of C, and β' the initial velocity of P; x the distance of P from O after a time t; μ the absolute force of attraction.

Then $\quad x = a + \beta t + \dfrac{\beta' - \beta}{\mu^{\frac{1}{2}}} \sin(\mu^{\frac{1}{2}} t) + (a' - a)\cos(\mu^{\frac{1}{2}} t).$

Riccati; *Bonon. Institut.* Tom. VI. p. 138; 1783.

(22) The circumstances remaining the same as in the preceding problem, except that the force is repulsive; to find the position of P at any time.

$$x = a + \beta t - \{\mu^{\frac{1}{2}}(a - a') + (\beta - \beta')\}\frac{\epsilon^{\mu^{\frac{1}{2}} t}}{2\mu^{\frac{1}{2}}} - \{\mu^{\frac{1}{2}}(a - a') - (\beta - \beta')\}\frac{\epsilon^{-\mu^{\frac{1}{2}} t}}{2\mu^{\frac{1}{2}}}.$$

Riccati; *Ib.* p. 151.

(23) Supposing the centre C to move along OA with a uniform acceleration, attracting directly as the distance; to determine the place of P at any time.

Let f represent the increment of C's velocity in each unit of time, and β its velocity at the commencement of the motion; then, the notation remaining the same as in the two preceding problems,

$$x = a - \frac{f}{\mu} + \beta t + \tfrac{1}{2} f t^2 + \frac{\beta' - \beta}{\mu^{\frac{1}{2}}} \sin (\mu^{\frac{1}{2}} t) + \left(a' - a + \frac{f}{\mu}\right) \cos (\mu^{\frac{1}{2}} t).$$

Riccati; *Ib.* p. 168.

(24) The circumstances and notation remaining the same as in the preceding example, except that the force is repulsive; to find the place of P at any time.

$$\begin{aligned} x = a + \frac{f}{\mu} + \beta t + \tfrac{1}{2} f t^2 \\ - \left\{\mu^{\frac{1}{2}} \left(a - a' + \frac{f}{\mu}\right) + (\beta - \beta')\right\} \frac{\epsilon^{\mu^{\frac{1}{2}} t}}{2\mu^{\frac{1}{2}}} \\ - \left\{\mu^{\frac{1}{2}} \left(a - a' + \frac{f}{\mu}\right) - (\beta - \beta')\right\} \frac{\epsilon^{-\mu^{\frac{1}{2}} t}}{2\mu^{\frac{1}{2}}}. \end{aligned}$$

Riccati; *Ib.* p. 182.

(25) A particle is attached by a straight elastic string to a centre of repulsive force, the intensity of which varies as the distance: the string is at first at its natural length: to find the greatest distance from the centre of force to which the particle will proceed, and the time of returning to its natural length.

Let m be the mass of the particle, μ the absolute accelerating force, a the natural length of the string, $ma\lambda$ the modulus of elasticity: then the required distance and time are respectively equal to

$$\frac{\lambda + \mu}{\lambda - \mu} a, \quad \frac{2\pi}{(\lambda - \mu)^{\frac{1}{2}}}.$$

(26) Two bodies hang at rest from a fixed point at the lower end of a fine elastic string: supposing one of them to drop off, to find the subsequent motion of the other.

Let W be the weight of the body which sticks to the string, W' that of the body which falls off, a the natural length and λ the modulus of elasticity of the string: then, at the end of any time t from the commencement of the motion, the depth of the weight W below the upper end of the string is equal to

$$a + \frac{a}{\lambda} \cdot \left[W + W' \cos \left\{\left(\frac{\lambda g}{W a}\right)^{\frac{1}{2}} t\right\}\right].$$

(27) Two particles, connected by a fine elastic string, are moving in the same direction with equal velocities, viz. in the direction of the string, their distance being the natural length of the string: if the hinder particle be suddenly stopped, to find how far the other particle will move before it begins to return.

Let a be the length of the string, m the mass and u the initial velocity of each particle, λ the modulus of elasticity of the string: then the required distance is equal to $u\left(\frac{am}{\lambda}\right)^{\frac{1}{2}}$.

Griffin: *Solutions of the Examples on the Motion of a Rigid Body*, p. 111.

(28) A particle is placed at a given distance from a uniform thin plate of infinite extent, every particle of which attracts with a force varying inversely as the square of the distance; to find the time in which the particle will arrive at the surface of the plate.

Let k denote the thickness of the plate, ρ its density, and a the initial distance of the particle from it: then

$$\text{the time} = \left(\frac{a}{\pi\rho k}\right)^{\frac{1}{2}}.$$

(29) A particle is placed at a given distance from a thin circular lamina of uniform density, in a line passing through its centre and perpendicular to its plane: to find the velocity which it will acquire by moving to the lamina, the attractive force of each molecule of the lamina varying inversely as the square of the distance.

Let a be the radius of the circular lamina, k its thickness, ρ its density, b the given distance; then, the unit of attraction being the attraction of a unit of mass at a unit of distance, and V being the velocity required,

$$V^2 = 4\pi\rho k\,\{a + b - (a^2 + b^2)^{\frac{1}{2}}\}.$$

(30) A particle is placed at a small distance from the centre of a thin ring of uniform density and thickness, every molecule of

which repels with a force varying inversely as the square of the distance; to determine the position of the particle at any time, and the period of its oscillations.

Let l be the initial distance of the particle from the centre of the ring, a the radius of the ring, k the area of a section, ρ the density, and x the distance of the particle from the centre at the end of a time t. Then, the repulsion of a unit of the ring's mass at a unit of distance being taken as the unit of repulsion,

$$x = l \cos\left\{\frac{(\pi\rho k)^{\frac{1}{2}}}{a}\, t\right\} \text{ and the period of an oscillation} = \left(\frac{\pi}{\rho k}\right)^{\frac{1}{2}} a.$$

(31) Two cannons (each free to recoil) differ only in weight and in the weight of the ball: assuming that, at any instant during the explosion, the explosive force depends only on the space occupied by the vapour of the gunpowder; to compare (1) the emerging velocities of the balls, and (2) also the emerging velocities of balls fired from the same cannon when it is free to recoil, and when it is absolutely fixed.

(1) Let m, m_1, be the masses of the balls, and M, M_1, of the respective cannons, and let v, v_1, be the emerging velocities of the balls: then

$$v : v_1 :: \frac{(m_1 + M_1)^{\frac{1}{2}}}{(mM_1)^{\frac{1}{2}}} : \frac{(m + M)^{\frac{1}{2}}}{(m_1 M)^{\frac{1}{2}}}.$$

(2) Let m, M, be the respective masses of the ball and cannon, and v, v_1, the emergent velocities of the ball fired from the cannon when free and when fixed: then

$$v : v_1 :: M^{\frac{1}{2}} : (m + M)^{\frac{1}{2}}.$$

Cayley: *Messenger of Mathematics*, Vol. v. p. 43.

(32) A mass M attached to the end A of a chain AC is placed (with the chain) on a horizontal plane, in such wise that a portion AB of the chain forms a straight line, the remaining portion BC being heaped up at B: the mass M is then set in motion in the direction B to A with a given velocity, and so moves in a straight line, dragging the chain: to determine the motion.

Let m denote the mass of a unit of length of the chain; let CA be equal to a initially and to $a+x$ at the end of the time t, x being not greater than $l-a$, l denoting the whole length of the chain. Let V be the initial velocity of M. Then

$$(M+ma)\,x+\tfrac{1}{2}mx^2=(M+ma)\,Vt,$$

and, when the whole chain is in motion, the mass and the chain will move with a uniform velocity equal to

$$V.\frac{M+ma}{M+ml}.$$

Cayley: *Messenger of Mathematics*, Vol. V. p. 48.

SECT. 2. *Motion in Resisting Media.*

The retardation, experienced by a material particle in traversing a resisting medium of variable density, depends at any point of its path upon the density of the medium and the velocity of the particle, and will therefore be some function of these quantities. The nature of this function can be ascertained only by experiment. In mathematical investigations, for the sake of simplicity and as a probable approximation to the truth, the function is assumed to be of the form $k\rho\Omega$, where ρ denotes the density of the medium and Ω some function of the velocity of the particle; and where k is an invariable coefficient depending upon the nature of the particular medium in respect to the tenacity and the friction of its constituent molecules.

For the earliest mathematical development of the theory of the resistance of media to the motion of bodies, we are indebted to the labours of Newton and Wallis. The profound researches of Newton on this theory were published in the year 1687 in the second book of the *Principia*. In the same year, after the publication of Newton's investigations, Wallis, who had independently arrived at valuable conclusions on the subject, communicated his reflections to the Royal Society, which were inserted

in the *Philosophical Transactions* for the year 1687. There is a paper by Leibnitz on the question of resisting media in the *Acta Erudit. Lips.* ann. 1689, in which he developes opinions which he declares to have been communicated by him twelve years before to the Royal Academy of Sciences. Huyghens also has discussed certain points of the theory at the end of his *Discours de la Cause de la Pesanteur*, published in the year 1690. Finally, all which these philosophers had communicated to the scientific world either with or without demonstration, was investigated analytically by Varignon in a series of papers in the *Mémoires de l'Acad. des Sciences de Paris*, for the years 1707, 1708, 1709, and 1710. There is an elaborate paper by Bouguer in the *Mém. de l'Acad. des Sciences de Paris*, 1731, p. 390, in which he investigates the motion of a particle in resisting media which are themselves in motion.

(1) A particle, acted upon by no forces, is projected with a given velocity in a resisting medium of uniform density, where the resistance varies directly as the velocity; to determine the velocity and the space described at the end of any time.

For the motion of the particle we have

$$v\frac{dv}{dx} = -\mu v,$$

where μ is some constant quantity; hence

$$\frac{dv}{dx} = -\mu,$$

$$v = C - \mu x.$$

Let β denote the initial velocity, and let the initial value of x be zero; then $\beta = C$, and therefore

$$v = \beta - \mu x;$$

whence, v being equal to $\dfrac{dx}{dt}$, we have

$$dt = \frac{dx}{\beta - \mu x},$$

$$t = C - \frac{1}{\mu}\log(\beta - \mu x).$$

But $t = 0$ when $x = 0$; and therefore

$$0 = C - \frac{1}{\mu}\log\beta;$$

hence
$$t = \frac{1}{\mu}\log\frac{\beta}{\beta - \mu x},$$

$$\epsilon^{\mu t} = \frac{\beta}{\beta - \mu x},$$

$$x = \frac{\beta}{\mu}(1 - \epsilon^{-\mu t}),$$

and therefore
$$v = \beta\epsilon^{-\mu t}.$$

Newton; *Principia*, Lib. II. Prop. 1 and 2. Leibnitz; *Acta Erudit. Lips.* ann. 1689. Varignon; *Mém. de l'Acad. des Sciences de Paris*, ann. 1707, p. 391.

(2) A body falls towards a centre of attractive force, which varies as the inverse cube of the distance, in a medium of which the density varies also as the inverse cube, and of which the resistance varies as the square of the velocity; to find the velocity at any distance from the centre.

Let x represent the distance of the particle from the centre after a time t, and let a be the initial distance. Let k denote the force of resistance at a unit of distance for a unit of velocity, and μ the absolute force of attraction.

Then, for the motion of the particle,

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^3} + \frac{k}{x^3}\frac{dx^2}{dt^2},$$

$$2\frac{dx}{dt}\frac{d^2x}{dt^2} = -\frac{2\mu}{x^3}\frac{dx}{dt} + \frac{2k}{x^3}\frac{dx^3}{dt^3},$$

$$\frac{d}{dt}\frac{dx^2}{dt^2} = \mu\frac{d}{dt}\frac{1}{x^2} - k\frac{dx^2}{dt^2}\frac{d}{dt}\frac{1}{x^2}.$$

Assume $\frac{dx^2}{dt^2} = w$ and $\frac{1}{x^2} = z$; then

$$\frac{dw}{dt} = \mu \frac{dz}{dt} - kw \frac{dz}{dt},$$

$$dw + kwdz = \mu dz,$$

$$d\left(\epsilon^{kz} w\right) = \mu \epsilon^{kz} dz,$$

$$\epsilon^{kz} w = C + \frac{\mu}{k} \epsilon^{kz}.$$

Hence, putting for w and z their values,

$$\epsilon^{\frac{k}{x^2}} v^2 = C + \frac{\mu}{k} \epsilon^{\frac{k}{x^2}}.$$

But $x = a$ when $v = 0$; hence

$$0 = C + \frac{\mu}{k} \epsilon^{\frac{k}{a^2}},$$

$$\epsilon^{\frac{k}{x^2}} v^2 = \frac{\mu}{k} \left(\epsilon^{\frac{k}{x^2}} - \epsilon^{\frac{k}{a^2}}\right),$$

$$v^2 = \frac{\mu}{k} \left\{1 - \epsilon^{-k\left(\frac{1}{x^2} - \frac{1}{a^2}\right)}\right\}.$$

(3) To determine the motion of a particle, not acted upon by any force, when the resistance varies as any power of the velocity.

For the determination of the relation between the velocity and the space,

$$v \frac{dv}{dx} = -kv^n,$$

$$kdx = -\frac{dv}{v^{n-1}},$$

$$kx = C + \frac{1}{(n-2)\, v^{n-2}}.$$

Let $x = 0$ and $v = \beta$ initially; then

$$0 = C + \frac{1}{(n-2)\, \beta^{n-2}},$$

$$(n-2)\,kx = \frac{1}{v^{n-2}} - \frac{1}{\beta^{n-2}} \quad\text{.....................(1).}$$

Again, for the relation between the velocity and the time,

$$\frac{dv}{dt} = -kv^n, \quad kdt = -\frac{dv}{v^n},$$

$$kt = C + \frac{1}{(n-1)\,v^{n-1}}.$$

But $v = \beta$ when $t = 0$; hence

$$0 = C + \frac{1}{(n-1)\,\beta^{n-1}},$$

$$(n-1)\,kt = \frac{1}{v^{n-1}} - \frac{1}{\beta^{n-1}} \quad\text{..................(2).}$$

If between (1) and (2) we eliminate v, we shall obtain a relation between s and t.

Varignon; *Mém. de l'Acad. des Sciences de Paris*, 1707, p. 404.

(4) A particle acted on by gravity falls from a given altitude in a medium of uniform density, where the resistance varies as the square of the velocity; on arriving at the lowest point of its descent it is reflected upwards with the velocity which it has acquired in its fall; after reaching its greatest altitude it again descends and is again reflected; and so on perpetually: to determine the altitude of ascent after any number of reflections.

Let the maximum altitudes of the particle be represented by $a_1, a_2, a_3, \ldots$, a_1 being the altitude from which it originally falls. Let c denote the volume of the particle, and ρ, ρ', the density of the particle and of the fluid.

For the descent down any of the altitudes there is

$$v\frac{dv}{dx} = \frac{cg\rho - cg\rho'}{c\rho} - kv^2,$$

or $$v\frac{dv}{dx} = g' - kv^2, \text{ where } g' = \left(1 - \frac{\rho'}{\rho}\right)g,$$

$$\frac{vdv}{g' - kv^2} = dx,$$

$$-\frac{1}{2k}\log(g' - kv^2) = x + C;$$

but, the origin of x being the highest point,

$$-\frac{1}{2k}\log g' = C;$$

hence $$\frac{1}{2k}\log\frac{g'}{g' - kv^2} = x,$$

and therefore, if v_n denote the velocity acquired down the n^{th} altitude,

$$\log\frac{1}{1 - \frac{k}{g'}v_n^2} = 2ka_n,$$

$$\frac{k}{g'}v_n^2 = 1 - \epsilon^{-2ka_n} \text{ ....(1).}$$

For the ascent up the $(n+1)^{th}$ altitude, the origin of x being the lowest point,

$$v\frac{dv}{dx} = -g' - kv^2, \qquad \frac{vdv}{g' + kv^2} = -dx,$$

$$\frac{1}{2k}\log(g' + kv^2) = C - x,$$

$$\frac{1}{2k}\log(g' + kv_n^2) = C,$$

$$\frac{1}{2k}\log g' = C - a_{n+1},$$

$$\frac{1}{2k}\log\left(1 + \frac{k}{g'}v_n^2\right) = a_{n+1},$$

$$\frac{k}{g'}v_n^2 = \epsilon^{2ka_{n+1}} - 1 \text{ ....(2).}$$

Hence, from (1) and (2),

$$\epsilon^{2ka_{n+1}} + \epsilon^{-2ka_n} = 2:$$

assume $\epsilon^{2ka_n} = u_n$, and we have

$$u_{n+1} + \frac{1}{u_n} = 2,$$

$$u_n u_{n+1} + 1 = 2u_n.$$

Putting $u_n = v_n + 1$, we get

$$(v_n + 1)(v_{n+1} + 1) + 1 = 2(v_n + 1),$$

$$v_n v_{n+1} + v_{n+1} - v_n = 0, \qquad \frac{1}{v_{n+1}} - \frac{1}{v_n} = 1,$$

$$\Delta \frac{1}{v_n} = 1, \quad \frac{1}{v_n} = n + C,$$

$$\frac{1}{u_n - 1} = n + C, \quad u_n - 1 = \frac{1}{n + C}, \quad u_n = \frac{n + 1 + C}{n + C}.$$

But $\dfrac{1}{u_1 - 1} = 1 + C$; hence

$$u_n = \frac{n + \dfrac{1}{u_1 - 1}}{n - 1 + \dfrac{1}{u_1 - 1}} = \frac{nu_1 - n + 1}{(n-1)u_1 - n + 2}.$$

Or, putting for u_n, u_1, their values,

$$\epsilon^{2ka_n} = \frac{n\epsilon^{2ka_1} - n + 1}{(n-1)\epsilon^{2ka_1} - n + 2},$$

$$a_n = \frac{1}{2k}\log\frac{n\epsilon^{2ka_1} - n + 1}{(n-1)\epsilon^{2ka_1} - n + 2}.$$

If a_1 be equal to infinity,

$$a_n = \frac{1}{2k}\log\frac{n}{n-1}.$$

Euler; *Mechan.* Tom. I. p. 192.

(5) To determine the centripetal force in order that a particle may always descend to a given centre in the same time from whatever distance it commences its motion; the density of the medium in which the particle moves being known at every point in its path, and the resistance varying as the square of the velocity.

The equation of motion is

$$v\frac{dx}{dv} = -p + kv^2,$$

where p denotes the centripetal force, and k the density at any point. Multiplying by

$$2\epsilon^{-2\int k dx} dx,$$

the equation becomes

$$d\left(\epsilon^{-2\int k dx} v^2\right) = -2\epsilon^{-2\int k dx} p dx.$$

Integrating, $$\epsilon^{-2\int k dx} v^2 = C - 2\int \epsilon^{-2\int k dx} p dx.$$

Let a denote the initial distance of the particle from the centre of force; then, the velocity being initially equal to zero,

$$\epsilon^{-2\int k dx} v^2 = A - X,$$

where $$A = 2\int_0^a \epsilon^{-2\int k dx} p dx \text{ and } X = 2\int_0^x \epsilon^{-2\int k dx} p dx.$$

Therefore $$v = (A - X)^{\frac{1}{2}} \epsilon^{\int k dx},$$

$$dt = -\frac{\epsilon^{-\int k dx} dx}{(A - X)^{\frac{1}{2}}}.$$

Now, since X, k, are both functions of x, it is clear that we may assume

$$\epsilon^{-\int k dx} dx = \frac{dX}{P}$$

where P is a function of X alone: hence

$$dt = -\frac{dX}{P(A - X)^{\frac{1}{2}}},$$

and the whole time of descent, since $X = 0$ when $x = 0$, will be equal to

$$-\int_a^0 \frac{\epsilon^{-\int k dx} dx}{(A - X)^{\frac{1}{2}}} = -\int_A^0 \frac{dX}{P(A - X)^{\frac{1}{2}}};$$

and, since the value of this integral is to be the same for all values of a and therefore of A, the differential

$$\frac{dX}{P(A-X)^{\frac{1}{2}}}$$

must be of no dimensions in X, dX, and A. Hence P, which clearly cannot involve a, must be equal to $\frac{X^{\frac{1}{2}}}{\beta}$, where β is some constant quantity; and therefore

$$\beta\frac{dX}{X^{\frac{1}{2}}}=\epsilon^{-\int k dx}dx:$$

hence, X and x being simultaneously equal to zero,

$$2\beta X^{\frac{1}{2}}=\int_0^x \epsilon^{-\int k dx}dx,\quad 4\beta^2 X=\left\{\int_0^x \epsilon^{-\int k dx}dx\right\}^2,$$

$$8\beta^2\int_0^x(\epsilon^{-2\int k dx}p\,dx)=\left\{\int_0^x \epsilon^{-\int k dx}dx\right\}^2,$$

$$4\beta^2\epsilon^{-2\int k dx}p\,dx=\epsilon^{-\int k dx}dx\int_0^x \epsilon^{-\int k dx}dx,$$

$$p=\frac{1}{4\beta^2}\epsilon^{\int k dx}\int_0^x \epsilon^{-\int k dx}dx.$$

If k be equal to zero, which corresponds to a perfect vacuum,

$$p=\frac{1}{4\beta^2}\int_0^x dx=\frac{x}{4\beta^2},$$

or the centripetal force varies as the distance.

If the medium be uniform, or k a constant quantity,

$$p=\frac{1}{4\beta^2}\epsilon^{kx}\int_0^x \epsilon^{-kx}dx$$

$$=\frac{1}{4\beta^2}\epsilon^{kx}\cdot\frac{1}{k}(1-\epsilon^{-kx})=\frac{1}{4\beta^2 k}(\epsilon^{kx}-1).$$

Euler; *Mechan.* Tom. I. p. 220.

(6) A centre of force C, (fig. 120), moves along the straight line OA with a uniform velocity, repelling with a force varying directly as the first power of the distance: a particle P is moving along the same straight line OA in a medium the resistance of which varies as the velocity; having given the initial position of C, and both the initial position and the initial velocity of P, to find the position of P at any time.

Let β be the uniform velocity of C, a its initial distance from O; x the distance of P from O at the end of any time t; μ the absolute force of repulsion; k the resistance of the medium for a unit of velocity. Then, the distance between C and P at the time t being $a+\beta t-x$, we have for the motion of P, so long as it continues to proceed in the direction OA,

$$\frac{dv}{dt}=-\mu\,(a+\beta t-x)-kv,$$

v being the velocity of P at the time t, estimated in the direction OA. Supposing the particle to be moving in the direction AO, then, v being also estimated in this direction, we should have

$$\frac{dv}{dt}=\mu\,(a+\beta t-x)-kv,$$

an equation deducible from the former one by the substitution of $-v$ in place of v: hence the former equation applies to the motion of the particle under all circumstances, the quantity v being supposed to involve the direction-sign of the velocity implicitly.

But $v=\dfrac{dx}{dt}$, and therefore

$$\frac{d^2x}{dt^2}=-\mu\,(a+\beta t-x)-k\frac{dx}{dt},$$

$$\frac{d^2x}{dt^2}+k\frac{dx}{dt}=\mu\,(x-a-\beta t);$$

and therefore, putting $x - a - \frac{k\beta}{\mu} - \beta t = z$,

$$\frac{d^2z}{dt^2} + k\frac{dz}{dt} = \mu z.$$

Assume $z = A\epsilon^{\rho t}$, A and ρ being constants; then, substituting for z in the differential equation, we have

$$\rho^2 + k\rho = \mu,$$

$$4\rho^2 + 4k\rho + k^2 = 4\mu + k^2,$$

$$2\rho = \quad -k \pm (4\mu + k^2)^{\frac{1}{2}}:$$

hence the complete integral is

$$z = C\epsilon^{\gamma t} + C'\epsilon^{\gamma' t},$$

where γ, γ', are the two values of ρ obtained by the solution of the quadratic. Hence for the position of P at any time we have

$$x = a + \frac{k\beta}{\mu} + \beta t + C\epsilon^{\gamma t} + C'\epsilon^{\gamma' t}.$$

Suppose a', β', to be the initial values of x, $\frac{dx}{dt}$; then clearly

$$a' = a + \frac{k\beta}{\mu} + C + C', \quad \beta' = \beta + \gamma C + \gamma' C';$$

from which two equations the values of the constants C and C' are immediately determined.

For further information on the subject of the rectilinear motion of a particle in a resisting medium under the action of a centre of force moving according to any assigned law, the reader is referred to Riccati; *De motu rectilineo corporis attracti aut repulsi a centro mobili; Disquisitio quarta. Comment. Bonon.* Tom. VI. p. 212; 1783.

(7) A particle, projected with a velocity of 1000 feet a second, loses half its velocity by passing through 3 inches of a resisting medium, in which the resistance is uniform; to find the time of passing through this space.

The required time $= 3000^{\text{th}}$ part of a second.

(8) A particle, acted on by gravity, falls in a medium the resistance of which varies as the velocity: to find the terminal velocity[1] of the particle.

If k denote the resistance corresponding to a unit of velocity, the required velocity is equal to $\frac{g}{k}$.

Newton: *Principia*, Lib. II., Prop. 3.

(9) A particle is projected with a given velocity, towards a centre of force attracting inversely as the cube of the distance, in a medium of which the density varies inversely as the square of the distance from the centre of force; to determine the velocity of the particle at any distance from the centre, the resistance for a given density varying as the square of the velocity.

If β denote the velocity of projection, μ the absolute attracting force; k the retarding force of the medium, at a unit of distance from the centre of force, for a unit of velocity; a the initial distance of the particle, and x its distance corresponding to a velocity v,

$$\epsilon^{-\frac{2k}{a}} v^2 - \epsilon^{-\frac{2k}{x}} \beta^2 = \frac{\mu}{2k^2}\left(\frac{a-2k}{a}\,\epsilon^{-\frac{2k}{x}} - \frac{x-2k}{x}\,\epsilon^{-\frac{2k}{a}}\right).$$

(10) A particle is projected with a given velocity in a uniform medium, in which the resistance varies as the square root of the velocity; to find what time will elapse before the particle is reduced to rest.

If β be the velocity of projection, and k the resistance for a unit of velocity,

$$\text{the required time} = \frac{2\beta^{\frac{1}{2}}}{k}.$$

[1] "Terminal velocity" is a term, first used by Huyghens (*Discours sur la Cause de la Pesanteur*, p. 170), signifying the ultimate velocity of a particle descending in a resisting medium to an indefinitely great depth.

(11) If t denote the time in which a particle falling from rest will acquire a certain velocity, and τ the time in which, when projected vertically upwards, it will lose the same velocity; the motion in both cases taking place in a medium of uniform density, where the resistance varies as the square of the velocity; to investigate the relation between t and τ.

If k denote the resistance for a unit of velocity,

$$\log \tan (\tfrac{1}{4}\pi + k^{\frac{1}{2}} g^{\frac{1}{2}} \tau) = 2g^{\frac{1}{2}} k^{\frac{1}{2}} t.$$

(12) A particle, attracted to a centre of constant attractive force, moves directly towards it from rest, through a medium of which the resistance varies as the square of the velocity directly, and as the distance from the centre inversely; to find the velocity for any position of the particle during its approach towards the centre, and to ascertain its distance from the centre when its velocity is a maximum.

If f denote the constant central force, v the velocity for any distance x from the centre, a the initial value of x, k the resistance when x and v are each equal to unity, and x' the central distance when v is a maximum;

$$v^2 = \frac{2f}{1-2k} x^{2k} (a^{1-2k} - x^{1-2k}), \quad x' = (2k)^{\frac{1}{1-2k}} a.$$

(13) One particle begins to fall from the higher extremity of a vertical line, at the same instant in which another is projected upwards with a given velocity; the particles move in a uniform medium in which the resistance varies as the velocity; to find the time in which they will meet.

Let a denote the length of the vertical line, β the velocity with which the lower particle is projected upwards, k the resistance for a unit of velocity, and t the required time; then

$$t = \frac{1}{k} \log \frac{\beta}{\beta - ka}.$$

(14) A particle is projected vertically upwards in a medium in which the resistance is equal to kv^2; if V be the velocity of

projection, to find the particle's velocity V_1, when it again arrives at the point of projection.

$$V_1^2 = \frac{gV^2}{g + kV^2}.$$

(15) A particle, of which the elasticity is e, falls from rest from an altitude a in a uniform medium, the resistance of which is kv^2; and, impinging upon a perfectly hard horizontal plane, rises and falls alternately; to determine the whole space described before the motion ceases.

$$\text{The required space} = a + \frac{1}{k} \log \frac{1 - e^2 \epsilon^{-2ka}}{1 - e^2}.$$

Bordoni; *Memorie della Societa Italiana*, 1816, p. 162.

CHAPTER III.

FREE CURVILINEAR MOTION OF A PARTICLE.

SECT. 1. *Forces acting in any directions in one Plane.*

LET a particle, moving in a plane curve under the action of any accelerating forces, be referred to two fixed co-ordinate axes in the plane of its motion. Let x, y, be its co-ordinates at the end of a time t from an assigned epoch; and X, Y, the sum of the resolved parts of the accelerating forces parallel to the axes of x, y. Then the circumstances of the motion will be completely represented by the equations

$$\frac{d^2x}{dt^2} = X, \quad \frac{d^2y}{dt^2} = Y \dots\dots\dots\dots (A.)$$

The method of resolving parallel to fixed axes the accelerating forces which act upon a particle, and thus reducing the determination of the circumstances of its motion to the formulæ for rectilinear acceleration, was first given by Maclaurin, *Treatise of Fluxions*, Vol. I. Art. 465, et sq., published in the year 1742. Before this time all problems in curvilinear motion were solved by the method of the tangential and normal resolutions, which, although more immediately suggested by the physical conception of the motion, is not generally so convenient in analysis as that of Maclaurin. The great work of Euler on Mechanics, which appeared in 1736, proceeds altogether by the ancient method of resolution. We shall devote the third section of this chapter to the illustration of the ancient equations of motion.

(1) A particle acted on by gravity is describing a path $KABL$, (fig. 121); having given the resolved part of the velocity at A at right angles to the chord AB, to find the resolved part at B taken in the same direction.

Let u be the given resolved part of the velocity at A; v the velocity at right angles to AB at any point of the path corre-

sponding to a time t from leaving A, and x the perpendicular distance of the particle from AB at this time; also let α be the inclination of AB to the horizon. Then, for the motion of the particle,

$$x = ut - \tfrac{1}{2} g \cos \alpha \,.\, t^2,$$

and $$v = u - g \cos \alpha \,.\, t.$$

Now at the point B, $x = 0$, and therefore

$$u - \tfrac{1}{2} g \cos \alpha \,.\, t = 0, \quad g \cos \alpha \,.\, t = 2u;$$

hence for the value of v at B we have

$$v = -u.$$

Thus we see that the velocity of the particle at B, resolved at right angles to AB, is equal to the similarly resolved part at A, but of an opposite direction.

(2) A particle revolves in a parabola about a centre of force situated at the point of intersection of the directrix with the axis; to find the force at any point of the path of the particle.

Take the centre of force as the origin of co-ordinates, the axis of the parabola as the axis of x, and the directrix as the axis of y; let $4m$ be the principal parameter, r the distance of the particle at any time from the origin, and F the force estimated repulsively.

The equations of motion will be

$$\frac{d^2x}{dt^2} = \frac{Fx}{r}, \qquad \frac{d^2y}{dt^2} = \frac{Fy}{r} \;\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The equation to the parabola will be

$$y^2 = 4m\,(x - m) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2);$$

hence $$y\frac{dy}{dt} = 2m\frac{dx}{dt} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

$$y\frac{d^2y}{dt^2} + \frac{dy^2}{dt^2} = 2m\frac{d^2x}{dt^2};$$

and therefore, from (1),

$$F(2mx - y^2) = r\frac{dy^2}{dt^2} \;\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Again, eliminating F between the equations (1), we have

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0;$$

integrating, and adding a constant c which will represent twice the area described in a unit of time about the centre of force, we obtain

$$x\frac{dy}{dt}-y\frac{dx}{dt}=c;$$

and therefore, by (3),

$$(2mx-y^2)\frac{dy}{dt}=2mc;$$

hence, from (4),

$$F=\frac{4m^2c^2r}{(2mx-y^2)^3}=\frac{c^2r}{2m\,(2m-x)^3}, \text{ by (2).}$$

(3) **A particle, urged towards a** plane by a force varying as the perpendicular distance from it, is projected at right angles to the plane from a given point in it with a given velocity; to determine the force which must act at the same time on the particle parallel to the plane, in order that it may move in a given parabola the axis of which is in the plane, and to find the co-ordinates of the particle at any epoch of the motion in terms of the time.

Let the initial place of the particle be taken as the origin of co-ordinates, the axis of the parabola as the axis of x, and a straight line at right angles to the plane through the origin as the axis of y. Then, since the required force must evidently act parallel to the axis of x, we have, by Maclaurin's Equations (A),

$$\frac{d^2x}{dt^2}=X\text{..........................(1)},$$

$$\frac{d^2y}{dt^2}=-\mu y\text{.......................(2)},$$

where X is the required force and μ a constant quantity. Also the equation to the parabola will be

$$y^2=4mx\text{............................(3)}.$$

Differentiating this equation,

$$y\frac{dy}{dt}=2m\frac{dx}{dt},$$

$$\frac{dy^2}{dt^2}+y\frac{d^2y}{dt^2}=2m\frac{d^2x}{dt^2}=2mX, \text{ by (1), ...(4)}.$$

The integral of (2) is

$$y = C \sin (\mu^{\frac{1}{2}} t + \epsilon).$$

Let V be the velocity of projection; then

$$V = C\mu^{\frac{1}{2}} \cos \epsilon.$$

Also $y = 0$, initially, and therefore

$$0 = C \sin \epsilon.$$

Hence
$$y = \frac{V}{\mu^{\frac{1}{2}}} \sin (\mu^{\frac{1}{2}} t) \quad\text{.........................(5).}$$

From (5) we easily see that

$$\frac{dy^2}{dt^2} = V^2 - \mu y^2 \quad\text{.........................(6).}$$

From (2), (4), (6),

$$2mX = V^2 - 2\mu y^2,$$

$$X = \frac{V^2}{2m} - \frac{\mu}{m} y^2 = \frac{V^2}{2m} - 4\mu x, \text{ by (3), ...(7),}$$

which determines the required force.

Also, from (1) and (7),

$$\frac{d^2x}{dt^2} = \frac{V^2}{2m} - 4\mu x,$$

$$\frac{d^2}{dt^2}\left(x - \frac{V^2}{8\mu m}\right) + 4\mu\left(x - \frac{V^2}{8\mu m}\right) = 0;$$

the integral of this equation is

$$x - \frac{V^2}{8\mu m} = C \cos (2\mu^{\frac{1}{2}} t + \epsilon):$$

since $x = 0$ and $\frac{dx}{dt} = 0$, initially, this integral is easily reduced to the form

$$x = \frac{V^2}{8\mu m} \operatorname{vers} (2\mu^{\frac{1}{2}} t) \quad\text{..................(8).}$$

From the expressions (5) and (8) for y and x it appears that the particle oscillates continually in a portion of the parabola cut

off by a double ordinate at a distance $\frac{V^2}{4\mu m}$ from the vertex; and that the period of a complete oscillation is $\frac{\pi}{\mu^{\frac{1}{2}}}$.

(4) A particle, attracting with a force which varies directly as the distance, moves uniformly in a given straight line in a given plane; to determine the motion of another particle which is in the given plane, the initial circumstances of the latter particle being given.

Let the initial position of the attracting particle be taken as the origin of co-ordinates: and let x', y', be the co-ordinates of the attracting, and x, y, of the attracted particle at any time t: then the equations of motion will be

$$\frac{d^2x}{dt^2} = \mu^2 (x' - x), \quad \frac{d^2y}{dt^2} = \mu^2 (y' - y),$$

μ^2 denoting the absolute force of attraction.

But, if α, β, denote the resolved parts of the velocity of the attracting particle parallel to the axes of x, y, which are by the hypothesis invariable,

$$x' = \alpha t, \quad y' = \beta t,$$

and therefore

$$\frac{d^2x}{dt^2} = \mu^2 (\alpha t - x) \dots\dots\dots\dots\dots\dots\dots\dots(1),$$

$$\frac{d^2y}{dt^2} = \mu^2 (\beta t - y) \dots\dots\dots\dots\dots\dots\dots\dots(2).$$

From the equation (1),

$$\frac{d^2}{dt^2}(x - \alpha t) + \mu^2 (x - \alpha t) = 0.$$

The integral of this equation is

$$x = A \cos (\mu t) + B \sin (\mu t) + \alpha t \dots\dots\dots\dots\dots(3);$$

where A, B, are arbitrary constants. Differentiating we have

$$\frac{dx}{dt} = -A\mu \sin (\mu t) + B\mu \cos (\mu t) + \alpha \dots\dots\dots\dots(4).$$

Let a, m, be the initial values of x, $\frac{dx}{dt}$; then, from (3) and (4),

$$a = A,\ m = B\mu + \alpha,$$

and therefore, from (3),

$$x = a\cos(\mu t) + \frac{m-\alpha}{\mu}\sin(\mu t) + \alpha t.$$

In precisely the same way, b and n being the initial values of y and $\frac{dy}{dt}$,

$$y = b\cos(\mu t) + \frac{n-\beta}{\mu}\sin(\mu t) + \beta t.$$

(5) A particle is moving in a plane under the action of a force always perpendicular to a line drawn from the particle to a fixed point in the plane: to find the law of the force in order that the angular velocity of the particle about the point may be constant, and to determine the path described.

Let O be the fixed point in the plane, P the position of the particle at any time t, θ the inclination of OP to a fixed line Ox in the plane of the motion, $OP = r$, $G =$ the force. Then, by formulæ proved in systematic treatises on the motion of particles[1],

$$\frac{d^2r}{dt^2} - r\frac{d\theta^2}{dt^2} = 0 \quad \text{.........................} \quad (1),$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = G \quad \text{.........................} \quad (2).$$

Let $\frac{d\theta}{dt} = \omega$; ω being a constant: then

$$\frac{d^2r}{dt^2} = \omega^2 r,$$

$$r = \alpha e^{\omega t} + \beta e^{-\omega t},$$

α and β being constants.

Hence $$G = 2\omega\frac{dr}{dt} = 2\omega^2(\alpha e^{\omega t} - \beta e^{-\omega t}),$$

and therefore

$$r^2 - \frac{G^2}{4\omega^4} = (\alpha e^{\omega t} + \beta e^{-\omega t})^2 - (\alpha e^{\omega t} - \beta e^{-\omega t})^2 = 4\alpha\beta,$$

$$G = 2\omega^2(r^2 - 4\alpha\beta)^{\frac{1}{2}},$$

which gives the law of the force.

[1] Tait and Steele: *Dynamics of a Particle*, 2nd *Edition*, p. 9.

Since $\omega t = \theta$, we have

$$r = \alpha e^{\theta} + \beta e^{-\theta}$$

for the equation to the path.

(6) An imperfectly elastic particle, subject to the action of gravity, is projected from an assigned point in a horizontal plane with a given velocity and in a given direction; to find the velocity of incidence and of reflection, and also the total range with the corresponding time of flight, after the particle has described by rebounding any number of parabolic arcs.

Let e be the elasticity of the particle; u_x the velocity at each end of the x^{th} parabolic arc, and α_x the inclination of the curve at these points to the horizon; t_x the time which elapses before the x^{th} impact; s_x the distance of the point of x^{th} impact from the initial position of the particle.

By the theory of impact we have

$$u_{x+1} \cos \alpha_{x+1} = u_x \cos \alpha_x \dots\dots\dots\dots (1),$$

$$u_{x+1} \sin \alpha_{x+1} = e u_x \sin \alpha_x \dots\dots\dots\dots (2);$$

and, by the properties of the motion of projectiles,

$$\Delta t_x = \frac{2}{g} u_{x+1} \sin \alpha_{x+1} \dots\dots\dots\dots (3),$$

$$\Delta s_x = u_{x+1} \cos \alpha_{x+1} \Delta t_x \dots\dots\dots\dots (4).$$

From (1) it is evident that

$$u_x \cos \alpha_x = u_1 \cos \alpha_1 \dots\dots\dots\dots (5),$$

where u_1 is the given velocity and α_1 the given angle of projection.

Again, from (2), putting $u_x \sin \alpha_x = v_x$, we have

$$\Delta v_x = (e - 1) v_x;$$

and therefore, integrating,

$$v_x = C e^x,$$

where C is an arbitrary constant. But $x = 1$, $v_x = v_1$, simultaneously; hence $Ce = v_1$, and therefore

$$v_x = v_1 e^{x-1}, \qquad u_x \sin \alpha_x = u_1 \sin \alpha_1 e^{x-1} \dots\dots\dots\dots (6).$$

From (5) and (6) we get

$$\tan\alpha_x = \tan\alpha_1 \, . \, e^{x-1}, \qquad u_x = u_1\cos\alpha_1 \, . \, \{1 + \tan^2\alpha_1 \, . \, e^{2(x-1)}\}^{\frac{1}{2}},$$

by which the circumstances of the projection are determined for each of the parabolic paths.

Again, from (3) and (6),

$$\Delta t_x = \frac{2}{g} u_1 \sin\alpha_1 e^x \quad \text{.....................} \quad (7);$$

integrating and adding a constant,

$$t_x = \frac{2u_1 \sin\alpha_1}{g} \frac{e^x}{e-1} + C;$$

but $t_0 = 0$; hence

$$0 = \frac{2u_1 \sin\alpha_1}{g} \frac{1}{e-1} + C,$$

and therefore $$t_x = \frac{2u_1 \sin\alpha_1}{g} \frac{1-e^x}{1-e}.$$

Again, from (4) and (7),

$$\Delta s_x = \frac{2}{g} u_1 \sin\alpha_1 \, . \, e^x \, . \, u_{x+1} \cos\alpha_{x+1},$$

and therefore, by (5),

$$\Delta s_x = \frac{u_1^2 \sin 2\alpha_1}{g} e^x,$$

whence, integrating and observing that $s_0 = 0$, we shall have

$$s_x = \frac{u_1^2 \sin 2\alpha_1}{g} \frac{1-e^x}{1-e}.$$

Bordoni; *Memorie della Societa Italiana*, Tom. XVII. P. I. p. 191; 1816.

(7) A particle is projected obliquely from a point A at an angle α with the horizon, so as to hit a point B, AB being inclined at an angle β to the horizon; and the velocity of projection is such that the particle, moving uniformly with this velocity, would describe the straight line AB in n seconds; to find the time of flight.

$$\text{The required time} = n \frac{\cos\beta}{\cos\alpha}.$$

(8) To find the angle at which a body must be projected from a point in a given inclined plane, in order to impinge upon the plane at right angles; the plane of projection being vertical and at right angles to the inclined plane.

If $\alpha =$ the inclination of the given plane to the horizon, the angle which the direction of projection must make with this plane is equal to

$$\tan^{-1}\left(\frac{\cot\alpha}{2}\right).$$

(9) Two projectiles, thrown from the same point, rise to the same height and pass through a common point: if a, a', be the horizontal ranges of the two projectiles, to find the horizontal distance of the common point from the point of projection.

The required horizontal distance is equal to

$$\frac{aa'}{a+a'}.$$

(10) A particle is projected horizontally from a point A: P and Q are points in its path such that P's horizontal distance from A and Q's vertical distance below P are each equal to a: also, the particle's motion at Q is inclined to the horizon at an angle α: to find Q's horizontal distance from A.

The required distance is equal to $a \cot \dfrac{\alpha}{2}$.

(11) A spherical particle, of which e is the elasticity, is projected with a velocity v at an angle of elevation α, and, at the instant of attaining its greatest altitude, strikes horizontally a similar and equal particle falling downwards with a velocity $\frac{1}{2}v$; to find the distance of the particles from each other at the end of t seconds after the collision.

$$\text{The distance required} = \tfrac{1}{2}vt\,(1 + 4e^2\cos^2\alpha)^{\frac{1}{2}}.$$

(12) If two particles be projected from the same point, at the same instant, with velocities v, v', and at angles of elevation α, α'; to find the time which elapses between their transits through the other point which is common to both their paths.

$$\text{The required time} = \frac{2}{g}\,\frac{vv'\sin(\alpha \sim \alpha')}{v\cos\alpha + v'\cos\alpha'}.$$

(13) If α be the angle between the two tangents at the extremities of any arc of the parabolic path of a particle acted on by gravity; v, v', the velocities at these two points, and v_1 the velocity at the vertex; to find the time through the arc.

$$\text{The required time} = \frac{vv' \sin \alpha}{gv_1}.$$

(14) A cannon being pointed towards the top of a tower, the ball is seen to strike, after t seconds, a point of the tower on the same horizontal line with the cannon: the cannon being reloaded with a different charge, and raised to twice its former angle of elevation, the ball is observed to strike the top of the tower after τ seconds: to find the distance of the tower from the cannon.

The required distance is equal to

$$\tfrac{1}{2} gt^2 \cdot \left(\frac{\tau^2 + t^2}{\tau^2 - t^2}\right)^{\frac{1}{2}}.$$

(15) If a projectile pass through three points (a, b), (a', b'), (a'', b''); to find the equation connecting these co-ordinates, the point of projection being the origin, and the axis of x being horizontal.

The required equation is

$$\frac{\frac{b}{a}}{(a-a')(a-a'')} + \frac{\frac{b'}{a'}}{(a'-a'')(a'-a)} + \frac{\frac{b''}{a''}}{(a''-a)(a''-a')} = 0.$$

(16) The eyes of three observers are in the line in which the plane of a projectile's motion intersects a horizontal plane, the distances of the eyes from a given point of this line being a, $a_{,}$, $a_{,,}$ respectively; the greatest angular elevation of the projectile is, as seen by each observer, respectively, $\tan^{-1}\alpha$, $\tan^{-1}\alpha_{,}$, $\tan^{-1}\alpha_{,,}$; to find the greatest height it attains above the plane.

The greatest height is equal to

$$\alpha\alpha_{,}\alpha_{,,} \cdot \frac{a(\alpha_{,} - \alpha_{,,}) + a_{,}(\alpha_{,,} - \alpha) + a_{,,}(\alpha - \alpha_{,})}{a^2(\alpha_{,} - \alpha_{,,}) + a_{,}^2(\alpha_{,,} - \alpha) + a_{,,}^2(\alpha - \alpha_{,})}.$$

(17) A particle describes an ellipse under the action of a force at right angles to the axis major; to find the force at any point of the path.

Let a, b, be the semiaxes major and minor, y the distance of the particle at any point of its path from the axis major, β the velocity of the particle parallel to the axis major, which will remain invariable during the whole motion. Then

$$\text{the force required} = \frac{b^4\beta^2}{a^2y^3}.$$

If $b = a$, or the ellipse become a circle,

$$\text{the force} = \frac{a^2\beta^2}{y^3}.$$

Riccati; *Comment. Bonon.* Tom. IV. p. 149; 1757.
Newton; *Principia,* Lib. I. sect. 2, prop. 8.

(18) If a particle be acted on by a vertical force so as to describe the common catenary, to determine the force and the velocity at any point.

If β = the horizontal velocity of the particle, then, the equation to the catenary being

$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right),$$

the required force and velocity are respectively equal to $\frac{\beta^2}{c^2} \cdot y$ and $\frac{\beta}{c} \cdot y$.

(19) A particle describes the arc of a cycloid under the action of a force parallel to its base; to find the law of the force.

If the equations to the cycloid be

$$x = a \operatorname{vers} \theta, \qquad y = a(\theta + \sin\theta),$$

and F, β, denote the force required and the velocity parallel to the axis of the cycloid,

$$\frac{1}{F} = \frac{a}{\beta^2} \sin\theta \operatorname{vers}\theta.$$

(20) A particle is projected with a given velocity parallel to a given straight line towards which it is always attracted with a force proportional to its perpendicular distance from it; to determine the position of the particle at any time and the equation to its path.

Let A (fig. 122) be the initial position of the particle; Ox the given straight line; draw yAO at right angles to Ox; let Ox, Oy, be the axes of x, y; P the position of the particle after a time t; let $OM=x$, $PM=y$; $AO=b$, $\beta=$ the velocity of projection; and let μ^2 be the absolute force of attraction. Then

$$x=\beta t,\quad b\cos\frac{\mu x}{\beta}=y=b\cos(\mu t).$$

Riccati; *Comment. Bonon.* Tom. IV. p. 155; 1757.

(21) A particle is projected from a point $x=0$, $y=b$, with a velocity β parallel to the axis of x, and is subject to the action of a force, tending towards the axis of x, parallel to the axis of y, and varying inversely as the square of the distance; to find the equation to the path of the particle.

Let μ denote the attracting force at a unit of distance; then the equation to the path will be

$$\left(\frac{2\mu}{b}\right)^{\frac{1}{2}}\frac{x}{\beta}=\tfrac{1}{2}b\left\{\pi-\operatorname{vers}^{-1}\frac{2y}{b}\right\}+(by-y^2)^{\frac{1}{2}}.$$

Riccati; *Comment. Bonon.* Tom. IV. p. 159; 1757.

(22) A particle is projected from O (fig. 123) with a given velocity in the direction Oy, and is acted on by a central force, which attracts directly as the distance, while the centre of force moves uniformly with a given velocity along Ox at right angles to Oy; to determine the position of the particle when its motion first becomes parallel to Ox.

Let μ^2 denote the absolute force; a the initial distance of the centre of force from O; β the velocity with which the particle is projected, β' the uniform velocity of the centre of force along

Ox, and x', y', the co-ordinates of the required position of the particle; then

$$x' = a + \frac{\beta'}{2\mu}(\pi - 2), \quad y' = \frac{\beta}{\mu}.$$

(23) A particle moves in one plane under the action of a force of constant magnitude, the direction of which has a uniform angular motion: to find the position of the particle at any time.

Let the initial position of the particle be the origin of rectangular co-ordinates, the axis of x coinciding initially with the force's direction: let u, v, be the initial components of the particle's velocity along the axes; let f be the magnitude of the force: then, x, y, being the co-ordinates of the particle at the end of any time t,

$$x = ut + \frac{f}{\omega^2} \operatorname{vers} \omega t, \quad y = vt + \frac{f}{\omega^2}(\omega t - \sin \omega t).$$

Andrew Bell; *Cambridge and Dublin Mathematical Journal*, Vol. I. p. 282.

(24) A particle, attracted towards two centres of force the intensities of which vary directly as the distance, is placed at a given distance from its position of equilibrium: to find its distance from its position of equilibrium at the end of any time.

Let a be the given distance, μ and μ' the two absolute forces: then the required distance at the end of a time t is equal to

$$a \cos \{(\mu + \mu')^{\frac{1}{2}} t\}.$$

Abraham Schnée; *Nouvelles Annales de Mathématiques*, 2^{me} *Série*, Tom. II. p. 451.

(25) A particle, which is placed at rest initially in a given position, is acted on by two forces, one central and repulsive, varying as the distance from the centre, the other constant, acting in parallel lines; to determine the position of the particle at any time and the equation to its path.

Let the centre of the central force be taken as the origin of co-ordinates, and let the directions of the axes be so chosen that the direction of the constant force makes an angle of 45° with each of them. Then, if a, b, be the co-ordinates of the initial position of the particle, μ^2 the absolute force of repulsion, and f the constant force, we shall have, putting $f = 2^{\frac{1}{2}}\mu^2 m$,

$$\frac{x+m}{a+m} = \tfrac{1}{2}\left(\epsilon^{\mu t} + \epsilon^{-\mu t}\right) = \frac{y+m}{b+m}.$$

(26) A particle acted on by two forces, each tending to a fixed point, moves with a constant velocity in a path such that the product of the distances of the particle from the two fixed points is invariable: if one of the forces vary as the distance, to find the law of variation of the other.

At a distance r, one of the forces being μr, the other is $\frac{\mu c^4}{r^3}$, where c^2 denotes the product of the distances of the particle from the two centres.

(27) Four equal particles, attracting directly as the distance, are fixed at the corners of a square; to find the path of a particle projected from the centre of the square in any direction in the plane of the square.

Let the centre of the square be taken as the origin of co-ordinates, and let the axes be at right angles to the two pairs of opposite sides of the square. Then, if $2m$, $2n$, be the resolved parts of its velocity of projection parallel to the axes of x, y, respectively, the path of the free particle will be a portion of the line represented by the equation

$$\frac{x}{m} = \frac{y}{n}.$$

(28) A particle describes a cycloid under the action of a force which, in every position of the particle, is directed towards the centre of the corresponding generating circle: to determine the law of the force and the motion of the centre of force.

The centre of the generating circle moves uniformly and the force is constant.

Mackenzie and Walton; *Solutions of the Cambridge Problems* for 1854.

(29) A particle describes an ellipse under the simultaneous action of given central forces, tending towards the two foci and varying inversely as the square of the distance: to find the differential relation between the time and the eccentric anomaly.

If a denote the semi-axis major, μ, μ', the absolute forces, ϕ the eccentric anomaly, and t the time,

$$a^3 \frac{d\phi^2}{dt^2} = \frac{\mu}{(1 - e\cos\phi)^2} + \frac{\mu'}{(1 + e\cos\phi)^2}.$$

Cayley; *Messenger of Mathematics*, Vol. v. p. 194.

(30) A particle, subject to the action of gravity, is projected with a velocity of given magnitude in a given plane from a given point in it: to find the locus of the path of the particle.

The required locus is the surface of an elliptic paraboloid, the axis of which is vertical, and the point of projection is the umbilicus of the surface.

(31) A heavy particle, having been projected at a given angle to the inclined plane AB (fig. 124), proceeds to ascend this plane by bounding in a series of parabolic arcs; to determine the angles of incidence and reflection after any number of impacts.

Let ι be the inclination of the plane AB to the horizon; α_x the angle of reflection in the x^{th} arc, β_{x-1} the angle of incidence in the $(x-1)^{\text{th}}$; and e the elasticity of the particle. Then

$$\tan\alpha_x = \frac{(1-e)\,e^{x-1}\tan\alpha_1}{1 - e - 2(1 - e^{x-1})\tan\iota\tan\alpha_1} = e\tan\beta_{x-1}.$$

Bordoni; *Memorie della Societa Italiana*, Tom. XVII. P. I. p. 191; 1816.

(32) A ball, of which the elasticity is e, is projected with a velocity V in a direction making an angle $\alpha + \iota$ with the

horizon, and rebounds from a plane inclined to the horizon at an angle ι and passing through the point of projection. To determine the relation between R_x, R_{x+1}, R_{x+2}, three consecutive ranges upon the inclined plane after x, $x+1$, $x+2$, rebounds respectively, and to find the distance, from the point of projection, of the point on the plane at which hopping ceases.

If $\cot\beta = (1-e)\cot\iota$,

$$R_{x+2} - (e+e^2)\,R_{x+1} + e^3 R_x = 0,$$

and the required distance is equal to

$$\frac{2V^2 \sin\beta \sin\alpha \cos(\alpha+\beta)}{g \sin\iota \,.\, \cos^2\beta}.$$

SECT. 2. *Central Forces.*

Let the force which acts on a particle tend always towards a fixed centre, which we will take as the origin of co-ordinates. Let F denote the force at any distance from the centre; x, y, the co-ordinates of the particle at the end of a time t reckoned from an assigned epoch, r its distance from the centre of force, and θ the inclination of this distance to any fixed line in the plane of x, y. Then, by Maclaurin's Equations, the plane of co-ordinates being identical with the plane of the motion,

$$\frac{d^2x}{dt^2} = -\frac{Fx}{r}, \qquad \frac{d^2y}{dt^2} = -\frac{Fy}{r}.$$

From these equations may be obtained the following formulæ:

$$r^2 d\theta = h\,dt \quad\text{.................................}\quad \text{(I)},$$

$$v = \frac{h}{p} \quad\text{.................................}\quad \text{(II)},$$

$$v^2 = h^2\left\{\left(\frac{d}{d\theta}\frac{1}{r}\right)^2 + \frac{1}{r^2}\right\} \quad\text{....................}\quad \text{(III)},$$

$$v^2 = v'^2 - 2\int_{r'}^{r} F\,dr \quad\text{......................}\quad \text{(IV)},$$

$$F = \frac{h^2}{r^2}\left\{\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right\} \quad \text{(V)},$$

$$F = \frac{h^2}{p^3}\frac{dp}{dr} \quad \text{(VI)},$$

$$F = r\frac{d\theta^2}{dt^2} - \frac{d^2r}{dt^2} \quad \text{(VII)}.$$

In these formulæ h represents twice the area swept out by the radius vector about the centre of force in a unit of time, p the perpendicular from the centre upon the tangent at any point of the orbit, v the velocity of the particle, and v', r', any simultaneous values of v, r. If the central force, instead of being attractive as we have been supposing, be repulsive, we must replace F in these formulæ by $-F$.

The equation (I) shews that the area swept out by the radius vector varies as the time, and either of the equations (II) and (III) that the velocity at any point of the orbit varies inversely as the perpendicular from the centre of force upon the tangent to the orbit at that point: these two propositions were first established by Newton[1]. The equation (IV) shews that the velocity of the particle at any point of its path depends only upon the distance of the point from the centre, the velocity of projection, and the prime radius vector, whatever be the course which it may have pursued; the discovery of this proposition is likewise due to Newton[2]. The formula (V), by which the path of the particle may be determined when we know the law of the central force and conversely, Ampère[3] ascribes to Binet. The formula (VI) was communicated without demonstration to John Bernoulli by De Moivre in the year 1705; a proof of the formula was returned to him by Bernoulli in a letter dated Basle, Feb. 1706. The formula (VII) was given much about the same time by Clairaut[4] and by Euler[5], and signifies that the

[1] *Principia*, Lib. I. Prop. 1. [2] *Ib.* Lib. I. Prop. 40.

[3] *Annales de Gergonne*, Tom. XX. p. 53.

[4] *Théorie de la Lune*, p. 2; the first edition of which appeared in 1752, from a MS. sent to St Petersburg in 1750.

[5] *Nov. Comment. Petrop.* 1752, 1753, p. 164.

acceleration of the radius vector is equal to the excess of the centrifugal above the attractive force.

(1) To find the law of the force by which a particle may be made to describe the Lemniscata of James Bernoulli, the centre of force coinciding with the node, and to investigate the time of describing one of the ovals.

The polar equation to the Lemniscata is

$$r^2 = a^2 \cos 2\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

hence $$\frac{1}{r} = \frac{1}{a(\cos 2\theta)^{\frac{1}{2}}}, \quad \frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{\sin 2\theta}{a(\cos 2\theta)^{\frac{3}{2}}},$$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{2}{a(\cos 2\theta)^{\frac{1}{2}}} + \frac{3(\sin 2\theta)^2}{a(\cos 2\theta)^{\frac{5}{2}}} = \frac{3}{a(\cos 2\theta)^{\frac{5}{2}}} - \frac{1}{a(\cos 2\theta)^{\frac{1}{2}}},$$

and therefore

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{3}{a(\cos 2\theta)^{\frac{5}{2}}} = \frac{3a^4}{r^5}.$$

Hence, by the formula (V),

$$F = \frac{3a^4h^2}{r^7}.$$

Again, by the formula (I) and the equation (1), we have

$$hdt = r^2 d\theta = a^2 \cos 2\theta \, d\theta,$$

and therefore, if P denote the required periodic time,

$$P = \frac{a^2}{h}\int_{-\frac{1}{4}\pi}^{+\frac{1}{4}\pi} \cos 2\theta \, d\theta = \frac{a^2}{h}.$$

Let μ denote the value of F when $r = 1$; then we have

$$\mu = 3a^4h^2, \quad h = \frac{\mu^{\frac{1}{2}}}{3^{\frac{1}{2}}a^2}, \quad P = \frac{3^{\frac{1}{2}}a^4}{\mu^{\frac{1}{2}}}.$$

(2) A particle moves in an equiangular spiral under the action of a force tending towards the pole; to find the law of the force and the velocity at any point of the orbit.

If β be the invariable angle, r the radius vector, and p the perpendicular from the pole upon the tangent,

$$p = r \sin \beta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Differentiating with respect to r, we have $\frac{dp}{dr} = \sin\beta$, and therefore, from (VI),

$$F = \frac{h^2}{p^3}\sin\beta = \frac{h^2}{r^3\sin^2\beta}, \text{ by (1)}, \quad\ldots\ldots\ldots\ldots\ldots(2).$$

Let c be the velocity corresponding to a given radius vector r'; then, by (II) and (1),

$$h = cr'\sin\beta.$$

Hence, from (2), $$F = \frac{c^2r'^2}{r^3},$$

and, from (II) and (1),

$$v = \frac{cr'\sin\beta}{r\sin\beta} = \frac{cr'}{r}.$$

(3) **A particle describes an equilateral hyperbola round a centre of force at the centre**; to find the law of the force and the angle which the particle will describe about the centre in a given time after leaving the apse.

The equation to the hyperbola being

$$\left(\frac{1}{r}\right)^2 = \frac{\cos 2\theta}{a^2}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

we have $$\frac{1}{r}\frac{d}{d\theta}\left(\frac{1}{r}\right) = -\frac{\sin 2\theta}{a^2}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$\frac{1}{r}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \left\{\frac{d}{d\theta}\left(\frac{1}{r}\right)\right\}^2 = -\frac{2\cos 2\theta}{a^2};$$

and therefore, from (2),

$$\frac{1}{r}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{r^2}{a^4}\sin^2 2\theta = -\frac{2\cos 2\theta}{a^2},$$

and thence, by (1),

$$\frac{1}{r}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{r^2}{a^4} - \frac{1}{r^2} = -\frac{2}{r^2},$$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{r^3}{a^4}.$$

Hence, by the formula (V),

$$F = -\frac{h^2 r}{a^4}.$$

The negative sign shews that the force must be repulsive; let $-\mu$ be the absolute force, that is, the value of F at a unit of distance. Then

$$\mu = \frac{h^2}{a^4}, \quad F = -\mu r.$$

Putting for h its value $a^2\mu^{\frac{1}{2}}$ in the formula (I), and $\frac{a^2}{\cos 2\theta}$ for r^2, we get

$$\frac{d\theta}{\cos 2\theta} = \mu^{\frac{1}{2}} dt, \quad \frac{\cos 2\theta\, d\theta}{1 - \sin^2 2\theta} = \mu^{\frac{1}{2}} t, \quad \frac{d \sin 2\theta}{1 - \sin^2 2\theta} = 2\mu^{\frac{1}{2}} dt:$$

integrating, and supposing the time to be reckoned from apsidal passage, we have

$$\tfrac{1}{2} \log \frac{1 + \sin 2\theta}{1 - \sin 2\theta} = 2\mu^{\frac{1}{2}} t,$$

whence, writing μ' in place of $4\mu^{\frac{1}{2}}$, we obtain

$$\sin 2\theta = \frac{e^{\mu' t} - 1}{e^{\mu' t} + 1}.$$

(4) A particle is revolving in a parabola about a centre of force at the focus, and, when it arrives at a given distance from the focus, the absolute force is suddenly doubled; to determine the nature of the subsequent path of the particle.

Let $4m$ be the latus rectum of the parabola, r the radius vector at any point, and p the perpendicular from the focus upon the tangent. Then, by the nature of the parabola,

$$\frac{1}{p^2} = \frac{1}{mr}, \quad \frac{2}{p^3} \frac{dp}{dr} = \frac{1}{mr^2},$$

and therefore, by (VI),

$$F = \frac{h^2}{2mr^2}.$$

But, after the absolute force has been doubled, we shall have for the motion

$$F = \frac{h^2}{mr^2},$$

and therefore, by (VI),

$$\frac{h^2}{mr^2} = \frac{h^2}{p^3}\frac{dp}{dr}, \quad \frac{1}{mr^2} = \frac{1}{p^3}\frac{dp}{dr}.$$

Integrating, we have

$$\frac{1}{mr} = C + \frac{1}{2p^2}.$$

Let c be the value of r at the instant when the absolute force is doubled; then, p being then common both to the parabola and to the new path, we have

$$\frac{1}{mc} = C + \frac{1}{2mc}, \quad C = \frac{1}{2mc},$$

and therefore, for the equation to the new path, there is

$$\frac{1}{mr} = \frac{1}{2mc} + \frac{1}{2p^2}, \quad \frac{mc}{p^2} = \frac{2c}{r} - 1,$$

which is the equation to an ellipse, $2c$ being the major and $2(mc)^{\frac{1}{2}}$ the minor axis.

Since the ellipse touches the parabola when $r = c$, the semi-axis major, it follows from the nature of the ellipse that the point of contact is an extremity of the semi-axis minor, and therefore that the axis major of the ellipse is parallel to the tangent at the point $r = c$ of the parabola. But the sine of the angle of inclination of the tangent of the parabola at this point to its axis is $\frac{p}{r} = \frac{m^{\frac{1}{2}}}{r^{\frac{1}{2}}}$, when $r = c$, that is, $= \frac{m^{\frac{1}{2}}}{c^{\frac{1}{2}}}$, and therefore the inclination of the major axis of the ellipse to the axis of the parabola is $\sin^{-1}\left(\frac{m^{\frac{1}{2}}}{c^{\frac{1}{2}}}\right)$.

(5) A particle is describing a curve about a certain centre of force, the velocity of the particle varying inversely as the n^{th} power of its distance from the centre of force; to find the law of the force and the equation to the path.

We shall have, μ denoting some constant quantity,

$$v = \frac{\mu}{r^n}.$$

Hence, from (IV), there is

$$\frac{\mu^2}{r^{2n}} = v'^2 - 2\int_{r'}^{r} F dr,$$

and therefore, differentiating,

$$F = \frac{n\mu^2}{r^{2n+1}},$$

which determines the law of the force.

Again, from (III), there is

$$\frac{\mu^2}{r^{2n}} = h^2 \left\{ \left(\frac{d}{d\theta} \frac{1}{r} \right)^2 + \frac{1}{r^2} \right\},$$

$$\left(\frac{\mu^2 - h^2 r^{2n-2}}{r^{2n}} \right)^{\frac{1}{2}} = h \frac{d}{d\theta} \frac{1}{r},$$

$$(n-1)\, d\theta = - \frac{dr^{n-1}}{\left(\frac{\mu^2}{h^2} - r^{2n-2} \right)^{\frac{1}{2}}},$$

$$(n-1)\, \theta = C + \cos^{-1} \frac{hr^{n-1}}{\mu}.$$

Suppose $\theta = 0$, when $r = a$; then, k denoting a constant quantity,

$$(n-1)\, \theta = \cos^{-1}(kr^{n-1}) - \cos^{-1}(ka^{n-1}).$$

Riccati; *Comment. Bonon.* Tom. IV. p. 184.

(6) If the force vary as the n^{th} power of the distance, and a particle be projected from an apsidal distance, with a velocity of which the square is equal to $1 - \epsilon$ times the square of the velocity in a circle, described about the same centre of force, the radius of the circle being equal to the apsidal distance; to find the equation to the orbit, ϵ being a very small quantity.

Let a be the apsidal distance; then $r = a - x$, where x is a small quantity, because the path of the particle, as is evident from the initial circumstances, will be nearly circular. Then, approximately,

$$\frac{1}{r} = \frac{1}{a - x} = \frac{1}{a}\left(1 + \frac{x}{a}\right), \quad \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{1}{a^2}\frac{d^2 x}{d\theta^2}.$$

Also, μ denoting the absolute force of attraction,

$$\frac{Fr^2}{h^2} = \frac{\mu r^{n+2}}{h^2} = \frac{\mu}{h^2}(a-x)^{n+2} = \frac{\mu a^{n+2}}{h^2}\left\{1 - \frac{(n+2)\,x}{a}\right\}.$$

Hence, by the formula (V),

$$\frac{1}{a^2}\frac{d^2x}{d\theta^2} + \frac{x}{a^2} + \frac{1}{a} - \frac{\mu a^{n+2}}{h^2}\left\{1 - \frac{(n+2)\,x}{a}\right\} = 0,$$

$$\frac{d^2x}{d\theta^2} + \left\{1 + (n+2)\frac{\mu a^{n+3}}{h^2}\right\}x + a - \frac{\mu a^{n+4}}{h^2} = 0 \;\ldots\ldots\ldots\ldots(1).$$

Let V be the velocity of projection, and v the velocity in a circle, described about the same centre of force, the radius of the circle being a; then

$$V^2 = (1-\epsilon)\,v^2 = (1-\epsilon)\,\mu a^{n+1} \;\ldots\ldots\ldots\ldots\ldots(2).$$

But, by the formula (II),

$$h^2 = a^2 V^2,$$

because the motion is initially at right angles to the radius vector, and a, V, are the initial values of the radius vector and of the velocity. Hence, from (2),

$$h^2 = \mu(1-\epsilon)\,a^{n+3},\quad \frac{\mu}{h^2} = \frac{1}{(1-\epsilon)\,a^{n+3}},$$

and therefore, from (1), the product of ϵ and x being neglected,

$$\frac{d^2x}{d\theta^2} + (n+3)\,x - \epsilon a = 0,$$

or
$$\frac{d^2}{d\theta^2}\left(x - \frac{\epsilon a}{n+3}\right) + (n+3)\left(x - \frac{\epsilon a}{n+3}\right) = 0.$$

The integral of this equation is evidently

$$x - \frac{\epsilon a}{n+3} = A \sin\{(n+3)^{\frac{1}{2}}\theta + \beta\},$$

where A and B are constant quantities.

Let $\theta = 0$ when $x = 0$: then

$$A \sin\beta = -\frac{\epsilon a}{n+3}.$$

Again, by the hypothesis, $\frac{dr}{d\theta} = 0$ when $r = a$, and therefore $\frac{dx}{d\theta} = 0$ when $x = 0$: hence $\cos\beta = 0$.

The integral therefore becomes

$$x - \frac{\epsilon a}{n+3} = -\frac{\epsilon a}{n+3} \cos\{(n+3)^{\frac{1}{2}}\theta\},$$

and therefore the polar equation to the orbit is

$$r = a - \frac{\epsilon a}{n+3} \operatorname{vers}\{(n+3)^{\frac{1}{2}}\theta\} \dots\dots\dots\dots\dots\dots (3).$$

Differentiating the equation (3), we get for the determination of apses,

$$\frac{dr}{d\theta} = -\frac{\epsilon a}{(n+3)^{\frac{1}{2}}} \sin\{(n+3)^{\frac{1}{2}}\theta\} = 0;$$

whence $$(n+3)^{\frac{1}{2}}\theta = \lambda\pi,$$

where λ is any integer: let θ', θ'', be the values of θ for two consecutive apsidal distances; then

$$(n+3)^{\frac{1}{2}}\theta' = \lambda\pi, \quad (n+3)^{\frac{1}{2}}\theta'' = (\lambda+1)\pi,$$

and therefore, ϕ denoting the angle between any two consecutive apses,

$$\phi = \theta'' - \theta' = \frac{\pi}{(n+3)^{\frac{1}{2}}}.$$

(7) A particle is attached to one end of a very fine and slightly extensible string, of which the other end is fixed and which rests at its natural length in a straight line on a smooth horizontal plane: if the particle be projected at right angles to the string, to determine the extension of the string at any time and the path of the particle.

Let a, c, denote the initial length of the string and the initial velocity of the particle. Let $r = a + x$ represent the length of the string at the end of any time t, the corresponding angular co-ordinate being θ, and the initial position of the string the prime radius vector. Let m denote the mass of the particle, and p the

tension necessary to stretch the string to a length $a+\beta$. Then, the extension being, by Hooke's law, proportional to the tension, we have for the tension at the time t the expression $\frac{px}{\beta}$. Hence, by the formula (VII), we have

$$\frac{d^2r}{dt^2}=r\frac{d\theta^2}{dt^2}-\frac{px}{m\beta},$$

and therefore, by (I),

$$\frac{d^2r}{dt^2}=\frac{h^2}{r^3}-\frac{px}{m\beta}.$$

But, from the circumstances of the projection of the particle, it is clear by the formula (II) that $h=ac$; hence

$$\frac{d^2r}{dt^2}=\frac{a^2c^2}{r^3}-\frac{px}{m\beta},$$

$$\frac{d^2x}{dt^2}=\frac{a^2c^2}{(a+x)^3}-\frac{px}{m\beta}.$$

Multiplying by $2\frac{dx}{dt}$ and integrating,

$$\frac{dx^2}{dt^2}=C-\frac{a^2c^2}{(a+x)^2}-\frac{px^2}{m\beta}:$$

but, initially, $x=0$, $\frac{dx}{dt}=0$; hence $C=c^2$, and therefore

$$\begin{aligned}\frac{dx^2}{dt^2}&=c^2-\frac{a^2c^2}{(a+x)^2}-\frac{px^2}{m\beta}\\&=c^2-c^2\left(1-\frac{2x}{a}\right)-\frac{px^2}{m\beta}\\&=\frac{2c^2x}{a}-\frac{px^2}{m\beta}\quad\text{......................... (1)},\end{aligned}$$

where small quantities of the first order only are retained, $\frac{x}{\beta}\cdot x$ being of the first order because x and β are of the same order. Extracting the root, we have

$$\frac{dt}{dx}=\left(\frac{m\beta}{p}\right)^{\frac{1}{2}}\left(2\frac{m\beta c^2}{ap}x-x^2\right)^{-\frac{1}{2}};$$

whence, integrating and bearing in mind that $x=0$ when $t=0$,

$$t=\left(\frac{m\beta}{p}\right)^{\frac{1}{2}}\text{vers}^{-1}\left(\frac{pax}{mc^2\beta}\right),$$

or
$$x=\frac{mc^2\beta}{pa}\text{vers}\left\{\left(\frac{p}{m\beta}\right)^{\frac{1}{2}}t\right\},$$

which gives the extension of the string at any time during the motion.

We proceed now to the determination of the equation to the path of the particle. From (I) and (1) there is

$$\frac{dx^2}{d\theta^2}=\frac{r^4}{h^2}\left(\frac{2c^2x}{a}-\frac{px^2}{m\beta}\right)$$

$$=\frac{(a+x)^4}{a^2c^2}\left(\frac{2c^2x}{a}-\frac{px^2}{m\beta}\right)$$

$$=2ax-\frac{pa^2x^2}{mc^2\beta}, \text{ approximately,}$$

$$=\frac{pa^2}{mc^2\beta}\left(2\frac{mc^2\beta}{pa}x-x^2\right),$$

$$\frac{d\theta}{dx}=\left(\frac{mc^2\beta}{pa^2}\right)^{\frac{1}{2}}\left(2\frac{mc^2\beta}{pa}x-x^2\right)^{-\frac{1}{2}}$$

Integrating, and remembering that $\theta=0$, $x=0$, simultaneously,

$$\theta=\left(\frac{mc^2\beta}{pa^2}\right)^{\frac{1}{2}}\text{vers}^{-1}\frac{pax}{mc^2\beta},$$

$$r=a+x=a+\frac{mc^2\beta}{pa}\text{vers}\left\{\left(\frac{pa^2}{mc^2\beta}\right)^{\frac{1}{2}}\theta\right\} \quad \text{............ (2)},$$

which is the polar equation to the orbit.

From this equation we have, for the determination of apses,

$$\frac{dx}{d\theta}\propto\sin\left\{\left(\frac{pa^2}{mc^2\beta}\right)^{\frac{1}{2}}\theta\right\}=0:$$

hence the angle between consecutive apses is

$$\frac{\pi c}{a}\left(\frac{m\beta}{p}\right)^{\frac{1}{2}}.$$

The force varies jointly as the distance of the particle from the centre of force and the reciprocal of the cube of its distance from the tangent to the ellipse at the centre of force.

(12) A particle revolves in a circle about a centre of force at a point in its circumference; to determine the force and the velocity at any point of the path.

If μ denote the absolute force,

$$F=\frac{\mu}{r^5}, \quad v^2=\frac{\mu}{2r^4}.$$

Newton; *Princip.* Lib. I. Prop. 7. Riccati; *Comment. Bonon.* Tom. IV. p. 175.

(13) A particle is moving about a centre of force, its velocity at each point of its path varying inversely as its distance from the centre of force; to determine the path of the particle.

The path will be a logarithmic spiral.

Riccati; *Ib.* p. 184.

(14) A body, attracted towards a centre of force which varies inversely as the square of the distance, is projected with a velocity equal to the velocity in a circle at an equal distance, and in a direction making an angle of 45° with the radius vector: to find the magnitude and position of the orbit described.

The orbit will be an ellipse, the point of projection being an extremity of the minor axis and the centre of force a focus. If the prime radius vector be a, the axis major $= 2a$ and the axis minor $= a\sqrt{2}$.

(15) If the force vary inversely as the seventh power of the distance, and a particle be projected from an apse with a velocity which is to the velocity in a circle at the same distance as 1 to $\sqrt{3}$; to find the equation to the curve described.

If the apsidal distance be taken as the prime radius vector, and be denoted by a, the equation to the curve described will be

$$r^2 = a^2 \cos 2\theta.$$

(16) A particle, projected in a given direction with a given velocity, is attracted towards a centre of force: the velocity of the particle, at every distance from the centre, is to that of a particle describing a circle, of which the distance is a radius, about the same centre of force, as 1 to $\sqrt{2}$: to find the orbit described and the law of the force.

Let S (fig. 125) be the centre of force, P the point of projection; let $SP=a$, $\beta=$ the angle between PS and the direction PT of projection.

Draw PA at right angles to SP and equal to $a\cot\beta$: join SA. The orbit described is a circle, of which $SA \doteq a \operatorname{cosec}\beta$ is the diameter, and the law of force is that of the inverse fifth power.

(17) A particle moves in an equilateral hyperbola about a centre of force at the centre; to find the locus of the point to which the particle must move from the curve under the action of the force to acquire the velocity in the curve.

If a be the semi-axis of the equilateral hyperbola in which the particle is moving, the required locus will also be an equilateral hyperbola having a semi-axis equal to $2^{\frac{1}{2}}.a$, the centres of the two hyperbolas being coincident and their axes being in the same straight lines.

(18) One end of an indefinitely fine elastic string, extended to its natural length, is fixed, and to the other is attached a particle of matter; the particle is projected at right angles to the string with an angular velocity such that, if it were revolving in a circle with this angular velocity, the length of the string must have been stretched to twice its natural length; to find the apsidal distances of the particle.

If a be the natural length of the string, there are two apsidal distances, viz. a, and the real root of the equation

$$2r^3 - 2ar^2 - a^2r - a^3 = 0.$$

(19) In a curve, described by a body about a centre of force, the angle between the radius vector and the tangent varies as the time: to find the curve and the law of the force.

If F = the force, and β, h, ω, be certain constants, then

$$F = \beta^2 . h . e^{-\frac{\omega r^2}{h}} . \frac{\omega r^2 + h}{r^3},$$

and the differential equation to the path is

$$\theta = \int \frac{dr}{r\,(\beta^2 . e^{-\frac{\omega r^2}{h}} - 1)^{\frac{1}{2}}}.$$

(20) A particle, attracted towards a point by a force equal to $\frac{r}{m^2} + \frac{h^2}{r^3}$, is projected from an apse at the distance $m^{\frac{1}{2}} h^{\frac{1}{2}}$, h being twice the area described in a unit of time; to find the polar equation to the orbit, and the time of describing any angle about the centre of force.

If θ be the angle described about the centre of force in any time t after the projection,

$$r^2 = \frac{mh}{1 + \theta^2}, \quad t = m \tan^{-1} \theta.$$

(21) At a distance a from a centre of force, a particle is projected at an angle of 45° to the distance, and with a velocity, which is to that in a circle described about the centre of force at the same distance, as $2^{\frac{1}{2}}$ to $3^{\frac{1}{2}}$: the central force at any distance r is equal to $\frac{2a^2 n}{r^5} + \frac{n}{r^3}$: to find the equation to the orbit.

If the angular co-ordinate be estimated from the initial radius vector, the equation to the orbit will be

$$r = a\,(1 - \theta).$$

(22) A particle is projected at a distance a from a centre of force, at right angles to the distance: the force is repulsive and of constant intensity: the initial velocity is that which would be acquired in moving from the centre of force to the point of projection under the influence of the force: determine the orbit.

If r be the distance of the particle at any time from the centre of force and θ the inclination of r to the initial distance a,

$$\left(\frac{a}{r}\right)^3 = \left(\cos\frac{3\theta}{2}\right)^2.$$

(23) A particle, acted on by a central force varying as any function of the distance, is projected at an apse with a velocity nearly equal to that requisite for a circular orbit; to find the distance, from the centre of force, of the apse at which the particle next arrives.

Let the force at any distance r be equal to $\frac{1}{r^2}\,\phi\left(\frac{1}{r}\right)$, where $\phi\left(\frac{1}{r}\right)$ is any function of $\frac{1}{r}$; let a be the initial distance of the particle from the centre of force, a' the distance of the apse at which it next arrives, and let the velocity of projection be to the velocity in a circle about the same centre as 1 to $1+m$; then

$$a' = a\left\{1 - \frac{4m\phi\left(\frac{1}{a}\right)}{\phi\left(\frac{1}{a}\right) - \frac{1}{a}\phi'\left(\frac{1}{a}\right)}\right\}.$$

(24) To find the whole action of a planet, regarded as a particle of given mass, in a complete revolution about the Sun.

If μ be the absolute force of the Sun's attraction, and a the mean distance of the planet, the required action is equal to

$$2\pi\,(\mu a)^{\frac{1}{2}}.$$

(25) If the mass of a planet were distributed over its orbit, so that the part of the mass distributed over any portion of the orbit should be proportional to the time which the planet occupies in describing that portion; to find the centre of gravity of the whole mass so distributed.

The centre of gravity would be at the point midway between the centre of the orbit and the focus which is not occupied by the Sun.

(26) A particle describes a circular orbit about a centre of force at the centre of the circle; to find the condition that the form of the orbit may be stable or unstable.

If $P =$ the central force, $u =$ the reciprocal of the radius vector, and $\frac{1}{a} =$ the radius of the circle, the form of the orbit will be stable or unstable according as

$$\frac{d \log P}{d \log u}, \quad \text{when } u = a,$$

is less or not less than 3.

SECT. 3. *Tangential and Normal Resolutions.*

The method of the solution of the general problem of the curvilinear motion of a particle in one plane, by the principle of the tangential and normal resolutions, is expressed by the equations

$$v \frac{dv}{ds} = T \text{..............................(A)},$$

$$\frac{v^2}{\rho} = N \text{..............................(B)},$$

where v denotes the velocity at any point of the path, ds an element of the curve, ρ the radius of curvature, T the sum of the resolved parts of the forces which act on the particle estimated in the direction of its motion, and N the sum of the resolved parts along the normal on the concave side of the curve in the neighbourhood of the particle.

(1) A particle is projected with a given velocity and in a given direction, and is acted upon by a constant force in parallel lines; to determine the path of the particle.

Let the axis of x be taken so as to pass through the initial place of the particle, and let the axis of y be taken parallel to the constant force which acts towards the axis of x. Let f

denote the constant force. Then, the tangential resolved part being $-f\dfrac{dy}{ds}$, and the normal one being $f\dfrac{dx}{ds}$, we have for the motion of the particle,

$$v\frac{dv}{ds} = -f\frac{dy}{ds} \qquad\qquad (1),$$

$$\frac{v^2}{\rho} = f\frac{dx}{ds} \qquad\qquad (2).$$

Integrating (1), $\qquad v^2 = C - 2fy.$

Let V be the initial velocity; then, y being zero initially, $V^2 = C$, and therefore

$$v^2 = V^2 - 2fy.$$

Hence, substituting this expression for v^2 in (2),

$$\frac{1}{\rho}(V^2 - 2fy) = f\frac{dx}{ds};$$

but
$$\rho = -\frac{\dfrac{ds^3}{dx^3}}{\dfrac{d^2y}{dx^2}};$$

hence
$$-\frac{d^2y}{dx^2}(V^2 - 2fy) = f\frac{ds^2}{dx^2} = f\left(1 + \frac{dy^2}{dx^2}\right),$$

$$(V^2 - 2fy)\frac{d}{dx}\left(1 + \frac{dy^2}{dx^2}\right) - \left(1 + \frac{dy^2}{dx^2}\right)\frac{d}{dx}(V^2 - 2fy) = 0.$$

Integrating, we have $\qquad C\left(1 + \dfrac{dy^2}{dx^2}\right) = V^2 - 2fy,$

where C is an arbitrary constant. Let β be the angle which the direction of projection makes with the axis of x; then

$$C(1 + \tan^2\beta) = V^2:$$

hence
$$V^2\left(1 + \frac{dy^2}{dx^2}\right) = \sec^2\beta\,(V^2 - 2fy),$$

$$V^2\frac{dy^2}{dx^2} = V^2\tan^2\beta - 2fy\sec^2\beta,$$

$$Vdy = (V^2\tan^2\beta - 2fy\sec^2\beta)^{\frac{1}{2}}\,dx;$$

whence, by integration,

$$C - V(V^2 \tan^2 \beta - 2fy \sec^2 \beta)^{\frac{1}{2}} = fx \sec^2 \beta.$$

But $x = 0$, $y = 0$, simultaneously; hence

$$C - V^2 \tan \beta = 0,$$

and therefore

$$V^2 \tan \beta - V(V^2 \tan^2 \beta - 2fy \sec^2 \beta)^{\frac{1}{2}} = fx \sec^2 \beta.$$

Clearing the equation of radicals, and simplifying;

$$y = \tan \beta \,.\, x - \frac{f \sec^2 \beta}{2V^2} x^2.$$

Euler; *Mechan.* Tom. I. p. 232.

(2) A particle, always acted on by a force in parallel lines, describes a given curve; to determine the nature of the force, the velocity and direction of projection being given.

Let the force be represented by Y, which we will suppose to act towards the axis of x parallel to the axis of y. The equations of motion will be

$$v\frac{dv}{ds} = -Y\frac{dy}{ds} \text{ .... (1)},$$

$$\frac{v^2}{\rho} = Y\frac{dx}{ds} \text{ .... (2)}.$$

Eliminating Y, we have

$$\frac{1}{v}\frac{dv}{ds} = -\frac{1}{\rho}\frac{dy}{dx} = \frac{\frac{dy}{dx}\frac{d^2y}{dx^2}}{\frac{ds^3}{dx^3}},$$

$$\frac{1}{v}\frac{dv}{dx} = \frac{\frac{dy}{dx}\frac{d^2y}{dx^2}}{\frac{ds^2}{dx^2}} = \frac{\frac{dy}{dx}\frac{d^2y}{dx^2}}{1 + \frac{dy^2}{dx^2}}.$$

Integrating, we get

$$\log v = C + \tfrac{1}{2}\log\left(1 + \frac{dy^2}{dx^2}\right) = C + \log\frac{ds}{dx}.$$

Let V denote the initial velocity, and β the angle which the direction of projection makes with the axis of x; then

$$\log V = C + \log \sec \beta,$$

and therefore
$$\log \frac{v}{V} = \log \frac{\frac{ds}{dx}}{\sec \beta},$$

$$v = V \cos \beta \frac{ds}{dx}.$$

Substituting this value of v in (2), we have

$$Y = \frac{V^2 \cos^2 \beta}{\rho} \frac{ds^3}{dx^3}.$$

Euler; *Mechan.* Tom. I. p. 240.

(3) A particle describes a given curve about a centre of force; to determine the motion of the particle and the law of the force.

Let ABP (fig. 126) be the path of the particle, S the centre of force; P the position of the particle at any time; PT a tangent at the point P, and SY perpendicular to PT. Let F denote the force along PS, and ϕ the angle SPT; then, the direction of the motion at P being towards B,

$$v \frac{dv}{ds} = -F \cos \phi \quad \text{.......................(1)},$$

$$\frac{v^2}{\rho} = F \sin \phi \quad \text{.......................(2)}.$$

Now, since
$$ds \cos \phi = dr,$$

and
$$\rho \sin \phi = \frac{rdr}{dp} \sin \phi = p \frac{dr}{dp},$$

where p denotes SY, we have, by (1) and (2),

$$vdv = -Fdr \quad \text{..........................(3)},$$

$$v^2 = Fp \frac{dr}{dp} \quad \text{.......................(4)}.$$

Eliminating F between (3) and (4),

$$\frac{dv}{v} = -\frac{dp}{p}, \quad \log v = C - \log p.$$

Let V, P, be the initial values of v, p; then

$$\log V = C - \log P,$$

and therefore
$$\log \frac{v}{V} = \log \frac{P}{p}, \quad v = \frac{V.P}{p} \quad\text{.................}\quad (5).$$

Again, if t denote the time of the motion,

$$\frac{ds}{dt} = v = \frac{V.P}{p}, \text{ by (5)},$$

$$pds = VPdt;$$

but pds is equal to dh', where h' represents twice the area swept out by the radius vector in its motion from some assigned position; hence

$$dh' = VPdt, \quad h' = VPt \quad\text{....................}\quad (6),$$

the area being supposed to commence with the time.

Again, by (2), we have

$$F = \frac{v^2}{\rho \sin \phi} = \frac{v^2}{p}\frac{dp}{dr} = \frac{V^2P^2}{p^3}\frac{dp}{dr}, \text{ by (5)}, \quad\text{..........}\quad (7).$$

Suppose now that h represents twice the area swept out in a unit of time; then, since, by (6), h is equal to VP, we have, by (6), (5), (7),

$$h' = ht \quad\text{..............................}\quad (8),$$

$$v = \frac{h}{p} \quad\text{..............................}\quad (9),$$

$$F = \frac{h^2}{p^3}\frac{dp}{dr} = \frac{h^2 r}{\rho p^3} \quad\text{.......................}\quad (10).$$

The formulæ (8) and (9) were given by Newton[1]. The formula (10) was discovered by De Moivre in the year 1705, by whom it was communicated without demonstration to John Bernoulli. A proof of the formula was obtained by Bernoulli and forwarded

[1] *Principia*, Lib. I. Prop. 1.

to De Moivre in a letter dated Basle, Feb. 16, 1706. Demonstrations were afterwards given by Keill[1], and by Hermann[2]. See De Moivre's *Miscell. Analyt.* Lib. VIII., and John Bernoulli, *Opera*, Tom. I. p. 477.

Integrating the equation (3) we get another expression for the velocity,

$$v^2 = V^2 - 2\int_R^r F dr,$$

where R denotes the prime radius vector. This result shews that the velocity of the particle depends only upon its distance from the centre of force, and not upon the path described; a theorem proved by Newton[3].

Euler; *Mechan.* Tom. I. p. 240.

(4) Bodies are projected with the same velocity and from the same point but in different directions, and describe curves about a centre of force: to find the locus of the centres of the circles of curvature to the different orbits, at the point of projection.

The locus is a straight line cutting at right angles the distance between the point of projection and the centre of force.

SECT. 4. *Motion in Resisting Media.*

(1) A particle acted on by gravity is projected in a uniform medium, of which the resistance varies as the velocity, with a given velocity and at a given angle of inclination to the horizon; to find after what interval of time the particle will arrive at its greatest altitude.

Let k be the resistance for a unit of velocity, u the velocity and α the angle of projection, and let y be the height through which the particle has ascended at the end of the time t. Then

[1] *Phil. Trans. Num.* 317; 1708.
[2] *Phoronomia*, p. 70.
[3] *Princip.* Lib. I. Prop. 40.

$$\frac{d^2y}{dt^2} = -g - k\frac{ds}{dt}\frac{dy}{ds} = -g - k\frac{dy}{dt},$$

$$\log\left(g + k\frac{dy}{dt}\right) = C - kt:$$

but $\frac{dy}{dt} = u \sin \alpha$ when $t = 0$; hence

$$\log(g + ku \sin \alpha) = C,$$

and therefore

$$\log \frac{g + ku \sin \alpha}{g + k\frac{dy}{dt}} = kt.$$

When y is a maximum, $\frac{dy}{dt} = 0$, and therefore the required value of t will be equal to

$$\frac{1}{k}\log(1 + \frac{k}{g} u \sin \alpha).$$

(2) A particle, moving in a resisting medium, is acted on by a given force the direction of which is always parallel to a fixed line; to find the resistance in order that any proposed curve may be described, and conversely.

Let the position of the particle be referred to two rectangular axes Ox, Oy, (fig. 127) the latter of which is parallel to the fixed line; let $OM = x$, $PM = y$, $AP = s$; P being the position of the particle at any time, and APB the curve of its motion; also let Y denote the accelerating force at P, v the velocity of the particle, and R the resistance of the medium.

Then, by the equations of tangential and normal resolution given in the preceding section, we have

$$v\frac{dv}{ds} = -R + Y\frac{dy}{ds} \quad\text{..................... (1)},$$

$$\frac{v^2}{\rho} = Y\frac{dx}{ds} \quad\text{.......................... (2)};$$

where ρ denotes the radius of curvature at P. But

$$\rho = \frac{\frac{ds^3}{dx^3}}{\frac{d^2y}{dx^2}};$$

hence, from (2),

$$v^2 = \frac{\dfrac{ds^2}{dx^2}}{\dfrac{d^2y}{dx^2}} Y;$$

differentiating, we have, since $\dfrac{ds^2}{dx^2} = 1 + \dfrac{dy^2}{dx^2}$,

$$v\frac{dv}{dx} = Y\frac{dy}{dx} + \frac{1}{2}\frac{ds^2}{dx^2}\frac{d}{dx}\left\{\frac{Y}{\dfrac{d^2y}{dx^2}}\right\},$$

$$v\frac{dv}{ds} = Y\frac{dy}{ds} + \frac{1}{2}\frac{ds}{dx}\frac{d}{dx}\left\{\frac{Y}{\dfrac{d^2y}{dx^2}}\right\};$$

hence, from (1), $$R = -\frac{1}{2}\frac{ds}{dx}\frac{d}{dx}\left\{\frac{Y}{\dfrac{d^2y}{dx^2}}\right\},$$

which gives the resistance if the curve be given, and conversely.

The solution of this problem, which Newton had given in the first edition of the *Principia*, involved certain errors, which at the suggestion of John Bernoulli were afterwards corrected.

COR. If the resistance vary as the square of the velocity for a uniform density; then, Q denoting the density generally, we have

$$R = Qv^2 = Q\rho\frac{dx}{ds}Y, \text{ by (2)},$$

and therefore $$Q = \frac{R}{Y}\frac{\dfrac{d^2y}{dx^2}}{\dfrac{ds^2}{dx^2}}$$

$$= -\frac{1}{2}\frac{dx}{ds}\frac{\dfrac{d^2y}{dx^2}}{Y}\frac{d}{dx}\left\{\frac{Y}{\dfrac{d^2y}{dx^2}}\right\}$$

$$= -\frac{1}{2}\frac{dx}{ds}\frac{d}{dx}\log\left\{\frac{Y}{\dfrac{d^2y}{dx^2}}\right\},$$

which gives the density at any point within the medium; or, if the density be given, determines the curve.

COR. It is evident that $\rho \dfrac{dx}{ds}$ is equal to half the chord of curvature at P parallel to Oy, or in the direction of the force Y; let q denote this chord of curvature. Then

$$v^2 = 2Y \cdot \tfrac{1}{4}q;$$

and therefore the space due to the velocity, supposing the force to continue constant, is equal to one-fourth of the chord of curvature.

Newton; *Princip.* Lib. II. Prop. 10. John Bernoulli; *Act. Erudit. Lips.* 1713; *Opera,* Tom. I. p. 514.

(3) A particle moves in a resisting medium under the action of a given force tending towards a fixed centre; to determine the law of resistance when the path of the particle is given, and conversely.

Let APB (fig. 128) be the path of the particle, S the centre of force; let $AP = s$, $SP = r$, $p =$ the perpendicular from S upon the tangent at P, $v =$ the velocity at P; let ρ be the radius of curvature, P the central force, and R the resistance of the medium.

Then, by the equations of tangential and normal resolution, we have

$$v \frac{dv}{ds} = -R - P \frac{dr}{ds} \dots\dots\dots\dots (1),$$

$$\frac{v^2}{\rho} = \frac{p}{r} P \dots\dots\dots\dots (2).$$

Since ρ is equal to $r \dfrac{dr}{dp}$, we have, from (2),

$$v^2 = p \frac{dr}{dp} P \dots\dots\dots\dots (3);$$

and therefore, differentiating with respect to s,

$$v\frac{dv}{ds}=\frac{1}{2}\frac{d}{ds}\left(p\frac{dr}{dp}P\right)$$
$$=\frac{1}{2}\frac{d}{ds}\left\{\left(p^3\frac{dr}{dp}P\right)\frac{1}{p^2}\right\}$$
$$=-\frac{1}{p^3}\frac{dp}{ds}\left(p^3\frac{dr}{dp}P\right)+\frac{1}{2p^2}\frac{d}{ds}\left(p^3\frac{dr}{dp}P\right)$$
$$=-\frac{dr}{ds}P+\frac{1}{2p^2}\frac{d}{ds}\left(p^3\frac{dr}{dp}P\right).$$

Hence, by substituting this value of $v\dfrac{dv}{ds}$ in (1), we have

$$R=-\frac{1}{2p^2}\frac{d}{ds}\left(p^3\frac{dr}{dp}P\right)\text{....................}(4);$$

which determines the law of resistance when the curve is known, and conversely.

COR. Supposing the resistance to vary as the density of the medium multiplied by the square of the velocity of the particle, we have, Q denoting the density,

$$R=Qv^2=Qp\frac{dr}{dp}P,\text{ by (3)},$$

and therefore, by (4),

$$Q=-\frac{1}{2}\frac{\dfrac{d}{ds}\left(p^3\dfrac{dr}{dp}P\right)}{p^3\dfrac{dr}{dp}P}=-\frac{1}{2}\frac{d}{ds}\log\left(p^3\frac{dr}{dp}P\right)\text{..............}(5),$$

which determines the law of the density when the curve is given, and conversely.

COR. From the equation (2) we have

$$v^2=\frac{p}{r}\rho P=\tfrac{1}{2}qP=2\left(\tfrac{1}{4}q\right)P,$$

where q denotes the chord of curvature through S. This result shews that the velocity at any point of the curve is that due to falling in vacuum towards the centre of force, continued constant, through a quarter of the chord of curvature.

COR. From (4) we have

$$p^3 \frac{dr}{dp} P = h^2 \epsilon^{-2\int Q ds}, \quad P = \frac{h^2 \dfrac{dp}{dr}}{p^3} \epsilon^{-2\int Q ds},$$

where h is some constant quantity. This formula gives the central force when the law of the density and the orbit are given. It is easily shewn that, if $\angle ASP = \theta$,

$$\frac{h^2}{p^3}\frac{dp}{dr} = \frac{h^2}{r^2}\left\{\frac{1}{r} + \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)\right\},$$

and therefore $$P = \frac{h^2}{r^2}\left\{\frac{1}{r} + \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)\right\} \epsilon^{-2\int Q ds}.$$

If $Q = 0$, we get $$P = \frac{h^2}{r^2}\left\{\frac{1}{r} + \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)\right\},$$

which is Binet's formula for the central force in vacuum.

Newton; *Principia*, Lib. II. Prop. 17, 18. John Bernoulli; *Opera*, Tom. IV. p. 347. Euler; *Mechan.* Tom. I. p. 428 et sq., p. 451 et sq.

(4) A particle is projected with a given velocity in a uniform medium, in which the resistance varies as the square of the velocity; the particle is acted on by gravity, and the direction of its projection makes a very small angle with the horizon; to determine approximately the equation to the portion of the path which lies above the horizontal plane passing through the point of projection.

Let Ox and Oy (fig. 129), horizontal and vertical respectively, be the axes of x and y, O being the point of projection; P the position of the particle at any time; let $OM = x$, $PM = y$, $v =$ the velocity at P, $s =$ the arc OP, $k =$ the resistance for a unit of velocity; then, by the tangential and normal resolutions,

$$v\frac{dv}{ds} = -kv^2 - g\frac{dy}{ds} \quad\text{......................(1)},$$

$$v^2 = g\frac{dx}{ds}\rho = -g\frac{1+p^2}{q} \quad\text{.................. (2)},$$

where $p = \frac{dy}{dx}$ and $q = \frac{dp}{dx}$. Hence, eliminating v between these two equations, we have

$$\frac{d}{ds}\frac{1+p^2}{q} + 2k\frac{1+p^2}{q} - 2\frac{dy}{ds} = 0,$$

$$\frac{d}{dx}\frac{1+p^2}{q} + 2k\frac{1+p^2}{q}\frac{ds}{dx} - 2p = 0,$$

$$\frac{d}{dx}\log\frac{1+p^2}{q} + 2k\frac{ds}{dx} - \frac{2pq}{1+p^2} = 0.$$

Integrating, we get

$$\log\frac{1+p^2}{q} + 2ks + \log C - \log(1+p^2) = 0,$$

C being some constant quantity; whence

$$\log\frac{C}{q} + 2ks = 0, \quad q = Ce^{2ks} \dots\dots\dots\dots\dots\dots (3).$$

Let u be the velocity, and α the angle of projection; then, initially, by (3) and (2),

$$q = C, \quad u^2 = -g\frac{1+\tan^2\alpha}{q},$$

and therefore $$C = -\frac{g}{u^2\cos^2\alpha}:$$

hence, by (3), $$q = -\frac{g}{u^2\cos^2\alpha}e^{2ks}:$$

but, the angle of projection being small, we may neglect all powers of p beyond the first, and therefore

$$s = \int_0^x (1+p^2)^{\frac{1}{2}}\,dx = \int_0^x dx = x \text{ nearly}:$$

hence $$q = -\frac{g}{u^2\cos^2\alpha}e^{2kx} \text{ nearly.}$$

Multiplying by dx, and integrating,

$$p = C - \frac{g}{2ku^2\cos^2\alpha}e^{2kx};$$

but $p = \tan\alpha$ when $x = 0$; hence

$$\tan\alpha = C - \frac{g}{2ku^2\cos^2\alpha},$$

and therefore $$p = \tan\alpha + \frac{g}{2ku^2\cos^2\alpha}(1 - e^{2kx}).$$

Integrating again,

$$y = C + x\tan\alpha + \frac{g}{2ku^2\cos^2\alpha}\left(x - \frac{1}{2k}e^{2kx}\right):$$

but $x = 0$, $y = 0$, simultaneously; hence

$$0 = C - \frac{g}{4k^2u^2\cos^2\alpha},$$

and therefore $$y = x\tan\alpha + \frac{g}{4k^2u^2\cos^2\alpha}(1 + 2kx - e^{2kx}).$$

Moreau; *Journal de l'Ecole Polytech.* Cahier XI. p. 215.

The general problem of the path of a projectile in a uniform resisting medium, where the resistance varies as the square of the velocity, was proposed by Keill as a trial of skill to John Bernoulli, by whom the challenge was received in February 1718. Keill, trusting to the complexity of the analysis, which had probably deterred Newton from attempting any regular solution of the problem in the second book of the *Principia*, was in hopes that the exertions of Bernoulli would prove unsuccessful. Bernoulli, however, having expeditiously effected a solution, not only of Keill's problem, but likewise of the more general one where the resistance varies as the n^{th} power of the velocity, expressed a determination not to publish his investigation until he had received intimation that his antagonist had himself been able to solve his own problem. He gave Keill till the following September to exercise his talents, declaring that if he received by that time no satisfactory communication, he should feel himself entitled to question the ability of his adversary. At the request of a common friend, Bernoulli consented to extend the interval to the first of November. It turned out, however, that Keill was unable to obtain a solution. At length Nicholas

Bernoulli, Professor of Mathematics at Padua, communicated to John Bernoulli a solution of Keill's problem, which the author afterwards extended to the more general one. Finally, on the 17th of November, information was received by John Bernoulli from Brook Taylor, to the effect that he had obtained a solution. John Bernoulli published his own analysis, together with that of his nephew Nicholas, in the *Acta Erudit. Lips.* 1719 mai., p. 216; see also his *Opera*, Tom. II. p. 393. For further information on this celebrated problem, the reader may avail himself of the labours of Euler[1], Borda[2], Legendre[3], Templehoff[4], and Moreau[5].

(5) A particle acted on by gravity moves in a semicircle in a medium where the resistance varies as the density and the square of the velocity; to find the law of the resistance and density.

Let OA, OB (fig. 130), be horizontal and vertical radii of the semicircle, OAx and OBy being the axes of x and y; let a = the radius of the circle, $OM = x$, and gravity $= g$. Then

$$R = \frac{3}{2}\frac{gx}{a}, \quad v^2 = g\,(a^2 - x^2)^{\frac{1}{2}}, \quad Q = \frac{3x}{2a}\,(a^2 - x^2)^{-\frac{1}{2}}.$$

Newton; *Princip.* Lib. II. Prop. 10, Ex. 1. John Bernoulli; *Act. Erudit. Lips.* 1713. Euler; *Mechan.* Tom. II. p. 392.

(6) A particle acted on by gravity moves in a parabola of any order; to find the law of resistance.

Let the equation to the parabola ABC (fig. 130) be $y = b - cx^n$, Oy being vertical; then

$$R = \frac{(n-2)\,(1 + n^2c^2x^{2n-2})^{\frac{1}{2}}\,g}{2n\,(n-1)\,cx^{n-1}}.$$

(7) A particle moves in an hyperbola of any order, one of the asymptotes of which is vertical; to find the law of the

[1] *Mém. de l'Acad. de Berlin*, 1753. [2] *Ib.* 1769.
[3] *Ib.* 1782. [4] *Ib.* 1788—89.
[5] *Journal de l'Ecole Polytech.* Cahier XI. p. 204.

density, the resistance varying as the product of the density and the square of the velocity.

Let APB (fig. 131) denote the path of the particle, Oy the vertical asymptote being taken as the axis of y, and Ox, which is horizontal, as the axis of x; then, if the equation to the hyperbola be

$$y = ax + \frac{\beta^{n+1}}{x^n},$$

we shall have
$$Q = \frac{\frac{1}{2}(n+2)\,x^n}{\{x^{2n+2} + (ax^{n+1} - n\beta^{n+1})^2\}^{\frac{1}{2}}}.$$

Euler; *Mechan.* Tom. II. p. 400.

(8) A particle moves in a circle about a centre of force in the circumference, the force being attractive and varying as any power of the distance; to determine the resistance of the medium and the law of the density, supposing the resistance to be equal to the product of the density and the square of the velocity.

Let P (fig. 132) be the position of the particle at any time, its motion taking place towards S the centre of force; let C be the centre of the circle; let $SP = r$, $\angle PSA = \theta$, the central force $= \mu r^n$; then

$$R = \frac{1}{4}\mu\,(5+n)\,r^n \sin\theta, \quad Q = \frac{1}{2}(n+5)\,\frac{\sin\theta}{r}.$$

(9) A particle moves in an equiangular spiral about a centre of force at the pole, the force varying as any power of the distance from the pole; to find the law of the resistance and of the density of the medium, the resistance being considered equal to the product of the density and the square of the velocity.

If α be the constant angle, r the radius vector at any point, μr^n the attractive central force, and the particle be so moving as to approach the centre of force;

$$R = \frac{1}{2}\mu\,(n+3)\,r^n \cos\alpha, \quad Q = \frac{1}{2}(n+3)\,\frac{\cos\alpha}{r}.$$

Newton; *Princip.* Lib. II. Prop. 15, 16. John Bernoulli; *Opera*, Tom. IV. p. 350.

(10) A particle moves in the circumference of a circle about a centre of force at the centre; the resistance of the medium in which the motion takes place is constant; to determine the law of the force, the velocity at any time of the motion, and the time which elapses, as well as the space which is described, before the particle is reduced to rest.

Let β be the initial velocity of the particle, a the radius of the circle, c the constant value of the resistance, s the arc described from the beginning of the motion, P the central force; then

$$v^2 = \beta^2 - 2cs, \quad P = \frac{1}{a}(\beta^2 - 2cs):$$

when the particle is reduced to rest,

$$s = \frac{\beta^2}{2c}, \quad t = \frac{\beta}{c}.$$

(11) A particle is moving along the curve of an equiangular spiral about a centre of force at the pole, so as to be approaching the pole; the motion takes place in a medium the resistance of which varies as any power of the distance from the pole; to find the law of the force.

Let α be the constant angle, β the initial velocity, a the initial distance, kr^n the resistance at a distance r, P the required force; then

$$P = \frac{(n+3)\,a^2\beta^2 \cos\alpha + 2k\,(r^{n+3} - a^{n+3})}{(n+3)\,r^3 \cos\alpha}.$$

Euler; *Mechan.* Tom. I. p. 442.

(12) A particle is projected with a velocity u, and at an inclination α to the horizon, in a uniform medium the resistance of which varies as the velocity; to determine the time which elapses before the direction of the motion is inclined to the horizon at an angle β.

If k represent the resistance for a unit of velocity, t will be found from the equation

$$g \cos\beta\,(\epsilon^{kt} - 1) = ku \sin(\alpha - \beta).$$

(13) Two particles, subject to the action of gravity, are simultaneously projected at equal angles of inclination to the horizon, and with equal velocities, the one in vacuum and the other in a medium the resistance of which varies as the velocity; to determine the relation between the times in which the particles describe two arcs so related to each other that the tangents at their extremities shall make equal angles with the horizon.

If k denote the resistance of the medium for a unit of velocity, and t_1, t_2 denote corresponding times in vacuum and in the medium; then

$$e^{kt_2} = 1 + kt_1.$$

(14) Having given the co-ordinates of the highest point of the path described by a particle acted on by gravity and projected in vacuum at a known angle of inclination to the horizon; to find the decrements of these co-ordinates when the particle is projected in a rare medium in which the resistance varies as the velocity.

Let a, b, be the given co-ordinates, k the resistance for a unit of velocity, and β the angle of projection; then

$$\delta a = -k\left(\frac{a^3 \tan\beta}{g}\right)^{\frac{1}{2}}, \quad \delta b = -k\left(\frac{8b^3}{9g}\right)^{\frac{1}{2}}.$$

SECT. 5. *Hodographs.*

If from any fixed point O a straight line OP be drawn, representing at any instant in magnitude and direction the velocity of a particle which is moving in any manner whatever, the locus of P is called the ***hodograph*** of the particle. The beautiful theory of the hodograph is due to Sir William Rowan Hamilton, by whom it was communicated to the Royal Irish Academy[1], December the 14th, 1846. See Hamilton's *Lectures on Quaternions*, 1853; and his *Elements of Quaternions*, 1866.

[1] *Proceedings of the Royal Irish Academy*, 1846—7, No. 58, p. 344. The perusal of Hamilton's article would be of great benefit to students interested in this branch of Dynamics.

(1) A particle describes a path in one plane under the action of a force the direction of which is perpendicular to a given straight line in the plane: to determine the nature of the hodograph.

Let v be the velocity of the particle at any time, β the component of v parallel to the given line, ϕ the inclination of the tangent of the particle's path to the given line. Then $v \cos \phi = \beta$, which, since β is a constant quantity, is the equation to the hodograph, which is consequently a straight line at right angles to the given straight line.

(2) A particle describes an equiangular spiral about a centre of force at the pole: to find the form of the hodograph.

The hodograph is also an equiangular spiral.

(3) Two particles are describing free paths in one plane, which are hodographs to one another; if the particles be always at corresponding points, to determine the forms of their paths and the nature of the forces acting on the particles.

The paths are conic sections: the forces are centric, varying as the distance from a common centre.

(4) A particle moves in a semicircle, under the action of gravity, in a medium where the resistance varies as the density and the square of the velocity; to find the hodograph.

If a be the radius of the semicircle, the polar equation to the hodograph, the prime radius vector being horizontal, is $r^2 = ag \cos \theta$.

(5) To prove that the whole accelerating force, which acts on a particle at any instant, is represented, both in direction and magnitude, by the element of the hodograph divided by the element of the time.

Hamilton: *Proceedings of the Royal Irish Academy*, 1846—7, No. 58, p. 345.

(6) To prove that the force is to the velocity, in any varied motion of a particle, as the element of the hodograph is to the corresponding element of the orbit.

Hamilton: *Ib.* p. 345.

(7) If a particle be describing an orbit about a fixed centre of force, to prove that half the chord of curvature of the hodograph (passing through or tending towards the fixed centre of force) is to the radius vector of the orbit as the element of the hodograph is to the element of the orbit.

Hamilton: *Ib.* p. 346.

(8) Under the circumstances of the preceding theorem, to prove that the radius of curvature of the hodograph is to the radius vector of the orbit, called the *vector of position*, as the rectangle under the same radius vector and the force is to the parallelogram under the vectors of position and velocity.

Hamilton: *Ib.* p. 347.

(9) If a particle describe an orbit about a fixed centre of force, which varies inversely as the square of the distance, to prove that the hodograph is a circle.

Hamilton: *Ib.* p. 347.

(10) If the hodograph of a particle, describing an orbit about a fixed centre of force, be a circle, to prove that the force must vary inversely as the square of the distance.

Hamilton: *Ib.* p. 347.

(11) If two circular hodographs, having a common chord, which passes through or tends towards a common centre of force, be cut perpendicularly by a third circle, to prove that the times of hodographically describing the intercepted arcs will be equal.

Hamilton: *Ib.* No. 63, p. 417.

CHAPTER IV.

CONSTRAINED MOTION OF A PARTICLE.

SECT. 1. *Constrained motion of particles without friction.*

LET a particle, under the action of any force in one plane, move within an indefinitely thin curvilinear tube APB (fig. 133). Let x, y, be the co-ordinates of the place P of the particle, after a time t from the commencement of the motion; and let $AP = s$, where A is some assigned point in the tube. Let X, Y, represent the resolved parts of the accelerating force acting on the particle parallel to the axes Ox, Oy, and S the resolved part along the tangent to the curve APB at P. Then the equation for the motion of the particle will be

$$\frac{d^2s}{dt^2} = X\frac{dx}{ds} + Y\frac{dy}{ds} = S \text{..................} (A),$$

or, by integration, v denoting the velocity of the particle at the point P,

$$v^2 \text{ or } \frac{ds^2}{dt^2} = 2\int(Xdx + Ydy) + C = 2\int Sds + C \text{......} (B),$$

where C is an arbitrary constant, introduced by the integration, which may be determined if we know the initial velocity and the initial position of the particle.

If the force acting on the particle be a central force; then, P representing its intensity at a distance r, we have, taking the centre of force as the origin of x, y,

$$Xdx + Ydy = Pdr,$$

and the formulæ (A), (B), become

$$\frac{d^2s}{dt^2} = P\frac{dr}{ds} \text{..........................} (C),$$

$$v^2 \text{ or } \frac{ds^2}{dt^2} = 2\int Pdr + C \text{....................} (D).$$

(1) A particle, acted on by gravity, descends from rest down a given circular arc, the tangent to which at the lowest point is horizontal; to compare the initial accelerating force estimated along the curve with that which would correspond to motion down the chord, when the arc is indefinitely diminished.

Let A (fig. 134) be the lowest point of the arc, P the initial position of the particle, T the intersection of the tangent at P with the tangent at A. Let F = the accelerating force at P down the arc PA, f = that down the chord PA, $\angle PAT = \theta$ $= \angle APT$, g = the force of gravity.

Then $$F = g \sin 2\theta, \quad f = g \sin \theta,$$

and therefore $$\frac{F}{f} = \frac{\sin 2\theta}{\sin \theta} = 2 \cos \theta,$$

and therefore, when the arc is indefinitely diminished, $F = 2f$.

> Saurin; *Mémoires de l'Académie des Sciences de Paris*, 1724, p. 70. Liouville; *Ib.* p. 128. Courtivron, *Ib.* 1744, p. 384.

(2) The highest point of the circumference of a circle in a vertical plane is connected by means of a chord with some other point in the curve; to determine the time in which a particle, under the action of gravity, will fall down this chord.

Let AB (fig. 135) be the diameter through the highest point A of the circle; AC the chord down which the particle descends. Join BC, and let P be the position of the particle after a time t from the commencement of its motion. Let $AP = s$, $AB = 2a$, $\angle BAC = \alpha$, $AC = l$. Then, the resolved part of g, the force of gravity, along AC, being $g \cos \alpha$, we have, for the motion of the particle,

$$\frac{d^2s}{dt^2} = g \cos \alpha.$$

Integrating, we get

$$\frac{ds}{dt} = gt \cos \alpha + C;$$

but $\frac{ds}{dt}=0$, $t=0$, simultaneously; hence $C=0$; and therefore

$$\frac{ds}{dt}=gt\cos\alpha.$$

Integrating again, and observing that $s=0$ when $t=0$, we have

$$s=\tfrac{1}{2}gt^2\cos\alpha.$$

Let T denote the whole time of descent down AC; then

$$l=\tfrac{1}{2}gT^2\cos\alpha;$$

but, from the geometry, $\quad l=2a\cos\alpha;$

hence $\quad 2a\cos\alpha=\tfrac{1}{2}gT^2\cos\alpha,$

and therefore $\quad T=2\left(\frac{a}{g}\right)^{\frac{1}{2}}.$

This result, being independent of the inclination of the chord to the vertical, shews that the descents down all such chords are performed in the same time; a proposition established by Galileo.

Wolff; *Elementa Matheseos Universæ*, Tom. II. p. 58.

(3) From one extremity of the horizontal diameter of a circle in a vertical plane, two chords are drawn subtending angles α, 2α, at the centre; given that the time down the latter chord is n times as great as that down the former, to find the value of α.

Let l, l', be the lengths of the two chords, and t, t', their times of description. Then, as in the preceding problem, it will be found that

$$l=\tfrac{1}{2}gt^2\cos\frac{\alpha}{2},\qquad l'=\tfrac{1}{2}gt'^2\cos\alpha,$$

and therefore $\quad \dfrac{l'}{l}=\dfrac{t'^2}{t^2}\,\dfrac{\cos\alpha}{\cos\frac{\alpha}{2}}=n^2\,\dfrac{\cos\alpha}{\cos\frac{\alpha}{2}}.$

But, r denoting the radius, it is evident from the geometry that

$$l=2r\sin\frac{\alpha}{2},\qquad l'=2r\sin\alpha:$$

hence
$$\frac{\sin \alpha}{\sin \frac{\alpha}{2}} = n^2 \frac{\cos \alpha}{\cos \frac{\alpha}{2}},$$

$$n^2 \cos \alpha = 2 \cos^2 \frac{\alpha}{2} = 1 + \cos \alpha,$$

and therefore $\cos \alpha = \frac{1}{n^2 - 1}, \quad \alpha = \cos^{-1} \frac{1}{n^2 - 1}.$

(4) A particle is placed anywhere within a thin rectilinear tube, and is acted on by a force tending towards a fixed centre, and varying directly as the distance; to find the time of an oscillation.

Let x be the distance of the particle, at any time t, from its position of equilibrium, r its corresponding distance from the centre of force, μ^2 the absolute force of attraction. Then, $\mu^2 r$ being the central force at the time t,

$$\frac{d^2x}{dt^2} = -\mu^2 r \frac{x}{r} = -\mu^2 x:$$

the integral of this equation is

$$x = A \cos (\mu t + \epsilon) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where A and ϵ are arbitrary constants.

Let a be the initial value of x: then, from (1),

$$a = A \cos \epsilon:$$

also, $\frac{dx}{dt}$ being initially zero, we have, from (1),

$$0 = A \sin \epsilon:$$

hence, substituting for $A \cos \epsilon$ and $A \sin \epsilon$ their values, the equation (1) is reduced to

$$x = a \cos \mu t.$$

Now, when x acquires its greatest negative value, $\mu t = \pi$; hence, T denoting the period of a complete oscillation, we have

$$T = \frac{\pi}{\mu}.$$

Euler; *Mechan.* Tom. II. p. 91, Cor. 4.

(5) A particle is constrained to move in a straight line, and is attached to one end of an indefinitely fine elastic string, the other end of which is fixed at a distance from the straight line equal to the unstretched length of the string; to find the time of a small oscillation.

Let a = the natural length of the string, m = the mass of the particle; s = its distance at any time t from its position of equilibrium, T = the tension of the string, and l = its length at the same time. Then, for the motion of the particle,

$$m\frac{d^2s}{dt^2} = -\frac{Ts}{l} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Again, by Hooke's law of extension,

$$l = a\left(1 + \frac{T}{\epsilon}\right) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where ϵ is a constant quantity depending upon the extensibility of the string.

But, s being a small quantity,

$$l = (a^2 + s^2)^{\frac{1}{2}} = a\left(1 + \frac{1}{2}\frac{s^2}{a^2}\right), \text{ nearly};$$

hence, from (2),

$$a\left(1 + \frac{1}{2}\frac{s^2}{a^2}\right) = a\left(1 + \frac{T}{\epsilon}\right), \quad T = \frac{\epsilon s^2}{2a^2};$$

and therefore, by (1),

$$m\frac{d^2s}{dt^2} = -\frac{\epsilon s^3}{2a^2 l} = -\frac{\epsilon s^3}{2a^3}, \text{ nearly};$$

multiplying by $2\dfrac{ds}{dt}$, and integrating, we get

$$m\frac{ds^2}{dt^2} = C - \frac{\epsilon s^4}{4a^3};$$

let c be the value of s when $\dfrac{ds}{dt} = 0$; then

$$0 = C - \frac{\epsilon c^4}{4a^3},$$

and therefore

$$m\frac{ds^2}{dt^2}=\frac{e}{4a^2}(c^4-s^4);$$

hence, supposing s to be diminishing as t increases,

$$2a\left(\frac{am}{e}\right)^{\frac{1}{2}}\frac{ds}{(c^4-s^4)^{\frac{1}{2}}}=-dt:$$

put $s=c\cos\phi$, and our equation becomes

$$\frac{2a}{c}\left(\frac{am}{e}\right)^{\frac{1}{2}}\frac{d\phi}{(1+\cos^2\phi)^{\frac{1}{2}}}=dt,$$

and therefore, 0, π, being the values of ϕ corresponding to the values c, $-c$, of s, the time of a complete oscillation will be equal to

$$\frac{2^{\frac{1}{2}}a}{c}\left(\frac{am}{e}\right)^{\frac{1}{2}}\int_0^{\pi}\frac{d\phi}{(1-\frac{1}{2}\sin^2\phi)^{\frac{1}{2}}},$$

an elliptic function of which the modulus is $\sin\frac{1}{4}\pi$.

(6) A particle moves from rest from a distance a along a thin spiral tube towards a centre of force at the pole attracting inversely as the square of the distance; to find the whole time which will elapse before the particle will arrive at the centre of force, the equation to the spiral being

$$\log\frac{c}{r}=\frac{\theta}{\alpha}.$$

Let μ denote the absolute force; then, by (D), we have, for the velocity at any time,

$$v^2=C-2\int\frac{\mu}{r^2}dr=C+\frac{2\mu}{r},$$

but $v=0$ when $r=a$; hence

$$v^2=2\mu\left(\frac{1}{r}-\frac{1}{a}\right),$$

or

$$\frac{dr^2}{dt^2}+r^2\frac{d\theta^2}{dt^2}=2\mu\left(\frac{1}{r}-\frac{1}{a}\right).$$

But, from the equation to the spiral,

$$d\theta = -a\frac{dr}{r};$$

hence $$\frac{dr^2}{dt^2}(1+a^2) = 2\mu\left(\frac{1}{r}-\frac{1}{a}\right);$$

extracting the square root, and taking the negative sign because r decreases with the increase of t, we get

$$(1+a^2)^{\frac{1}{2}}\frac{dr}{\left(\frac{1}{r}-\frac{1}{a}\right)^{\frac{1}{2}}} = -2^{\frac{1}{2}}\mu^{\frac{1}{2}}dt,$$

whence $$(1+a^2)^{\frac{1}{2}}\int\frac{dr}{\left(\frac{1}{r}-\frac{1}{a}\right)^{\frac{1}{2}}} = C-2^{\frac{1}{2}}\mu^{\frac{1}{2}}t.$$

But $$\int\frac{dr}{\left(\frac{1}{r}-\frac{1}{a}\right)^{\frac{1}{2}}} = a^{\frac{1}{2}}\int\frac{rdr}{(ar-r^2)^{\frac{1}{2}}}$$

$$= -a^{\frac{1}{2}}\int\frac{(\frac{1}{2}a-r)\,dr}{(ar-r^2)^{\frac{1}{2}}} + \tfrac{1}{2}a^{\frac{3}{2}}\int\frac{dr}{(ar-r^2)^{\frac{1}{2}}}$$

$$= -a^{\frac{1}{2}}(ar-r^2)^{\frac{1}{2}} + \tfrac{1}{2}a^{\frac{3}{2}}\operatorname{vers}^{-1}\frac{2r}{a}.$$

Hence we have

$$a^{\frac{1}{2}}(1+a^2)^{\frac{1}{2}}\left\{\tfrac{1}{2}a\operatorname{vers}^{-1}\frac{2r}{a}-(ar-r^2)^{\frac{1}{2}}\right\} = C-2^{\frac{1}{2}}\mu^{\frac{1}{2}}t.$$

Now $t=0$, $r=a$, simultaneously; hence

$$\tfrac{1}{2}\pi a^{\frac{3}{2}}(1+a^2)^{\frac{1}{2}} = C;$$

also, T denoting the whole time of the approach to the pole,

$$0 = C-2^{\frac{1}{2}}\mu^{\frac{1}{2}}T;$$

hence we have $$\tfrac{1}{2}\pi a^{\frac{3}{2}}(1+a^2)^{\frac{1}{2}} = 2^{\frac{1}{2}}\mu^{\frac{1}{2}}T,$$

$$T = \frac{\pi a^{\frac{3}{2}}(1+a^2)^{\frac{1}{2}}}{2^{\frac{3}{2}}\mu^{\frac{1}{2}}}.$$

(7) A particle descends by the action of gravity down a tube AO (fig. 136) in the form of a semi-cubical parabola of which

the axis Ox is vertical, and the cusp the lowest point; to investigate the time of falling from a given point A to the cusp O.

Let $OM = x$, $PM = y$; then, by (B), since $X = -g$, $Y = 0$,

$$\frac{ds^2}{dt^2} = -2gx + C.$$

Let h be the initial value of OM: then, $\frac{ds}{dt}$ being initially zero,

we have $$0 = -2gh + C,$$

and therefore $$\frac{ds^2}{dt^2} = 2g(h - x) \qquad (1).$$

Now the equation to the curve is $ay^2 = x^3$,

and therefore $$a^{\frac{1}{2}}y = x^{\frac{3}{2}}, \quad a^{\frac{1}{2}}dy = \tfrac{3}{2}x^{\frac{1}{2}}dx,$$

$$ady^2 = \tfrac{9}{4}xdx^2, \quad ads^2 = \tfrac{1}{4}(9x + 4a)\,dx^2.$$

Hence, by (1), there is

$$\frac{9x + 4a}{4a}\frac{dx^2}{dt^2} = 2g(h - x),$$

and therefore $$dt = -\frac{1}{2(2ag)^{\frac{1}{2}}}\left(\frac{9x + 4a}{h - x}\right)^{\frac{1}{2}} dx,$$

the negative sign being taken because x decreases as t increases.

Assume $z^2 = 9x + 4a$, and our equation becomes

$$dt = -\frac{1}{2(2ag)^{\frac{1}{2}}}\frac{z}{\left(h + \frac{4a}{9} - \frac{z^2}{9}\right)^{\frac{1}{2}}} \cdot \frac{2}{9}zdz$$

$$= -\frac{1}{3(2ag)^{\frac{1}{2}}}\frac{z^2dz}{(4a + 9h - z^2)^{\frac{1}{2}}}$$

$$= -\frac{1}{3(2ag)^{\frac{1}{2}}}\frac{z^2dz}{(\beta^2 - z^2)^{\frac{1}{2}}}, \text{ where } \beta^2 = 4a + 9h,$$

$$t = C - \frac{1}{3(2ag)^{\frac{1}{2}}}\int\frac{z^2dz}{(\beta^2 - z^2)^{\frac{1}{2}}}.$$

But $$\int \frac{z^2 dz}{(\beta^2 - z^2)^{\frac{1}{2}}} = -z(\beta^2 - z^2)^{\frac{1}{2}} + \int (\beta^2 - z^2)^{\frac{1}{2}} dz$$

$$= -z(\beta^2 - z^2)^{\frac{1}{2}} + \beta^2 \int \frac{dz}{(\beta^2 - z^2)^{\frac{1}{2}}} - \int \frac{z^2 dz}{(\beta^2 - z^2)^{\frac{1}{2}}}$$

$$= -\frac{1}{2} z(\beta^2 - z^2)^{\frac{1}{2}} + \frac{1}{2}\beta^2 \sin^{-1}\frac{z}{\beta};$$

hence $$t = C - \frac{1}{3(2ag)^{\frac{1}{2}}} \left\{ -\frac{1}{2} z(\beta^2 - z^2)^{\frac{1}{2}} + \frac{1}{2}\beta^2 \sin^{-1}\frac{z}{\beta} \right\};$$

but, initially, $x = h$, $z^2 = \beta^2$, and therefore

$$0 = C - \frac{1}{3(2ag)^{\frac{1}{2}}} \cdot \frac{1}{2}\beta^2 \frac{\pi}{2}:$$

hence, eliminating C, we have

$$t = \frac{1}{3(2ag)^{\frac{1}{2}}} \left\{ \frac{1}{2} z(\beta^2 - z^2)^{\frac{1}{2}} + \frac{1}{2}\beta^2 \cos^{-1}\frac{z}{\beta} \right\},$$

and, substituting for z and β their values,

$$t = \frac{1}{3(2ag)^{\frac{1}{2}}} \left\{ \frac{3}{2}(9x + 4a)^{\frac{1}{2}}(h - x)^{\frac{1}{2}} + \frac{1}{2}(9h + 4a)\cos^{-1}\left(\frac{9x + 4a}{9h + 4a}\right)^{\frac{1}{2}} \right\}.$$

When the particle arrives at the cusp, $x = 0$, and therefore the whole time of descent is equal to

$$\frac{1}{3(2ag)^{\frac{1}{2}}} \left\{ 3a^{\frac{1}{2}}h^{\frac{1}{2}} + \frac{1}{2}(9h + 4a)\cos^{-1}\frac{2a^{\frac{1}{2}}}{(9h + 4a)^{\frac{1}{2}}} \right\}.$$

(8) Two particles P, P' (fig. 137), of which the masses are m, m', are connected by a straight rigid rod without weight, and, being constrained to move in two straight grooves Aa, Aa', which are inclined to the horizon at given angles and are in the same vertical plane, make small oscillations; to find the length of the isochronous pendulum.

Let AP, AP', make angles α, α', with the vertical line through A, and let $\angle PAP'$ be equal to ι. Let T denote the action of the rod on each of the particles: the direction of this action will lie along the rod. Let $AP = x$, $AP' = x'$, $PP' = c$, $\angle P'Pa = \phi$, $\angle PP'a' = \phi'$.

Then, for the motion of the two particles, we have, resolving forces along AP and AP',

$$m\frac{d^2x}{dt^2} = mg\cos\alpha - T\cos\phi,$$

$$m'\frac{d^2x'}{dt^2} = m'g\cos\alpha' - T\cos\phi'.$$

Eliminating T, we get

$$m\cos\phi'\frac{d^2x}{dt^2} - m'\cos\phi\frac{d^2x'}{dt^2} = mg\cos\alpha\cos\phi' - m'g\cos\alpha'\cos\phi.$$

But from the geometry we evidently have

$$c\cos\phi = x'\cos\iota - x, \quad c\cos\phi' = x\cos\iota - x';$$

hence $$-m(x' - x\cos\iota)\frac{d^2x}{dt^2} + m'(x - x'\cos\iota)\frac{d^2x'}{dt^2}$$

$$= -mg\cos\alpha(x' - x\cos\iota) + m'g\cos\alpha'(x - x'\cos\iota)\ldots\ldots\ldots(1).$$

Let a, a', be the values of x, x', corresponding to a position of equilibrium; then, $\frac{d^2x}{dt^2}$ and $\frac{d^2x'}{dt^2}$ being both, for such a position, equal to zero, we have

$$0 = -m\cos\alpha(a' - a\cos\iota) + m'\cos\alpha'(a - a'\cos\iota)\ldots\ldots\ldots(2).$$

Assume $x = a + v$, $x' = a' + v'$, v and v' being by the hypothesis small quantities. Then, from the equation (1), as far as small quantities of the first order, we have, by the aid of (2),

$$-m(a' - a\cos\iota)\frac{d^2v}{dt^2} + m'(a - a'\cos\iota)\frac{d^2v'}{dt^2}$$

$$= -mg\cos\alpha(v' - v\cos\iota) + m'g\cos\alpha'(v - v'\cos\iota)\ldots\ldots(3).$$

Now, by the geometry,

$$c^2 = x^2 + x'^2 - 2xx'\cos\iota;$$

whence $$0 = x\delta x + x'\delta x' - (x\delta x' + x'\delta x)\cos\iota.$$

But $\delta x = v$, $\delta x' = v'$, $x = a + v$, $x' = a' + v'$; hence, neglecting small quantities higher than those of the first order,

$$0 = (a - a'\cos\iota)v + (a' - a\cos\iota)v'\ldots\ldots\ldots(4).$$

Let r represent the length of the isochronous pendulum; then

$$v = k \sin\left\{\left(\frac{g}{r}\right)^{\frac{1}{2}} t + \epsilon\right\}, \qquad v' = k' \sin\left\{\left(\frac{g}{r}\right)^{\frac{1}{2}} t + \epsilon\right\},$$

where k, k', ϵ, are constants: substituting these values of v, v', in the equations (3) and (4), we have

$$m\,(a' - a\cos\iota)\,\frac{k}{r} - m'\,(a - a'\cos\iota)\,\frac{k'}{r}$$

$$= k\,(m\cos\alpha\cos\iota + m'\cos\alpha') - k'\,(m'\cos\alpha'\cos\iota + m\cos\alpha),$$

and $$(a - a'\cos\iota)\,k + (a' - a\cos\iota)\,k' = 0.$$

Eliminating k and k' between these two equations,

$$\frac{m}{r}\,(a' - a\cos\iota)^2 + \frac{m'}{r}\,(a - a'\cos\iota)^2 = ma\cos\alpha\sin^2\iota + m'a'\cos\alpha'\sin^2\iota;$$

and therefore

$$r = \frac{m\,(a' - a\cos\iota)^2 + m'\,(a - a'\cos\iota)^2}{(ma\cos\alpha + m'a'\cos\alpha')\sin^2\iota} \quad \text{.........(5)}.$$

From P draw PO at right angles to AP, meeting $P'O$ drawn from P' at right angles to AP'. Then the projection of OP upon the line AP' is equal to $OP\sin\iota$, and the projection of PP' on the line AP' is equal to $a' - a\cos\iota$; but these two projections are evidently coincident; hence

$$OP^2\sin^2\iota = (a' - a\cos\iota)^2:$$

similarly $$OP'^2\sin^2\iota = (a - a'\cos\iota)^2.$$

Again, let G be the centre of gravity of m and m' in their position of equilibrium, and H the point at which a vertical line through G will cut a horizontal line through A; then we have

$$(m + m')\,GH = ma\cos\alpha + m'a'\cos\alpha'.$$

Hence, from (5), we obtain

$$r = \frac{m\,.\,OP^2 + m'\,.\,OP'^2}{(m + m')\,GH}.$$

(9) A particle slides down a plane of given length, inclined at an angle θ to a horizontal plane, and is reflected by the horizontal plane; to determine the value of θ in order that the

range on the horizontal plane may be the greatest possible, the particle being perfectly elastic.

$$\theta = \tan^{-1}(\sqrt{2}).$$

(10) The major axis of an ellipse is vertical; to find the radius vector by which a particle will descend in the shortest time from the upper focus to the curve.

If θ = the inclination of the required radius vector to the vertical line drawn downwards from the focus; then $\theta = 0$, if $e < \frac{1}{2}$; and $\theta = \cos^{-1}\left(\frac{1}{2e}\right)$, if $e > \frac{1}{2}$.

(11) The plane of an ellipse is inclined at any angle to a horizontal plane, in which its major axis lies: to determine the position of the line, drawn from a focus to the perimeter of the ellipse, down which a particle, acted on by gravity, will descend in the shortest time.

If θ denote the inclination of the required line to the distance of the focus from the more remote apse,

$$\cos\theta = \frac{1 - (8e^2 + 1)^{\frac{1}{2}}}{4e}.$$

(12) Two equal spherical particles of given elasticity are placed at two points in the circumference of a vertical circle, the radii of these two points making angles of 60° on each side of the radius which tends vertically downwards; to determine the sum of the chords of the arcs described by each particle before it ceases to move.

If a = the radius of the circle, and e = the common elasticity of the particles, the required space will be equal to

$$\frac{a}{1 - e}.$$

(13) A particle is placed within a thin rectilinear tube, and is attracted by a force tending towards a fixed point without the tube and varying as some function of the distance; to find the time of a small oscillation of the particle.

If $f(r)$ denote the intensity of the force at a distance r, and a be the perpendicular distance of the centre of force from the tube, the time of a small oscillation will be equal to

$$\frac{\pi a^{\frac{1}{2}}}{\{f(a)\}^{\frac{1}{2}}}.$$

(14) Two equal particles, attracting each other with forces varying inversely as the square of the distance, are constrained to move in two straight lines intersecting each other at right angles; supposing their velocities to be initially zero, to find the time in which each of them will arrive at the intersection of the two straight lines.

If a denote the initial distance between the particles, and μ the absolute attracting force of each, they will arrive simultaneously at the intersection of the straight lines in a time equal to

$$\frac{\pi a^{\frac{3}{2}}}{2(2\mu)^{\frac{1}{2}}}.$$

The particles would arrive simultaneously at the intersection of their paths for any other law of mutual attraction.

(15) A particle, acted on by gravity, is placed at any point in the arc of an inverted cycloid, of which the axis is vertical, and descends to the lowest point of the curve; to find the whole time of descent.

If a be the radius of the generating circle, the required time will be equal to

$$\pi \left(\frac{a}{g}\right)^{\frac{1}{2}}.$$

This result, being independent of the initial position of the particle, shews that the time of descent will be the same at whatever point in the curve the particle is placed. This elegant mechanical property of the Cycloid, from which it has received the name of a Tautochronous Curve, was first discovered by Huyghens, *Horolog. Oscill.* Pars II.

(16) A particle falls to the lowest point of a cycloid down any arc of the curve, the axis of the cycloid being vertical and its vertex downwards: to find the position of the particle when the vertical component of its velocity is greatest.

It is greatest when the particle has completed half its vertical descent.

(17) Two particles are connected by a fine inextensible string at full stretch in a narrow cycloidal tube, the axis of the cycloid being vertical and the vertex upwards: to find the tension of the string during the motion of the particles.

Let m, m', be the masses of the particles, c the length of the string, and $2a$ the length of the axis of the cycloid: then, throughout the motion, the required tension is equal to

$$\frac{c\,mm'g}{4a\,(m+m')}.$$

(18) Two equal molecules are connected together by a fine inelastic thread: one of them is placed on a smooth table, the other just over the edge, the thread being at full stretch at right angles to the edge: to find the whole interval of time from the commencement of the motion to the instant when the thread first becomes horizontal.

If c be the length of the string, the required time is equal to

$$\tfrac{1}{2}\left(\frac{c}{g}\right)^{\frac{1}{2}}\cdot(\pi+4).$$

(19) There are three fixed pegs in a horizontal line, the middle peg being equidistant from the other two: to the outside pegs are attached the ends of a fine inelastic thread, which hangs over the middle peg: beads of equal weights rest on the middle points of the two halves of the thread. Supposing the beads to be slightly displaced, in vertical directions, from their positions of equilibrium without slackening the thread, to find the length of a simple pendulum, isochronous with their subsequent oscillations.

Let c be the depth of either bead below the line of the pegs when the system is in a position of equilibrium, and α the corresponding inclination of each portion of the thread to the horizon: then the length of the pendulum is equal to $c\sec^2\alpha$.

(20) Two candles of equal weights are at rest in vertical positions, being attached to a perfectly flexible wire of insensible mass, which passes over a smooth pully: supposing one of them to be lighted, and to burn out before reaching the pully, in a time t, to find through what space the other candle will have descended by the end of this time.

The required descent is equal to

$$gt^2 \cdot \left(\frac{3}{2} - \log 4\right).$$

(21) A particle, attracted by a force tending towards a fixed point A, (fig. 138), and varying directly as the distance, describes the arc OP and the chord OP of a fixed smooth curve in the same time, whatever point P be chosen in the curve: the particle has no motion when at O. To find the nature of the curve.

The curve is the Lemniscata, the centre of which is at O and of which the axis is inclined to OA at an angle $\frac{\pi}{4}$.

Bonnet; *Liouville, Journal de Mathématiques*, Av., 1844.

(22) A particle falls under the action of gravity down an arc OB (fig. 139) of one of the loops of a Lemniscata, of which the axis OA is inclined at an angle of 45° to the horizon; to determine the time of the descent.

Let a denote the semi-axis of the corresponding equilateral hyperbola, θ the angle between the chord OB and the axis OA of the loop, and T the required time. Then

$$T = \left(\frac{8^{\frac{1}{2}}a}{g}\right)^{\frac{1}{2}} \cdot \left\{\tan\left(\tfrac{1}{4}\pi - \theta\right)\right\}^{\frac{1}{2}}.$$

This expression for the time is the same as that for the descent of a particle down the chord OB; a mechanical property of the Lemniscata which was discovered by Saladini, *Memorie dell' Istituto Nazionale Italiano*, Tom. I. parte 2.

(23) A spherical particle A impinges with a velocity u in a horizontal direction upon a spherical particle B, which is resting at the lowest point of an inverted cycloid, of which the axis is vertical; to determine the velocities of A and B after any number of impacts, the volumes of the particles being equal, while their masses differ in any proposed degree.

The velocities of A, B, after x impacts, will be respectively, e denoting their common elasticity and m, m', their masses,

$$\frac{m + m'(-e)^x}{m+m'}u, \qquad \frac{m - m(-e)^x}{m+m'}u.$$

SECT. 2. *Pressure of a moving Particle on immoveable plane Curves.*

The general value of the reaction of a curve against a particle, which is moving along the curve, is given by the formula

$$R = Y\frac{dx}{ds} - X\frac{dy}{ds} \pm \frac{1}{\rho}\frac{ds^2}{dt^2} = N \pm \frac{v^2}{\rho} \text{..............(A)},$$

where N represents the resolved part of the whole accelerating force on the particle estimated along the normal in an opposite direction to that in which the reaction R exerts itself, and ρ denotes the radius of curvature of the curve. In this formula the positive or the negative sign is to be taken according as the particle is moving on the concave or on the convex side of the curve.

This formula was first given by L'Hôpital[1] in the discussion of John Bernoulli's problem of the Curve of Equal Pressure.

When the expression for R becomes equal to zero, the particle

[1] *Mém. de l'Acad. des Sciences de Paris*, 1700, p. 9.

will either leave the curve or will move along it freely without experiencing any reaction; and the analytical condition

$$\frac{v^2}{\rho} = \mp N$$

shews that, on the commencement of free motion, the normal accelerating force and the centrifugal force of the particle must be equal and opposite.

(1) A particle, starting with a given velocity from the vertex of a parabola, of which the axis is vertical, descends down the convex side of the curve by the action of gravity; to find the reaction of the curve at any point of the descent.

The resolved part of the force of gravity along the normal in a direction opposite to the reaction is $g\frac{dy}{ds}$, and therefore by (A), the particle moving on the convex side of the curve,

$$R = g\frac{dy}{ds} - \frac{1}{\rho}\frac{ds^2}{dt^2}.$$

Now, the equation to the parabola being $y^2 = 4mx$,

$$\frac{dy}{ds} = \frac{m^{\frac{1}{2}}}{(m+x)^{\frac{1}{2}}} \text{ and } \rho = \frac{2}{m^{\frac{1}{2}}}(m+x)^{\frac{3}{2}}.$$

Also, if h be the altitude due to the initial velocity of the particle, we have

$$\frac{ds^2}{dt^2} = 2g(x+h).$$

Hence $$R = \frac{gm^{\frac{1}{2}}}{(m+x)^{\frac{1}{2}}} - \frac{gm^{\frac{1}{2}}}{(m+x)^{\frac{3}{2}}}(x+h)$$

$$= m^{\frac{1}{2}}g\frac{m-h}{(m+x)^{\frac{3}{2}}}.$$

If $h = m$, then the pressure during the whole motion will be equal to zero; and the particle will describe the parabola freely. If h were greater than m, since, from the nature of the case, R cannot have any negative value, the particle would

from the first proceed in a path different from the parabola in question. If, instead of supposing the particle to move on a mere curve, we were to conceive it to be moving within an indefinitely thin parabolic tube, R might be negative; and in fact always would be negative, supposing h to be greater than m, when the motion would be the same as if the particle were moving along the concave side of the parabolic curve.

Euler; *Mechan.* Tom. II. p. 64.

(2) A particle, starting from rest, descends down the convex side of a circle from a given point in its circumference; to find where it will leave the curve.

Let O (fig. 140) be the centre of the circle, AO being a vertical radius. Let P be the initial position of the particle, Q its point of departure; PM, QN, horizontal lines. Join OQ, and let $\angle AOQ = \phi$; let $a =$ the radius of the circle.

Then, the centrifugal force at Q being equal to the normal component of gravity, we have

$$\frac{v^2}{a} = g \cos\phi;$$

but, denoting MN by x,

$$v^2 = 2gx;$$

hence, putting $MO = c$,

$$2x = a\cos\phi = c - x, \quad x = \tfrac{1}{3}c.$$

Fontana; *Memorie della Societa Italiana*, 1782, p. 175.

(3) A particle is moving along the convex side of an equiangular spiral, towards the pole of which it is attracted by a force varying as any power of the distance; to determine the reaction of the curve at any time during the motion.

Let r be the distance of the particle from the pole at any time, μr^n the attractive force, α the constant angle between the curve and the radius vector, β the initial velocity, and a the initial value of r. Then, by the formula (A), N being equal to $\mu r^n \sin\alpha$, we have

$$R = \mu r^n \sin\alpha - \frac{v^2}{\rho} \text{.......................} (1).$$

Again, estimating the velocity v of the particle in a direction corresponding to an increase of r, and denoting by ds an element of its path, we have

$$v\frac{dv}{ds} = -\mu r^n \cos\alpha \text{.....................(2).}$$

Now, by the nature of the curve, $\cos\alpha\, ds = dr$; hence, from (2),

$$v\frac{dv}{dr} = -\mu r^n;$$

integrating and observing that β, a, are the initial values of v, r, we get

$$v^2 - \beta^2 = -\frac{2\mu}{n+1}(r^{n+1} - a^{n+1}) \text{............(3).}$$

Again, p denoting the perpendicular from the pole upon the tangent to the curve, we have, since $p = r\sin\alpha$,

$$\rho = r\frac{dr}{dp} = \frac{r}{\sin\alpha} \text{........................(4).}$$

From (1), (3), (4), we obtain

$$R = \mu r^n \sin\alpha - \frac{\sin\alpha}{r}\left\{\beta^2 - \frac{2\mu}{n+1}(r^{n+1} - a^{n+1})\right\}$$

$$= \mu\frac{n+3}{n+1}r^n \sin\alpha - \frac{\beta^2\sin\alpha}{r} - \frac{2\mu\sin\alpha}{(n+1)r}a^{n+1}.$$

Euler; *Mechan.* Tom. II. p. 86.

(4) An endless string POQ, fig. (141), passes through a small immoveable ring at O and lies on a horizontal table in the form of an isosceles triangle of which O is the vertex: at P and Q are two beads of equal mass, moveable along the string: supposing the beads to be projected with equal velocities along the external bisectors of the angles OPQ, OQP, respectively, to find the tension of the string during the motion.

From O draw ON at right angles to PQ, cutting it at N. Let c = half the length of the string. Draw OA parallel to NP, so that $OA = \dfrac{c}{2}$.

Then, since $OP+NP=c$, the path of P is a parabola of which O is the focus and A the vertex, the tangent to the path at P being, by the nature of the parabola, the external bisector of the angle OPQ.

Let v be the velocity of P, T the tension of the string, which acts on P in the directions PO and PN. Since PO and PN make equal angles with the tangent at P, v will be constant. Let R be the resultant pressure on P, which will act along the normal. Then, ρ being the radius of curvature of the path at P, and m the mass of a bead, $R=\dfrac{mv^2}{\rho}$. But $\rho=\dfrac{(2r)^{\frac{3}{2}}}{c}$, where $r=OP$. Hence $R=\dfrac{mc^{\frac{1}{2}}v^2}{(2r)^{\frac{3}{2}}}$.

Let $\theta=$ the inclination of the normal to either PN or PO: then $R=2T\cos\theta$. But, from the nature of the parabola, we may easily see that $\cos\theta=\left(\dfrac{c}{2r}\right)^{\frac{1}{2}}$: hence $R=T\left(\dfrac{2c}{r}\right)^{\frac{1}{2}}$, and therefore $T=\dfrac{mv^2}{4r}$. Thus we see that, throughout the motion, the tension of the string varies inversely as the distance of either bead from the ring.

(5) A particle attracted towards two centres of force, varying inversely as the square of the distance, moves in a hyperbolic groove, of which the foci are the centres of force; to find the pressure on the groove at any point, the particle being supposed to move on the concave side.

Let P (fig. 142) be the position of the particle at any time t: S, H, the foci of the hyperbola; let $SP=r$, $HP=r'$; let a be the transverse semi-axis; PT being a tangent at P, let $\angle SPT=\phi=\angle HPT$; let μ, μ', be the absolute forces towards S, H.

Resolving forces at right angles to the tangent at P, we have, by the equation (A),

$$R=\left(\frac{\mu'}{r'^2}-\frac{\mu}{r^2}\right)\sin\phi+\frac{v^2}{\rho}\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1):$$

also, for the value of v at any time, there is

$$v^2 = 2\int\left(-\frac{\mu'}{r'^2}dr' - \frac{\mu}{r^2}dr\right) + C$$

$$= \frac{2\mu'}{r'} + \frac{2\mu}{r} + C.$$

Let f, f', be the initial values of r, r', and β the initial value of v; then

$$\beta^2 = \frac{2\mu'}{f'} + \frac{2\mu}{f} + C,$$

and therefore $\quad v^2 = \frac{2\mu'}{r'} + \frac{2\mu}{r} - \frac{2\mu'}{f'} - \frac{2\mu}{f} + \beta^2.$

Hence, from (1), we have

$$R\rho = \left(\frac{\mu'}{r'^2} - \frac{\mu}{r^2}\right)\rho\sin\phi + \frac{2\mu'}{r'} + \frac{2\mu}{r} - \frac{2\mu'}{f'} - \frac{2\mu}{f} + \beta^2.$$

But $2\rho\sin\phi$ is equal to the chord of curvature through S, which, by the nature of the hyperbola, is equal to $\frac{2rr'}{a}$; hence

$$R\rho = \left(\frac{\mu'}{r'^2} - \frac{\mu}{r^2}\right)\frac{rr'}{a} + \frac{2\mu'}{r'} + \frac{2\mu}{r} - \frac{2\mu'}{f'} - \frac{2\mu}{f} + \beta^2$$

$$= \frac{\mu'(r+2a)}{ar'} - \frac{\mu(r'-2a)}{ar} - \frac{2\mu'}{f'} - \frac{2\mu}{f} + \beta^2$$

$$= \frac{\mu'}{a} - \frac{\mu}{a} - \frac{2\mu'}{f'} - \frac{2\mu}{f} + \beta^2$$

$$= \frac{\mu'(f-2a)}{af'} - \frac{\mu(f+2a)}{af} + \beta^2$$

$$= \frac{\mu' f}{af'} - \frac{\mu f'}{af} + \beta^2.$$

If the initial velocity be zero, and the particle be attracted at the commencement of its motion with equal intensity by the two centres of force; then $\beta = 0$, $\frac{\mu}{f^2} = \frac{\mu'}{f'^2}$, and therefore $R = 0$ during the whole motion. Hence the particle would under these circumstances describe the hyperbola freely.

(6) A particle, acted on by gravity, oscillates in a circular arc; to find the reaction of the curve at any point.

Let O (fig. 143) be the centre of the circle; P the position of the particle at any time; A the lowest point of the circle; let $\angle AOP = \theta$. Then, θ being the value of α when the velocity is zero, we have

$$R = g\,(3\cos\theta - 2\cos\alpha).$$

(7) The highest point of a circle, the plane of which is vertical, is given: a particle, starting at this point, slides down the convex side of the circle: to find the locus of the point where the particle leaves the circle.

The required locus is a straight line, passing through the highest point of the circle, and making an angle $\tan^{-1}(\sqrt{5})$ with the vertical.

(8) A particle is projected with a given velocity, at the highest point of a circle in a vertical plane, along the concave side of the curve; to determine the pressure on the curve at any point in its path.

Let AOB (fig. 144) be the vertical diameter, O being the centre of the circle; P the position of the particle at any time; let $OP = a$, $\angle AOP = \theta$; let β be the velocity of projection at A; then, for the pressure at P,

$$R = \frac{\beta^2}{a} + g\,(2 - 3\cos\theta).$$

Suppose that $R = 0$ initially; then $\dfrac{\beta^2}{a} = g$, and

$$\begin{aligned} R &= 3g \operatorname{vers}\theta \\ &= 6g, \qquad \text{when } \theta = \pi; \end{aligned}$$

which shews that, when the particle arrives at the lowest point, the reaction is six times the force of gravity.

Euler; *Mechan.* Tom. II. p. 65, Cor. 7.

(9) A particle falls through a narrow tube in the form of a cycloid, the axis of which is vertical and vertex upwards, the

initial position of the particle being close to the vertex: to find the pressure on the tube at any point in terms of the radius of curvature.

If a be the radius of the generating circle, and ρ the radius of curvature of the cycloid at any point, the pressure at that point is equal to

$$g\left(\frac{\rho}{2a}-\frac{4a}{\rho}\right).$$

(10) There is a rigid wire in the form of a catenary: a particle is projected along its concave side, at its lowest point, with a velocity which it would acquire by falling thence to the directrix: to find the pressure on the wire when the particle has risen through half its whole vertical ascent.

If W be the weight of the particle, the required pressure is equal to $\frac{10}{9}W$.

(11) A particle descends from rest down the convex side of a logarithmic curve, placed with its asymptote parallel to the horizon; to find where it leaves the curve.

Let P (fig. 145) be the initial position of the particle and Q the point where it leaves the curve: let $OM=h$, $ON=x$. Then, the equation to the curve being $y=\log_a x$, we have, putting $A=\log_e a$,

$$x=\frac{1}{A}\left\{Ah+(1+A^2h^2)^{\frac{1}{2}}\right\}.$$

Fontana; *Memorie della Societa Italiana*, 1782, p. 182.

(12) A particle descends from rest down the convex side of an ellipse of which the major axis is vertical, from a given point in the curve; to determine where it will leave the ellipse.

Let the highest point of the ellipse be taken as the origin of co-ordinates, the axis of x being vertical, and that of y horizontal. Let a, b, denote the semi-axes major and minor; h the distance, from the axis of y, of the initial position of the particle, and x

of the point at which it leaves the curve. Then the value of x will be a root of the cubic equation

$$(a^2 - b^2)(x^3 - 3ax^2) - 3a^2b^2x + a^3(b^2 + 2ah) = 0.$$

Fontana; *Ib.* p. 175.

(13) A particle descends from rest down the convex side of the Cissoid of Diocles, the asymptote of the Cissoid being vertical; the initial place of the particle being known, to find the point at which it will leave the curve.

Let P (fig. 146) be the initial position of the particle, and Q its place on leaving the curve: draw PS, QN, at right angles to Ox, Oy; let $PS = h$, $QN = x$. Then, a being the radius of the generating circle, the value of x will be a root of the cubic equation

$$x^3 - \frac{16a}{9}x^2 + \frac{64a^2 + 36h^2}{81}x - \frac{8ah^2}{9} = 0.$$

If the motion commence at the cusp O, $h = 0$, and therefore

$$x = \frac{8}{9}a.$$

Fontana; *Ib.* p. 181.

(14) A particle is projected with a given velocity along the convex side of a parabola from a given point of the curve: at the focus of the parabola there is a centre of attractive force which varies inversely as the square of the distance; to determine the reaction of the curve on the particle at any point of its path.

Let S (fig. 147) be the focus of the parabola; B the point from which the particle is projected; BT the tangent at B; P the position of the particle after any time; let $SP = r$, $SB = a$, $SA = m$, $\beta =$ the velocity of projection, $\mu =$ the absolute force towards S. Then, at P,

$$R = \left(\frac{m}{r^3}\right)^{\frac{1}{2}} \left(\frac{\mu}{a} - \frac{\beta^2}{2}\right).$$

(15) There is a centre of force at one extremity of the diameter of a semi-circle, the force being repulsive and varying as

the distance: to find the pressure exerted upon the curve by a particle which moves from rest from the centre of force along its concave side, and the time which elapses before it reaches the other extremity of the diameter.

If a = the radius of the circle, m = the mass of the particle, μ = the absolute force, and R = the pressure when the particle is at a distance r from the centre of force,

$$R = \frac{3\mu m r^2}{2a}.$$

The required time is infinite.

(16) A particle is attached to the end of a fine thread which just winds round the circumference of a circle, at the centre of which there is a repulsive force varying as the distance: to find the time of unwinding, and the tension of the string at any time.

If μ = the absolute force, and a = the radius of the circle, the time of unwinding is equal to $\frac{2\pi}{\sqrt{\mu}}$, and the tension at any time t is equal to $2\mu^{\frac{3}{2}} . a . t$.

(17) The major axis of an ellipse is vertical: to find the velocity with which a particle must be projected vertically upwards from the extremity of the minor axis along the interior of the elliptic arc, so that after quitting the curve it may pass through the centre.

If a, b, denote the semi-axes major and minor, the required velocity will be equal to

$$\left\{\frac{(8a^2 + b^2)\, g}{3a . 3^{\frac{1}{2}}}\right\}^{\frac{1}{2}}.$$

(18) A particle moves in a parabolic tube under the action of a repulsive force from the focus and a force parallel to the axis, each force varying as the focal distance of the particle: to find the pressure on the tube.

Let m be the focal distance of the vertex, μ and ν the abso-

lute forces, V the velocity at the vertex: then, when the particle's focal distance is r, the pressure is equal to

$$\frac{m^{\frac{1}{2}}}{2r^{\frac{3}{2}}}\{(3\mu-\nu)\,r^2-(\mu+\nu)\,m^2+V^2\}.$$

COR. If $\nu=3\mu$ and $V^2=(\mu+\nu)\,m^2$, the pressure is always zero and the particle would describe its path without constraint.

(19) A molecule moves in a narrow tube in the form of the lemniscate $r^2=a^2\cos 2\theta$ and is attracted towards the node by a force varying inversely as the seventh power of the distance: to find the law of the pressure exerted by the molecule on the tube.

The pressure varies directly as the distance of the molecule from the node.

(20) A particle moves along the convex side of an ellipse, under the action of two forces tending to the foci and varying inversely as the square of the distance, and a third force tending to the centre and varying as the distance; to find the reaction of the curve at any point.

Let R denote the reaction of the curve on the particle at any point, ρ the radius of curvature; f, f', the initial focal distances, and μ, μ', the corresponding absolute forces; μ'' the absolute force to the centre, $2a$ the axis major of the ellipse, and β the initial velocity. Then

$$R\rho=\frac{\mu f'}{af}+\frac{\mu' f}{af'}+\mu'' ff'-\beta^2.$$

If β', β'', β''', denote the velocities which the particle ought to have initially, in order to revolve freely round the three centres of force taken separately,

$$\beta'^2=\frac{\mu f'}{af},\quad \beta''^2=\frac{\mu' f}{af'},\quad \beta'''^2=\mu'' ff';$$

and therefore, when the forces are taken conjointly, it will revolve about them freely when

$$\beta^2=\beta'^2+\beta''^2+\beta'''^2.$$

SECT. 3. *Motion of a Particle on Rough Plane Curves.*

(1) A heavy particle slides on a rough cycloid, the base of which is horizontal and vertex downwards, starting from instantaneous rest at the highest point: to determine the coefficient of friction in order that the particle may come to rest at the vertex.

Let the tangent at the vertex be the axis of y, the axis of the curve being that of x. Let μ be the coefficient of friction, v the velocity at any point, R the normal reaction of the curve. Then for the motion we have, ρ denoting the radius of curvature, and m the mass of the particle,

$$mv\frac{dv}{ds} = -mg\frac{dx}{ds} + \mu R,$$

and

$$R = \frac{mv^2}{\rho} + mg\frac{dy}{ds},$$

whence

$$v\frac{dv}{ds} = -g\frac{dx}{ds} + \frac{\mu v^2}{\rho} + \mu g\frac{dy}{ds}.$$

But, by the properties of the cycloid,

$$x = a(1-\cos\theta), \quad y = a(\theta + \sin\theta), \quad s = 4a\sin\frac{\theta}{2}, \quad \rho = 4a\cos\frac{\theta}{2}:$$

hence $2v\,dv - \mu v^2 d\theta = -2ag\sin\theta d\theta + 2\mu ag(1+\cos\theta)\,d\theta$,

$$d(v^2 \,.\, e^{-\mu\theta}) = -2age^{-\mu\theta}\sin\theta d\theta$$
$$+ 2\mu age^{-\mu\theta}(1+\cos\theta)\,d\theta:$$

integrating from $\theta = 0$ to $\theta = \pi$, and bearing in mind that $v = 0$ at both limits, we have

$$0 = \mu\int_0^\pi e^{-\mu\theta}(1+\cos\theta)\,d\theta - \int_0^\pi e^{-\mu\theta}\sin\theta d\theta.$$

Integrating by parts we shall easily see that

$$\int_0^\pi e^{-\mu\theta}\sin\theta d\theta = \frac{1+e^{-\mu\pi}}{1+\mu^2}, \quad \int_0^\pi e^{-\mu\theta}\cos\theta d\theta = \frac{\mu}{1+\mu^2}\,.\,(1+e^{-\mu\pi}),$$

and therefore

$$1 - e^{-\mu\pi} + \frac{\mu^2 - 1}{\mu^2 + 1}(1 + e^{-\mu\pi}) = 0,$$

and therefore $\mu^2 e^{\mu\pi} = 1$, a relation by which the value of μ is determined.

(2) A heavy body is placed on a rough inclined plane, the inclination of which is greater than the angle of indifference, and is connected with a fine elastic string parallel to the plane and attached to a fixed point: if the body be initially at rest and the string of its natural length, to determine the circumstances of the resulting motion.

Let a be the natural length of the string, x its length at the end of any time t, α the angle of the plane, m the mass of the body, and λ the tension requisite to double the length of the string: then

$$x - a = \frac{mag}{\lambda} \,.\, (\sin\alpha - \mu\cos\alpha) \,.\, \text{vers}\left(\frac{\lambda}{ma}\right)^{\frac{1}{2}} t.$$

SECT. 4. *Inverse Problems on the Motion of a Particle along immoveable Plane Curves.*

(1) To find a curve EPF (fig. 148) such that, A and B being two given points in the same horizontal line, the sum of the times in which a particle will descend by the action of gravity down the straight lines AP, BP, may be the same whatever point of the curve P may be.

Bisect AB in O; let Ox, a vertical line, be the axis of x, and OAy, which is horizontal, the axis of y; let $AB = 2a$. Then, x, y, being the co-ordinates of P, the times down AP, BP, will be respectively equal to

$$\left\{\frac{x^2 + (a-y)^2}{\frac{1}{2}gx}\right\}^{\frac{1}{2}}, \quad \left\{\frac{x^2 + (a+y)^2}{\frac{1}{2}gx}\right\}^{\frac{1}{2}}.$$

Hence, k denoting the sum of the times,

$$(\tfrac{1}{2}k^2 gx)^{\frac{1}{2}} = \{x^2 + (a-y)^2\}^{\frac{1}{2}} + \{x^2 + (a+y)^2\}^{\frac{1}{2}} \ldots\ldots\ldots\ldots (1).$$

Putting $\frac{1}{2}k^2g = 4c$, and squaring both sides of the equation, we have

$$2cx = a^2 + x^2 + y^2 + \{x^2 + (a-y)^2\}^{\frac{1}{2}} \{x^2 + (a+y)^2\}^{\frac{1}{2}},$$
$$(2cx - a^2 - x^2 - y^2)^2 = \{x^2 + (a-y)^2\} \{x^2 + (a+y)^2\}.$$

Developing both sides of the equation, and simplifying, we shall readily find that

$$c^2x^2 - a^2cx - cx^3 - cxy^2 = -a^2y^2,$$

and therefore

$$y^2 = cx\,\frac{x^2 - cx + a^2}{a^2 - cx} \text{.....................} (2),$$

which is the equation to the required curve.

If we trace this curve, we shall find it to consist of a branch VOV' having an asymptote parallel to the axis of y, and of an oval EFP. The oval is the portion of the curve which corresponds to the problem which we are considering. The infinite branch VOV' would correspond to the condition that the times down AP', BP', shall have a constant difference: in which case we should have had, instead of the equation (1),

$$(\tfrac{1}{2}k^2gx)^{\frac{1}{2}} = \{x^2 + (a+y)^2\}^{\frac{1}{2}} - \{x^2 - (a-y)^2\}^{\frac{1}{2}};$$

whence, by the involution, we should have obtained the same equation (2). The curve has pretty much the shape of the Conchoid, although its equation is essentially different.

Fuss; *Mémoires de l'Acad. de St. Pétersb.* 1819.

(2) A particle, not acted on by any forces, is constrained to move within a narrow tube of such a form that the acceleration of the particle parallel to a given straight line is invariable; to determine the equation to the path of the particle.

Let the axis of x be taken parallel to the given line; let c be the constant acceleration of the particle parallel to the axis of x, and β its velocity within the tube, which will be invariable. Then

$$\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} = \beta^2 \text{......................} (1);$$

but, by the condition of the problem, $\frac{d^2x}{dt^2} = c$, and therefore, the axis of y being so chosen that $\frac{dx}{dt} = 0$ when $x = 0$,

$$2\frac{dx}{dt}\frac{d^2x}{dt^2} = 2c\frac{dx}{dt}, \qquad \frac{dx^2}{dt^2} = 2cx \ldots\ldots\ldots\ldots\ldots (2).$$

Eliminating dt between the equations (1) and (2), we get

$$2cx\left(1 + \frac{dy^2}{dx^2}\right) = \beta^2,$$

or, putting $\frac{\beta^2}{4c} = a$,

$$\frac{dy}{dx} = \left(\frac{2a - x}{x}\right)^{\frac{1}{2}};$$

whence, by integration, the position of the axis of x being supposed such that $x = 0$ when $y = 0$,

$$y = (2ax - x^2)^{\frac{1}{2}} + a \text{ vers}^{-1}\frac{x}{a};$$

which is the equation to a cycloid of which the axis is parallel to the given line.

There is an elaborate investigation by Euler, in the *Mémoires de l'Académie de St. Pétersb.*, Tom. x. p. 7, on the nature of the curve of constraint when the particle is subject to the action of gravity, and the direction of uniform acceleration is horizontal. A notice of this problem may be seen in the *Bulletin des Sciences de Bruxelles*, Tom. IX.

(3) To determine the curve down which a particle may descend by the action of gravity, so as to describe equal vertical spaces in equal times, the tangent to the curve at the point where the motion commences being vertical.

Let O (fig. 149) be the point where the motion commences, Ox the axis of x touching the required curve OA at O, Oy the axis of y at right angles to Ox; let $OM = x$, $PM = y$, $\beta =$ the invariable velocity of the particle parallel to Ox.

Then $\dfrac{ds^2}{dt^2} = C + 2gx$, C being a constant quantity,

$$\frac{ds^2}{dx^2}\frac{dx^2}{dt^2} = C + 2gx.$$

But $\dfrac{dx}{dt} = \beta$; hence

$$\beta^2 + \beta^2 \frac{dy^2}{dx^2} = C + 2gx.$$

But, when $x = 0$, $\dfrac{dy}{dx} = 0$; and therefore

$$\beta^2 = C, \qquad \beta^2 \frac{dy^2}{dx^2} = 2gx,$$

$$\frac{dy}{dx} = \frac{(2g)^{\frac{1}{2}}}{\beta} x^{\frac{1}{2}}, \qquad y = \frac{2\,(2g)^{\frac{1}{2}}}{3\beta} x^{\frac{3}{2}},$$

no constant being added because $x = 0$, $y = 0$, simultaneously. The required curve OA is therefore the semi-cubical parabola, O being the cusp, and Ox the axis.

This curve is called the Isochrone. It was proposed by Leibnitz[1], as a challenge to the disciples of Des Cartes, who, from an excessive attachment to the geometry of their master, affected to despise the methods of the Differential Calculus. No solution was communicated by any of the Cartesians. Huyghens alone successfully accepted the challenge, by whom a geometrical solution was given in the *Nouvelles de la République des Lettres, Octobre* 1687. The solution by Leibnitz appeared for the first time in the *Acta Erudit. Lips.* 1689, p. 196 et sq. The solutions both of Huyghens and of Leibnitz were synthetical. An analytical solution was given afterwards for the first time by James Bernoulli[2].

(4) A particle is projected with a given velocity from a point A (fig. 150) along a horizontal line AO towards a point O; to find the curve along which it must be constrained to move in order that it may approach the point O uniformly; the particle being acted on by gravity, and AO being a tangent to the required curve.

[1] *Nouvelles de la République des Lettres, Septembre* 1687.

[2] *Act. Erudit. Lips.* 1690, p. 217.

Let P be the position of the particle at any time; let $AO=a$, $OP=r$, $AOP=\theta$; $\beta=$ the velocity of the particle at its initial position A. For the motion of the particle at any point in its descent there is

$$\frac{dr^2}{dt^2}+r^2\frac{d\theta^2}{dt^2}=\beta^2+2gr\sin\theta,$$

or
$$\frac{dr^2}{dt^2}\left(1+r^2\frac{d\theta^2}{dr^2}\right)=\beta^2+2gr\sin\theta.$$

But, by the condition of the problem, $\dfrac{dr}{dt}=C$, a constant quantity: hence

$$C^2\left(1+r^2\frac{d\theta^2}{dr^2}\right)=\beta^2+2gr\sin\theta.$$

But, initially, $\theta=0$, $r\dfrac{d\theta}{dr}=0$; hence $C^2=\beta^2$, and therefore

$$\beta^2r^2\frac{d\theta^2}{dr^2}=2gr\sin\theta;$$

$$\frac{dr}{r^{\frac{1}{2}}}=\frac{\beta}{(2g)^{\frac{1}{2}}}\,\frac{d\theta}{(\sin\theta)^{\frac{1}{2}}}:$$

integrating, and observing that $\theta=0$, $r=a$, initially; we have

$$r^{\frac{1}{2}}-a^{\frac{1}{2}}=\frac{\beta}{2\,(2g)^{\frac{1}{2}}}\int_0^\theta\frac{d\theta}{(\sin\theta)^{\frac{1}{2}}},$$

which is an equation for the construction of the path of the particle.

The particle will move from A along $ABCO$ to the point O with a uniform velocity of approach; it will afterwards move from O along $OcBa$ with a uniform velocity of recession. When it has arrived at a, it will proceed uniformly along Oa produced.

This curve has been called the Paracentric Isochrone by Leibnitz, by whom the problem was originally proposed as a challenge to the mathematicians of the day, in the *Acta Erudit. Lips.* 1689, p. 198. Several years elapsed before the problem received a solution. At length James Bernoulli succeeded in obtaining one, which appeared in the *Acta Erudit. Lips.* 1694, p. 277. Solutions were shortly afterwards published by Leib-

nitz and John Bernoulli, in the *Acta Erudit. Lips.* 1694, p. 371, 394. The problem was afterwards generalized by Varignon in the *Mémoires de l'Académie des Sciences de Paris*, 1699, p. 9 et sq.

(5) To find the nature of the curve OPA (fig. 151) such that a particle acted on by gravity will descend down any arc OP in the same time as down its chord.

Let Ox be vertical, Oy horizontal, PM parallel to yO. Let $OP=r$, $\angle xOP=\theta$, arc $OP=s$, $OM=r\cos\theta$. Then, since the velocity acquired down the arc OP is the same as that which is due to falling freely down OM,

$$\frac{ds^2}{dt^2}=2gr\cos\theta,\qquad dt=\frac{1}{(2g)^{\frac{1}{2}}}\frac{ds}{(r\cos\theta)^{\frac{1}{2}}};$$

and therefore the whole time of descent down OP is equal to

$$\frac{1}{(2g)^{\frac{1}{2}}}\int_0^s\frac{ds}{(r\cos\theta)^{\frac{1}{2}}}.$$

But the time of descent down the chord OP is equal to

$$\left(\frac{2}{g}\right)^{\frac{1}{2}}\left(\frac{r}{\cos\theta}\right)^{\frac{1}{2}};$$

and, therefore, by the hypothesis,

$$\left(\frac{2}{g}\right)^{\frac{1}{2}}\left(\frac{r}{\cos\theta}\right)^{\frac{1}{2}}=\frac{1}{(2g)^{\frac{1}{2}}}\int_0^s\frac{ds}{(r\cos\theta)^{\frac{1}{2}}},$$

$$2\left(\frac{r}{\cos\theta}\right)^{\frac{1}{2}}=\int_0^s\frac{ds}{(r\cos\theta)^{\frac{1}{2}}}.$$

Differentiating both sides of the equation, we have

$$\left(\frac{\cos\theta}{r}\right)^{\frac{1}{2}}\frac{\cos\theta\,dr+r\sin\theta d\theta}{(\cos\theta)^2}=\frac{ds}{(r\cos\theta)^{\frac{1}{2}}},$$

$$\cos\theta\,dr+r\sin\theta\,d\theta=\cos\theta\,(dr^2+r^2d\theta^2)^{\frac{1}{2}}.$$

Squaring both sides and simplifying,

$$2\sin\theta\cos\theta\,rdrd\theta=r^2\cos 2\theta\,d\theta^2,$$

$$\frac{dr}{r}=\frac{\cos 2\theta}{\sin 2\theta}d\theta.$$

Integrating,

$$\log r^2 = \log a^2 + \log \sin 2\theta, \quad r^2 = a^2 \sin 2\theta,$$

where a^2 is some constant quantity.

From O draw OE, bisecting the angle xOy, and let $\angle POE = \phi$; then, since $\theta = \frac{1}{4}\pi - \phi$, we have

$$r^2 = a^2 \cos 2\phi;$$

which is the equation to the Lemniscata of James Bernoulli, O being the centre and A the vertex of the equilateral hyperbola. This very beautiful problem is due to Saladini.

Saladini; *Memorie dell' Istituto Nazionale Italiano*, Tom. I. parte 2. Fuss; *Mémoires de l'Acad. de St. Pétersb.* 1819.

(6) To find the equation to the tautochrone when a particle is acted on by any forces whatever in one plane.

A tautochrone is a curve along which a particle acted on by any assigned forces will arrive in the same time at a given point from whatever point in the curve its motion commences. Let A (fig. 152) be any assigned point, and E any point whatever in the curve AEB; then the time from E to A is to be independent of the position of E.

Let P be any point in AE; let $AP = s$, $AE = a$, S = the sum of the resolved parts of the accelerating forces on the particle along the tangent PT at the point P. Then, for the motion of the particle,

$$\frac{d^2 s}{dt^2} = -S, \quad \frac{ds^2}{dt^2} = C - 2\int S ds.$$

But, the particle being supposed to have no initial velocity,

$$0 = C - 2\int^a S ds;$$

and therefore

$$\frac{ds^2}{dt^2} = 2\int_s^a S ds \text{..........................} (1),$$

$$dt = -\frac{1}{2^{\frac{1}{2}}} \frac{ds}{\left(\int_s^a S ds\right)^{\frac{1}{2}}}.$$

The time from E to A is equal to

$$\frac{1}{2^{\frac{1}{2}}}\int_0^a \frac{ds}{\left(\int_s^a S\,ds\right)^{\frac{1}{2}}};$$

and this formula must be independent of a. Hence we must have

$$\int \frac{ds}{\left(\int_s^a S\,ds\right)^{\frac{1}{2}}} = \phi\left(\frac{s}{a}\right),$$

where $\phi\left(\frac{s}{a}\right)$ denotes some function of $\frac{s}{a}$. Hence

$$\frac{1}{\left(\int_s^a S\,ds\right)^{\frac{1}{2}}} = \frac{1}{a}\phi'\left(\frac{s}{a}\right), \quad \int_s^a S\,ds = \frac{a^2}{\left\{\phi'\left(\frac{s}{a}\right)\right\}^2};$$

and therefore, differentiating with respect to s,

$$-S\,ds = d\,\frac{a^2}{\left\{\phi'\left(\frac{s}{a}\right)\right\}^2}.$$

But, the tautochrone AB being an invariable curve, whatever be the value of a, it is manifest that a must not appear in this equation; hence

$$\left\{\phi'\left(\frac{s}{a}\right)\right\}^2 = -\frac{a^2}{As^2}, \text{ where } A \text{ is a constant quantity,}$$

and therefore $\qquad S = ks \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$

k being some constant quantity.

Hence, by (1) and (2),

$$\frac{ds^2}{dt^2} = k\,(a^2 - s^2),$$

and therefore, if τ denote the time of the motion from E to A,

$$\tau = -\frac{1}{k^{\frac{1}{2}}}\int_a^0 \frac{ds}{(a^2 - s^2)^{\frac{1}{2}}} = \frac{\pi}{2k^{\frac{1}{2}}}, \quad k = \frac{\pi^2}{4\tau^2}.$$

Hence we have, from (2),

$$S = \frac{\pi^2 s}{4\tau^2},$$

which is a differential equation to the tautochrone.

The direct problem of Tautochronism in the case when gravity is the accelerating force, was first considered by Huyghens, in his *Horologium Oscillatorium*, where he proves the inverted cycloid with its axis vertical to be tautochronous. The inverse problem was first considered by Newton, *Princip*. Lib. I. sect. 10. See also Euler, *Comment. Petrop*. 1729, and *Mechan*. Tom. II. p. 211.

(7) A particle is acted on by an attractive force, tending towards a fixed centre and varying as the distance ; to find the tautochrone.

Let μ denote the absolute force of attraction, r the radius vector at any point of the curve, p the perpendicular from the pole upon the tangent at the point, ϕ the inclination of the tangent to the radius vector.

Then, by the formula of the preceding general problem, we have, putting $\mu r \cos\phi$ for S,

$$\mu r \cos\phi = \frac{\pi^2 s}{4\tau^2},$$

whence
$$\mu d(r\cos\phi) = \frac{\pi^2 ds}{4\tau^2};$$

but $ds\cos\phi = dr$; hence we have

$$\mu r\cos\phi\, d(r\cos\phi) = \frac{\pi^2}{4\tau^2} r dr;$$

integrating, we get

$$\mu r^2 \cos^2\phi + C = \frac{\pi^2 r^2}{4\tau^2},$$

or
$$\mu(r^2 - p^2) + C = \frac{\pi^2 r^2}{4\tau^2}.$$

Let c be the value of r when $s = 0$, and therefore when $\phi = \frac{1}{2}\pi$; then also $p = c$, and consequently

$$C = \frac{\pi^2 c^2}{4\tau^2}.$$

Hence $$\mu(r^2-p^2)+\frac{\pi^2c^2}{4\tau^2}=\frac{\pi^2r^2}{4\tau^2},$$

$$\mu p^2=\left(\mu-\frac{\pi^2}{4\tau^2}\right)r^2+\frac{\pi^2c^2}{4\tau^2},$$

which is the differential equation to the curve.

Euler; *Mechan.* Tom. II. p. 208.

(8) An infinite number of similar curves originate at a given point; to determine the corresponding synchronous curve, or the curve which shall cut them in such a manner that a particle, acted on by gravity, may describe the intercepted arcs in equal times.

Let O (fig. 153) be the given point, and CPD the synchronous curve intercepting the arc OP of the curve OPQ, which is one of the similar curves. Let Ox, a vertical line, be taken as the axis of x, and Oy, at right angles to it, as the axis of y. Let $OM=x$, $PM=y$, $OP=s$. Then, if k denote the time down OP, which by the hypothesis is constant for every point P in the curve CPD, we have

$$k=\int_0^s\frac{ds}{(2gx)^{\frac{1}{2}}}=\int_0^x\frac{(1+p^2)^{\frac{1}{2}}}{(2gx)^{\frac{1}{2}}}\,dx\,\dots\dots\dots\dots\dots(1),$$

where p is equal to $\dfrac{dy}{dx}$.

Now, by the nature of similar curves, the equation to the curve OPQ is of the form

$$F\left(\frac{x}{a},\ \frac{y}{a}\right)=0,\ \text{ or } y=af\left(\frac{x}{a}\right)\dots\dots\dots\dots\dots(2),$$

where F, f, denote certain functions of the quantities to which they are prefixed, a being the value of the general parameter of the class of similar curves for the individual curve OPQ. Hence, assuming

$$x=a\tau,\ \text{and therefore } y=af(\tau),\ \text{by (2)},\ \dots\dots\dots(3),$$

we have from (1),

$$k=a^{\frac{1}{2}}\int_0^\tau\frac{(1+T'^2)^{\frac{1}{2}}}{(2g\tau)^{\frac{1}{2}}}\,d\tau=a^{\frac{1}{2}}\phi(\tau)\,\dots\dots\dots\dots\dots(4),$$

where $T = \frac{dy}{dx} = \frac{d}{d\tau} f(\tau)$, and $\phi(\tau)$ is some function of τ. Hence, from (3) and (4), there is

$$x = \frac{k^2\tau}{\{\phi(\tau)\}^2}, \quad y = \frac{k^2 f(\tau)}{\{\phi(\tau)\}^2} \quad \text{.....................(5)}.$$

Eliminating τ between these two last equations, we shall obtain an equation in x, y, the required equation to the synchronous curve.

If the integration indicated in the equation (1) can be effected, then it is needless to have recourse to the subsidiary symbol τ. We have merely in this case to eliminate, after the performance of the integration, the parameter a, by the aid of the equation (2). It rarely happens, however, that we can execute the operation of integration, and under these circumstances the equations (5) will enable us to construct the synchronous curve by the method of quadratures; a pair of values of x, y, and therefore a point in the synchronous curve, being ascertained approximately for every numerical value which we may assign to τ.

The problem of Synchronous Curves was first discussed by John Bernoulli, in the *Act. Erudit. Lips.* 1697, Mai, p. 206. The subject was afterwards investigated by Saurin, and by Euler[1].

(9) An assemblage of circles in the plane xOy, (fig. 153), all touch Ox at the point O; to determine the synchronous curve, Ox being vertical, and gravity the accelerating force; the descent being supposed to commence from O.

The equation to any one of the circles, its radius being a, will be

$$x^2 = 2ay - y^2,$$

or

$$y = a\left\{1 - \left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}}\right\}.$$

[1] *Mechan.* Tom. II. p. 47; *Mém. de l'Acad. de St. Pétersb.* 1819—1820, pp. 20, 35.

Adopting the notation of the preceding general problem, we have

$$f(\tau) = 1 - (1 - \tau^2)^{\frac{1}{2}},$$

$$T = \frac{df(\tau)}{d\tau} = \frac{\tau}{(1 - \tau^2)^{\frac{1}{2}}},$$

$$\phi(\tau) = \int_0^\tau \frac{(1 + T^2)^{\frac{1}{2}}}{(2g\tau)^{\frac{1}{2}}} d\tau = \frac{1}{(2g)^{\frac{1}{2}}} \int_0^\tau \frac{d\tau}{(\tau - \tau^3)^{\frac{1}{2}}}.$$

Hence $$x = \frac{2gk^2\tau}{\left\{\int_0^\tau \frac{d\tau}{(\tau - \tau^3)^{\frac{1}{2}}}\right\}^2}, \quad y = \frac{2gk^2\{1 - (1 - \tau^2)^{\frac{1}{2}}\}}{\left\{\int_0^\tau \frac{d\tau}{(\tau - \tau^3)^{\frac{1}{2}}}\right\}^2},$$

whence the required curve may be constructed by the method of quadratures. Euler; *Mechan.* Tom. II. p. 52.

(10) A particle, acted on by any assigned accelerating forces in one plane, moves along a curve from one given point to another; to determine the form of the curve in order that the whole time of the motion between the two points may be the least possible.

Let P (fig. 154) be any point of the required curve; let $OM = x$, $PM = y$; let A and B be the two given points; let $AP = s$; also let α, β, be the values of x at the points A, B. Then, v being the velocity of the particle at P,

$$dt = \frac{ds}{v} = \frac{(1 + p^2)^{\frac{1}{2}}}{v} dx, \text{ where } p = \frac{dy}{dx},$$

and the whole time from A to B will be equal to

$$\int_\alpha^\beta \frac{(1 + p^2)^{\frac{1}{2}}}{v} dx \quad\quad\quad\quad (1).$$

Assume $\frac{(1 + p^2)^{\frac{1}{2}}}{v} = V$; then, in order that the expression (1) may be a minimum, we have, by the Calculus of Variations, since V involves only p and v, of which the latter is a function of only x and y,

$$N - \frac{dP}{dx} = 0 \quad\quad\quad\quad (2),$$

where N, P, denote respectively the partial differential coefficients of V with regard to y, p; $\frac{dP}{dx}$ representing the total differential coefficient of P with respect to x.

Now $$N = -\frac{1}{v^2}(1+p^2)^{\frac{1}{2}}\frac{dv}{dy} \qquad (3),$$

where $\frac{dv}{dy}$ signifies the partial differential coefficient of v with regard to y: but

$$vdv = Xdx + Ydy \qquad (4),$$

where X, Y, represent the resolved parts of the whole accelerating force on the particle parallel to Ox, Oy; and therefore, in (3), $\frac{dv}{dy} = \frac{Y}{v}$. Hence

$$N = -\frac{Y}{v^3}(1+p^2)^{\frac{1}{2}} = -\frac{Y}{v^3}\frac{ds}{dx}.$$

Again, $$P = \frac{p}{v(1+p^2)^{\frac{1}{2}}} = \frac{1}{v}\frac{dy}{ds}.$$

Hence, substituting for N and P in (2), we have

$$\frac{Y}{v^3}\frac{ds}{dx} + \frac{d}{dx}\left(\frac{1}{v}\frac{dy}{ds}\right) = 0,$$

$$\frac{Y}{v^3}\frac{ds}{dx} - \frac{1}{v^2}\frac{dv}{dx}\frac{dy}{ds} + \frac{1}{v}\frac{d}{dx}\frac{dy}{ds} = 0:$$

but, from (4), $$\frac{dv}{dx} = \frac{1}{v}\left(X + Y\frac{dy}{dx}\right):$$

hence $$\frac{Y}{v^3}\frac{ds}{dx} - \frac{1}{v^3}\left(X + Y\frac{dy}{dx}\right)\frac{dy}{ds} + \frac{1}{v}\frac{d}{dx}\frac{dy}{ds} = 0;$$

and therefore, after a few obvious simplifications,

$$v^2\frac{d}{dx}\frac{dy}{ds} = X\frac{dy}{ds} - Y\frac{dx}{ds} \qquad (5).$$

If from this equation we eliminate v by the aid of (3), we shall obtain a differential equation of the second order, which is the equation to the required curve. The two arbitrary constants

introduced by the integration are to be determined from the conditions that the curve shall pass through the two given points A and B.

The equation (5) is equivalent to

$$\frac{v^2}{\rho} = Y\frac{dx}{ds} - X\frac{dy}{ds},$$

where ρ denotes the radius of curvature at the point P; a result which shews that the pressure on the curve due to the centrifugal force is equal to that which arises from the accelerating forces which act upon the particle.

The curve in question belongs to a class of mechanical curves called Brachistochrones, which are characterized by the general property that a particle, under the action of assigned accelerating forces, shall move along them between given limits in the least time possible.

The problem of the Brachistochrone between two given points, when gravity is the accelerating force, was proposed by John Bernoulli[1], as a challenge to the mathematicians of the day. Six months was the time allotted for its solution. Leibnitz[2] was immediately successful, and communicated his good fortune by letter to Bernoulli. No other solution however having made its appearance within the prescribed time, Bernoulli, in conformity with the desire of Leibnitz, consented to prorogue the term of the challenge to the following Easter, the results obtained by himself and Leibnitz being suppressed for that interval. A programme was accordingly published at Groningen, in January 1697, again announcing the problem and repeating the challenge. In consequence of this delay solutions were obtained by three other mathematicians: by Newton[3], anonymously; by James Bernoulli[4]; and by L'Hôpital[5]. The solution of Leibnitz was announced in the *Acta Erudit. Lips.* Mai. 1697, p. 203. The

[1] *Act. Erudit. Lips.* 1696, Jun. p. 269.
[2] *Commerc. Epistol.* Leibnitii et Bernoullii, Epist. 28.
[3] *Phil. Trans.* 1697, Num. 224, p. 389.
[4] *Act. Erudit. Lips.* Mai. 1697, p. 212.
[5] *Act. Erudit. Lips.* ib. 217.

conclusions of Newton, Leibnitz, and L'Hôpital were given without the analysis. John Bernoulli gave two different solutions, one direct and the other indirect. The latter was published in the *Acta Erudit. Lips.* Mai. 1697, p. 207; the former was not made public till the year 1718, in a Memoir on Isoperimetrical problems, in the *Mémoires de l'Académie des Sciences de Paris*, p. 136; see also his works, Tom. II. p. 266. A solution of the problem was afterwards given by Craig[1], who had merely seen Newton's result without consulting the analysis which had been given by John and James Bernoulli.

(11) To find the brachistochrone when a particle, acted on by a central attractive force, which varies inversely as the square of the distance, moves along a curve from one given point to another.

Let A (fig. 155) be the point where the motion commences, and B the point at which the particle is to arrive in the shortest time possible. Let P be any point in the brachistochrone, S the centre of force; let $SP = r$, $p =$ the perpendicular from S upon the tangent at P, $SA = a$, $\phi =$ the angle between SP and the tangent at P, $\mu =$ the absolute force of attraction, $v =$ the velocity, and $\rho =$ the radius of curvature at P. Then, since the pressure on the curve due to the centrifugal force must be equal to that due to the attraction, we have

$$\frac{v^2}{\rho} = \frac{\mu}{r^2} \sin \phi \qquad\qquad (1).$$

But
$$v^2 = C - 2 \int \frac{\mu}{r^2}\, dr = C + \frac{2\mu}{r},$$

or, since $v = 0$ when $r = a$,

$$v^2 = 2\mu \left(\frac{1}{r} - \frac{1}{a}\right) \qquad\qquad (2).$$

Also, the curve being convex towards S,

$$\rho = -r \frac{dr}{dp} \qquad\qquad (3).$$

[1] *Phil. Trans.* 1701, Vol. XXII. p. 746.

From (1), (2), (3), we have

$$\frac{2\mu\left(\frac{1}{a}-\frac{1}{r}\right)}{r\frac{dr}{dp}}=\frac{\mu}{r^2}\frac{p}{r},$$

$$2\left(\frac{1}{a}-\frac{1}{r}\right)=\frac{p}{r^2}\frac{dr}{dp},\quad 2\frac{dp}{p}=\frac{adr}{r(r-a)};$$

integrating, we get

$$2\log p=\log C+\log\frac{r-a}{r},$$

$$\log p^2=\log\left(C\frac{r-a}{r}\right);$$

but C must be a negative quantity, because, as will appear from the equation (2), a is greater than r; hence, putting $-A$ for C, we have, for the differential equation to the brachistochrone,

$$p^2=A\frac{a-r}{r}.$$

If from this equation we were to obtain, by integration, a relation between r and an angular co-ordinate θ, we should introduce another constant into the equation in addition to A. Both these constants would have to be determined by the conditions that the curve must pass through both A and B.

Euler; *Mechan.* Tom. II. p. 191.

(12) To find the inclination of a thin tube to the horizon, in order that a descending particle may describe the greatest horizontal space in a given time.

The required angle of inclination = 45°.

(13) A particle, having been placed at the point A, (fig. 156), moves along a thin tube APS towards a centre of attractive force at S, which varies as any function of the distance; to find the form of the tube in order that the time through any arc AP may be n times as great as through a portion Ap of the prime radius vector SA, Sp being equal to SP.

Let $SP=r$, $SA=a$, $\angle ASP=\theta$; then the equation to the curve APS will be

$$r=a\,.\,e^{-\frac{\theta}{(n^2-1)^{\frac{1}{2}}}}.$$

(14) A particle is projected with a given velocity from a point A (fig. 157) along a curve APO in which it is constrained to move, and is acted upon by a force always tending to O, and varying directly as the distance; to find the nature of this curve in order that the angular velocity of the radius vector OP may be invariable.

Let $AO=a$, $OP=r$, $\angle AOP=\theta$, $\mu^2=$ the absolute force of attraction, $\omega=$ the angular velocity of OP, $\beta=$ the initial velocity of the particle; then the equation to the curve will be

$$\left(\frac{\mu^2+\omega^2}{\omega^2}\right)^{\frac{1}{2}}\theta=\cos^{-1}\left\{\left(\frac{\mu^2+\omega^2}{\mu^2a^2+\beta^2}\right)^{\frac{1}{2}}r\right\}-\cos^{-1}\left\{\left(\frac{\mu^2+\omega^2}{\mu^2a^2+\beta^2}\right)^{\frac{1}{2}}a\right\}.$$

Euler; *Mechan.* Tom. II. p. 138.

(15) A particle is acted on by an attractive force, tending to a centre and varying inversely as the square of the distance; to find the tautochrone.

If τ denote the time of the motion, and the notation remain the same as in problem (7), the differential equation to the tautochrone will be

$$p^2=r^2-\frac{\pi^2}{2\mu c\tau^2}(r-c)r^3.$$

Euler; *Mechan.* Tom. II. p. 209.

(16) To find the tautochrone when the central attractive force is constant.

If f denote the constant central force, the equation to the tautochrone will be

$$p^2=\left(1+\frac{\pi^2c}{2f\tau^2}\right)r^2-\frac{\pi^2}{2f\tau^2}r^3.$$

Euler; *Mechan.* Tom. II. p. 210.

(17) An infinite number of straight lines originate at a single point and lie in one plane; to determine the synchronous curve, gravity being the accelerating force.

The given point being taken as the origin of co-ordinates, the axis of x extending vertically downwards, and that of y being horizontal; the synchronous curve will be a circle of which the equation is

$$x^2 + y^2 = \tfrac{1}{2}gk^2x,$$

where k denotes the common time of descent.

Euler; *Mém. de l'Acad. de St. Pétersb.* 1819, 1820, p. 22.

(18) There is an infinite number of cycloids, of which the bases all commence at the origin of co-ordinates, and coincide with the axis of y, which is horizontal; to find the synchronous curve, gravity being the accelerating force, and the motion commencing from the origin.

Let k denote the constant time of descent; then, the axis of x being vertical, the equation to the required curve depends upon the elimination of a between the two equations

$$x = a \text{ vers} \left\{k\left(\frac{g}{a}\right)^{\frac{1}{2}}\right\}, \qquad y = a \text{ vers}^{-1}\frac{x}{a} - (2ax - x^2)^{\frac{1}{2}},$$

and will cut all the cycloids at right angles.

John Bernoulli; *Act. Erudit. Lips.* 1697, Mai. p. 206.

(19) A particle, acted on by a central attractive force, which varies as the distance, moves along a curve from one given point to another; to find the nature of the curve when it is brachistochronous.

Let A (fig. 155) be the point at which the motion commences, and B the point at which the particle is to arrive in the shortest time possible. Let P be any point of the brachistochrone; let $SP = r$, $p =$ the perpendicular from S, the centre of force, upon the tangent at P, $SA = a$. Then the equation to the curve between p and r will be

$$p^2 = A(r^2 - a^2),$$

where A is a constant quantity, which is the equation to the hypocycloid.

If from this equation we were to obtain by integration a relation between r and an angular co-ordinate θ, we should have

another constant in the equation in addition to A. Both these constants would have to be determined by the conditions that the curve must pass through both A and B.

Euler; *Mechan.* Tom. II. p. 191.

SECT. 5. *Inverse Problems on the Pressure of a Particle on Smooth Fixed Curves.*

(1) A particle descends down a curve line in a vertical plane by the action of gravity; to find the nature of the curve in order that the pressure may be invariable.

Let OA (fig. 158) be the required curve; Ox, vertical, the axis of x, Oy, horizontal, the axis of y; P any point in the curve; let $OM = x$, $PM = y$, $OP = s$; let k be the constant pressure; β the initial velocity of the particle, O being its initial position. Then, by formula (A) of Sect. (II.), we have

$$k = g\frac{dy}{ds} + \frac{1}{\rho}\frac{ds^2}{dt^2} \qquad \text{(1)},$$

where ρ denotes the magnitude of the radius of curvature at P.

But $$\frac{ds^2}{dt^2} = \beta^2 + 2gx:$$

also, s being taken as the independent variable,

$$\frac{1}{\rho} = \frac{\dfrac{d^2y}{ds^2}}{\dfrac{dx}{ds}}:$$

hence, from (1), we have

$$k = g\frac{dy}{ds} + (2gx + \beta^2)\frac{\dfrac{d^2y}{ds^2}}{\dfrac{dx}{ds}},$$

$$\frac{k}{(2gx+\beta^2)^{\frac{1}{2}}}\frac{dx}{ds} = (2gx+\beta^2)^{\frac{1}{2}}\frac{d^2y}{ds^2} + \frac{g}{(2gx+\beta^2)^{\frac{1}{2}}}\frac{dy}{ds}\frac{dx}{ds}.$$

Integrating, we have

$$\frac{k}{g}(2gx+\beta^2)^{\frac{1}{2}}=(2gx+\beta^2)^{\frac{1}{2}}\frac{dy}{ds}+C,$$

$$\frac{dy}{ds}=\frac{k}{g}-\frac{C}{(2gx+\beta^2)^{\frac{1}{2}}},$$

where C is an arbitrary constant. Assume α to be the inclination of the curve to the vertical at the origin; then

$$\sin\alpha=\frac{k}{g}-\frac{C}{\beta};$$

and therefore,

$$\frac{dy}{ds}=\frac{k}{g}-\frac{\beta}{g}\frac{k-g\sin\alpha}{(2gx+\beta^2)^{\frac{1}{2}}}\ \dots\dots\dots\dots\dots\dots\ (2).$$

The relation between x and y may be obtained by a second integration, but the result is of little value in consequence of its complexity. For the investigation of the form of the curve which corresponds to the differential equation (2), the reader is referred to Whewell's *Dynamics*, part II. p. 95; or Earnshaw's *Dynamics*, p. 129.

The problem of the Curve of Equal Pressure, in the case of gravity, was first proposed by John Bernoulli[1], and solved by L'Hôpital[2]. Various problems of a similar character were afterwards discussed by Varignon[3].

Commerc. Epistolic. Leibnitii et Bernoullii, Epist. VII.

(2) A particle, acted on by gravity, descends from a point O (fig. 158) down a curve OA, which it presses at each point of its descent with a force varying as the square of its distance below the horizontal line through O; to find the nature of the curve OA, the initial velocity of the particle being zero.

Let the axes Ox, Oy, be taken vertical and horizontal; let k be the pressure on the curve when x is equal to unity. Then, by the formula (A) of Sect. (II.),

[1] *Act. Erudit. Suppl.* Tom. II. sect. vi. p. 291.
[2] *Mém. de l'Acad. des Sciences de Paris*, 1700, p. 9.
[3] *Mém. de l'Acad. des Sciences de Paris*, 1710, p. 196.

$$kx^2 = g\frac{dy}{ds} + \frac{2gx}{\rho};$$

but
$$\rho = -\frac{\left(1 + \frac{dx^2}{dy^2}\right)^{\frac{3}{2}}}{\frac{d^2x}{dy^2}};$$

hence
$$kx^2 = \frac{g}{\left(1 + \frac{dx^2}{dy^2}\right)^{\frac{1}{2}}} - \frac{2gx\frac{d^2x}{dy^2}}{\left(1 + \frac{dx^2}{dy^2}\right)^{\frac{3}{2}}},$$

$$kx^{\frac{3}{2}}\frac{dx}{dy} = \frac{gx^{-\frac{1}{2}}\frac{dx}{dy}}{\left(1 + \frac{dx^2}{dy^2}\right)^{\frac{1}{2}}} - \frac{2gx^{\frac{1}{2}}\frac{dx}{dy}\frac{d^2x}{dy^2}}{\left(1 + \frac{dx^2}{dy^2}\right)^{\frac{3}{2}}}:$$

integrating, we have
$$\tfrac{2}{5}kx^{\frac{5}{2}} = \frac{2gx^{\frac{1}{2}}}{\left(1 + \frac{dx^2}{dy^2}\right)^{\frac{1}{2}}},$$

no constant being added because the curve passes through the origin. Putting $\frac{5g}{k} = a^2$, we get
$$x^2\left(1 + \frac{dx^2}{dy^2}\right)^{\frac{1}{2}} = a^2,$$
$$(a^4 - x^4)^{\frac{1}{2}}\,dy = x^2dx,$$
which is the equation to the Elastic Curve of James Bernoulli[1].

Varignon; *Mémoires de l'Académie des Sciences de Paris*, 1710, p. 151.

(3) A particle, acted upon by a force parallel to the axis of x, is constrained to move along a given curve OPA (fig. 158); to find the law of the force in order that the curve may experience an invariable pressure.

Let k denote the constant pressure, β the velocity of the particle at O, which we will take as the origin of co-ordinates,

[1] *Act. Erudit. Lips.* 1694, p. 272; 1695, p. 538.

and X the force, at any point P of the curve, parallel to Ox. Then, by formula (A) of Section (II.) and formula (D) of Section (I.), we have

$$k = X\frac{dy}{ds} + \frac{1}{\rho}\{\beta^2 + 2\int_0^x Xdx\}.$$

Taking s as the independent variable, we have

$$\frac{1}{\rho} = \frac{\dfrac{d^2y}{ds^2}}{\dfrac{dx}{ds}};$$

and the equation becomes

$$k\frac{dx}{ds} = \beta^2\frac{d^2y}{ds^2} + X\frac{dx}{ds}\frac{dy}{ds} + 2\frac{d^2y}{ds^2}\int_0^x Xdx.$$

Multiplying by $\dfrac{dy}{dx}ds$, and integrating

$$k\int\frac{dx}{ds}\frac{dy}{ds}ds = \tfrac{1}{2}\beta^2\frac{dy^2}{ds^2} + \frac{dy^2}{ds^2}\int_0^x Xdx,$$

$$\int_0^x Xdx = -\tfrac{1}{2}\beta^2 + \frac{k}{\dfrac{dy^2}{ds^2}}\int\frac{dx}{ds}\frac{dy}{ds}ds;$$

and therefore, putting $\dfrac{dy}{dx} = p$,

$$\int_0^x Xdx = -\tfrac{1}{2}\beta^2 + k\left(\frac{1}{p^2}+1\right)\left\{\int\frac{pdx}{(1+p^2)^{\frac{1}{2}}} + C\right\}\text{............(1)}.$$

Differentiating with respect to x, we obtain the required expression for the force

$$X = \frac{k}{p}(1+p^2)^{\frac{1}{2}} - \frac{2k}{p^3}\frac{dp}{dx}\left\{\int\frac{pdx}{(1+p^2)^{\frac{1}{2}}} + C\right\}\text{............(2)}.$$

If we put $x = 0$, we have, from (1),

$$2k\left(\frac{1}{p^2}+1\right)\left\{\int\frac{pdx}{(1+p^2)^{\frac{1}{2}}} + C\right\} = \beta^2,$$

a condition which will determine the value of the arbitrary constant C.

Euler; *Mechan.* Tom. II. p. 101.

(4) A particle moves along a parabola OA, (fig. 158), of which the axis is Oy, under the action of a force always parallel to Ox, which is at right angles to Oy; to determine the law of the force in order that the particle may exert the same pressure on the curve during the whole of its motion.

Let Ox, Oy, be the co-ordinate axes, k the constant pressure, and $x^2 = ay$ the equation to the parabola. Then, by the formula for X given in the preceding problem, since $p = \dfrac{2x}{a}$, we have

$$X = \frac{ka}{2x}\left(1 + \frac{4x^2}{a^2}\right)^{\frac{1}{2}} - \frac{ka^2}{2x^3}\left\{\int \frac{2x\,dx}{(a^2 + 4x^2)^{\frac{1}{2}}} + C\right\}$$

$$= \frac{k}{2x}(a^2 + 4x^2)^{\frac{1}{2}} - \frac{ka^2}{2x^3}\{\tfrac{1}{2}(a^2 + 4x^2)^{\frac{1}{2}} + C\},$$

where C is a constant quantity,

$$= -\frac{ka^2}{2x^3}C - \frac{k}{4x^3}(a^2 - 2x^2)\cdot(a^2 + 4x^2)^{\frac{1}{2}} \ldots\ldots\ldots\ldots\ldots (1).$$

Again, by the formula (1) in the preceding problem,

$$\int_0^x X\,dx = -\tfrac{1}{2}\beta^2 + k\left(\frac{a^2}{4x^2} + 1\right)\{\tfrac{1}{2}(a^2 + 4x^2)^{\frac{1}{2}} + C\},$$

$$4x^2\int_0^x X\,dx = -2\beta^2x^2 + k(a^2 + 4x^2)\{\tfrac{1}{2}(a^2 + 4x^2)^{\frac{1}{2}} + C\}:$$

hence, putting $x = 0$, we get

$$0 = \tfrac{1}{2}a + C,$$

and therefore, by (1),

$$X = \frac{ka^3}{4x^3} - \frac{k}{4x^3}(a^2 - 2x^2)(a^2 + 4x^2)^{\frac{1}{2}}.$$

Euler; *Mechan.* Tom. II. p. 103.

(5) A particle descends from rest, under the action of gravity, from a point O down a curve OA, (fig. 159), which it presses, at each point of its descent, with a force varying as its perpendicular distance from the horizontal line through O; to find the nature of the curve OA.

Take Ox, Oy, the axes of co-ordinates, vertical and horizontal; let k be the pressure on the curve when x is equal to unity; then, putting $a = \frac{g}{k}$, the equation to the curve will be

$$x^2 = 6ay - y^2,$$

the equation to a circle of which OE the diameter is equal to $6a$.

Varignon; *Mém. de l'Acad. des Sciences de Paris*, 1710, p. 151.

(6) To find the curve when the pressure varies as the square root of the distance.

The equation to the curve is

$$y = 2a \operatorname{vers}^{-1} \frac{x}{2a} - (4ax - x^2)^{\frac{1}{2}},$$

which belongs to a cycloid OBA, (fig. 160), the radius of the generating circle being $2a$.

Varignon; *Ib.* p. 152.

(7) A particle, acted on by gravity, descends from rest down a curve; to find the nature of the curve in order that the pressure at any point due to the centrifugal force may vary as any positive power of the distance of the particle below the horizontal line passing through its initial position, the tangent to the curve at the initial position of the particle being supposed to coincide with the horizontal line.

Let k denote the pressure due to centrifugal force when x is equal to unity, the axis of x being vertical, as in fig. 158; then the differential equation to the curve will be, putting $\frac{g}{k} = a$,

$$(4n^2a^2 - x^{2n})^{\frac{1}{2}}\, dy = x^n dx.$$

Varignon; *Ib.* p. 156.

(8) To find the curve when the part of the pressure which is due to gravity varies as the n^{th} power of the depth of the descent, the tangent to the curve at the initial position of the particle being horizontal, and n being a positive quantity.

The notation being the same as in the preceding problem,

except that k denotes the pressure due to gravity alone when $x=1$, the differential equation to the curve will be

$$(a^2 - x^{2n})^{\frac{1}{2}}\, dy = x^n dx.$$

Varignon; *Ib.* p. 160.

(9) A particle descends from rest by the action of gravity down a curve line; to determine the nature of the curve when the part of the pressure due to centrifugal force bears a constant ratio to that due to gravity, the tangent to the curve at the initial position of the particle being horizontal.

Let Oy (fig. 158) be horizontal and Ox vertical; then, if $\frac{m}{n}$ denote the constant ratio, the differential equation to the curve will be

$$(a^{\frac{m}{n}} - x^{\frac{m}{n}})^{\frac{1}{2}}\, dy = x^{\frac{m}{2n}}\, dx,$$

where a is a constant quantity. If $m=n$, the curve will be an inverted cycloid of which the base is horizontal.

Varignon; *Ib.* p. 161.

(10) A particle, acted on by a force parallel to Ox, (fig. 160) moves from rest along the arc OB of a cycloid; to determine this force in order that the curve may always experience the same pressure.

If k denote the constant pressure, a the radius of the generating circle, X the required force, and $Om=x$; then

$$X = \frac{k}{3}\left(\frac{2a}{x}\right)^{\frac{1}{2}}.$$

Euler; *Mechan.* Tom. II. p. 104.

(11) A particle is projected along a smooth groove from a point which is half way between two centres of force of equal intensity, each varying inversely as the distance: to find what the form of the groove must be in order that the particle may move uniformly.

If a denote the initial distance of the particle from each centre, and r, r', the distances of any point in the curve from the two centres,

$$r \,.\, r' = a^2.$$

SECT. 6. *Motion of Particles acted on by smooth constraining lines moveable according to assigned geometrical conditions.*

Let Ox (fig. 161) be the axis of x, and Oy, at right angles to it, that of y. Let P be the position of the particle in the plane xOy at any time t from the commencement of the motion; let $OM = x$, $PM = y$. Let X, Y, denote the resolved parts of the accelerating forces on the particle parallel to the axes of co-ordinates, and X', Y', the resolved parts of the force of constraint at any point of its path. Then the circumstances of its motion will depend upon the differential equations

$$\frac{d^2x}{dt^2} = X + X', \quad \frac{d^2y}{dt^2} = Y + Y' \ldots\ldots\ldots\ldots\text{(A)}.$$

The complete solution of the general problem of the motion of the particle consists in the determination of x and y in terms of t. But the equations (A) involve, in addition to x and y, the two unknown quantities X' and Y'. Hence it appears that the general consideration of the motion affords us only two equations involving four unknown quantities. From this it is clear that the analytical expression of the conditions to which the motion of the constraining line in any particular problem is subject, must be virtually equivalent to two more equations involving only x, y, X', Y'.

Let r be the distance PO of the particle at the time t from the origin of co-ordinates, and let $\angle POx = \theta$; then, in case the action of the constraining line always takes place in a direction at right angles to OP, and F denote the sum of the resolved parts of the accelerating forces on the particle along OP, we may obtain from the equations (A) the formula

$$\frac{d^2r}{dt^2} - r\frac{d\theta^2}{dt^2} = F \ldots\ldots\ldots\ldots\ldots\ldots\text{(B)}.$$

The formula (B) was given by Ampère, *Annales de Gergonne*, Tom. XX. p. 37 et sq.

(1) A particle is fastened to one end of a straight thread which is supported on a smooth horizontal plane: the other end of the thread is constrained to move along the plane with a uniform velocity in a given straight line: to find the path of a particle.

Let Ox (fig. 162) be the given straight line, Oy a straight line, in the horizontal plane, at right angles to Ox: let P be the position of the particle and PQ of the thread at any time t. Draw PM at right angles to Ox. Let $OM=x$, $PM=y$, $OQ=x'$, $PQ=h$, $\angle PQO=\theta$. Then, R denoting the tension of the thread, we have, by the equations (A), since $X'=\frac{R}{m'}\cos\theta$, $Y'=-\frac{R}{m'}\sin\theta$, where m' denotes the mass of the particle,

$$\frac{d^2x}{dt^2}=\frac{R}{m'}\cos\theta, \qquad \frac{d^2y}{dt^2}=-\frac{R}{m'}\sin\theta.$$

Hence, eliminating R,

$$\sin\theta\frac{d^2x}{dt^2}+\cos\theta\frac{d^2y}{dt^2}=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (1):$$

but

$$x'=x+h\cos\theta=mt+n \;\ldots\ldots\ldots\ldots\ldots\ldots\; (2),$$

where m denotes the uniform velocity of Q, and n its initial distance from O: hence

$$\frac{d^2x}{dt^2}+h\frac{d^2}{dt^2}\cos\theta=0.$$

Also, by the geometry,

$$y=h\sin\theta \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (3),$$

and therefore

$$\frac{d^2y}{dt^2}=h\frac{d^2}{dt^2}\sin\theta:$$

hence, from (1),

$$\cos\theta\frac{d^2}{dt^2}\sin\theta-\sin\theta\frac{d^2}{dt^2}\cos\theta=0;$$

integrating, we obtain

$$\cos\theta\frac{d}{dt}\sin\theta-\sin\theta\frac{d}{dt}\cos\theta=\omega, \quad \text{or } \frac{d\theta}{dt}=\omega,$$

where ω is a constant quantity, which shews that the angular velocity of the thread PQ about Q is invariable.

Integrating again, $\theta = \alpha + \omega t$ ....(4),
α being the initial value of θ.

Hence we have, from (2) and (3),

$$x = mt + n - h\cos(\alpha + \omega t),$$
$$y = h\sin(\alpha + \omega t) \quad\text{..........................(5)},$$

which give the values of x and y at any time during the motion.

Eliminating t, we get, as the equation to the path of the particle in rectangular co-ordinates,

$$x = \frac{m}{\omega}\sin^{-1}\frac{y}{h} - (h^2 - y^2)^{\frac{1}{2}} + n - \frac{m\alpha}{\omega}.$$

The equations (2) and (4) however furnish us with the most convenient conception of the motion of the particle. In fact they shew that P's motion may be perfectly represented by supposing it to move with a uniform velocity ωh in the circumference of a circle of which the radius is h, and of which the centre moves along Ox with a uniform velocity m. The path of P is therefore a trochoid.

From (5), we have, by differentiation,

$$\frac{d^2y}{dt^2} = -h\omega^2\sin(\alpha - \omega t).$$

Hence, from (4), and the original equations of motion,

$$-h\omega^2\sin(\alpha + \omega t) = -\frac{R}{m'}\sin(\alpha + \omega t),$$

and therefore $\qquad \dfrac{R}{m'} = h\omega^2, \quad R = m'h\omega^2,$

which shews that the tension of the string is invariable.

This is an example of a class of curves called Tractories, which are traced by a material particle attached to one extremity of a string while the other is constrained to move along some assigned curve with a given velocity. The curve in which the end of the string is constrained to move is called the Directrix.

This problem formed the subject of a controversy between Fontaine and Clairaut; the solution given by Fontaine depended upon the assumption that the string would be always a tangent to the path of the particle, an hypothesis which Clairaut declared to be erroneous, and which, in fact, virtually involves a neglect of inertia. Fontaine's assumption would be admissible for the motion of a particle on a perfectly rough plane, where its motion would be destroyed the moment it was generated.

Clairaut; *Mémoires de l'Académie des Sciences de Paris*, 1736, p. 4. Euler; *Nova Acta Acad. Petrop.* 1784.

(2) A thin rectilinear tube, one point of which is fixed, is constrained to move in a horizontal plane with a uniform angular velocity: to find the motion of a particle sliding freely within the tube.

Let r denote the distance of the particle from the fixed point of the tube, and θ the angle through which the tube has revolved at the end of a time t: then by the formula (B), since no accelerating forces act on the particle,

$$\frac{d^2r}{dt^2} = r\frac{d\theta^2}{dt^2} = \omega^2 r,$$

where ω denotes the angular velocity of the tube.

The integral of this equation is

$$r = A\epsilon^{\omega t} + B\epsilon^{-\omega t}, \qquad (1),$$

where A and B are constants.

Let a, β, be the initial values of r, $\frac{dr}{dt}$; then, from (1),

$$a = A + B,$$

and

$$\beta = A\omega - B\omega;$$

and therefore

$$A = \frac{1}{2\omega}(\omega a + \beta),$$

$$B = \frac{1}{2\omega}(\omega a - \beta).$$

Hence, putting the values of A and B in (1), we see that

$$2\omega r = (\omega a + \beta)\,\epsilon^{\omega t} + (\omega a - \beta)\,\epsilon^{-\omega t},$$

which gives the value of r at any time during the motion.

The equation to the path of the particle is, putting $\theta = \omega t$,

$$2\omega r = (\omega a + \beta)\,\epsilon^{\theta} + (\omega a - \beta)\,\epsilon^{-\theta}.$$

John Bernoulli; *Opera*, Tom. IV. p. 248. Clairaut; *Mém. Acad. Paris*, 1742, p. 10.

(3) A particle is placed within a thin circular tube, which is constrained to revolve with a uniform angular velocity in a horizontal plane about a point in its circumference: to investigate the motion of the particle.

Let O (fig. 163) be the point about which the circle APO is constrained to revolve; C its centre at any time t, and P the position of the particle; R the action of the circle on the particle, which will take place in the direction PC. Let Ox, Oy, be the axes of co-ordinates, x, y, being the co-ordinates of P.

Let $\angle\, POx = \theta$, $\angle\, OPC = \phi = \angle\, COP$, $OP = r$, $OC = a$.

Then, since no accelerating forces act on the particle, we have, by the formulæ (A),

$$\frac{d^2x}{dt^2} = -R\cos(\theta - \phi),$$

$$\frac{d^2y}{dt^2} = -R\sin(\theta - \phi):$$

multiplying these equations by $\sin(\theta - \phi)$, $\cos(\theta - \phi)$, and subtracting, we have

$$\sin(\theta - \phi)\frac{d^2x}{dt^2} - \cos(\theta - \phi)\frac{d^2y}{dt^2} = 0,$$

or, since $x = r\cos\theta$, $y = r\sin\theta$,

$$\sin(\theta - \phi)\frac{d^2}{dt^2}(r\cos\theta) - \cos(\theta - \phi)\frac{d^2}{dt^2}(r\sin\theta) = 0.$$

But, from the geometry, it is evident that

$$r = 2a\cos\phi:$$

hence

$$\sin(\theta-\phi)\frac{d^2}{dt^2}(\cos\theta\cos\phi)-\cos(\theta-\phi)\frac{d^2}{dt^2}(\sin\theta\cos\phi)=0,$$

$$\sin(\theta-\phi)\frac{d^2}{dt^2}\{\cos(\theta+\phi)+\cos(\theta-\phi)\}$$

$$-\cos(\theta-\phi)\frac{d^2}{dt^2}\{\sin(\theta+\phi)+\sin(\theta-\phi)\}=0.$$

But, supposing ω to be the angular velocity of the diameter OCA of the circle about O, and $\angle AOx$ to be initially zero, it is clear that

$$\angle AOx \text{ or } \theta+\phi=\omega t \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Hence, putting ωt for $\theta+\phi$,

$$\sin(\theta-\phi)\frac{d^2}{dt^2}\cos(\theta-\phi)-\cos(\theta-\phi)\frac{d^2}{dt^2}\sin(\theta-\phi)$$

$$+\sin(\theta-\phi)\frac{d^2}{dt^2}\cos\omega t-\cos(\theta-\phi)\frac{d^2}{dt^2}\sin\omega t=0,$$

$$\frac{d}{dt}\left\{\sin(\theta-\phi)\frac{d}{dt}\cos(\theta-\phi)-\cos(\theta-\phi)\frac{d}{dt}\sin(\theta-\phi)\right\}$$

$$-\omega^2\sin(\theta-\phi)\cos\omega t+\omega^2\cos(\theta-\phi)\sin\omega t=0,$$

$$-\frac{d^2}{dt^2}(\theta-\phi)-\omega^2\sin(\theta-\phi-\omega t)=0.$$

But, by (1), we have $\theta=\omega t-\phi$; hence

$$2\frac{d^2\phi}{dt^2}+\omega^2\sin 2\phi=0.$$

Multiplying by $2\dfrac{d\phi}{dt}$, and integrating,

$$2\frac{d\phi^2}{dt^2}-\omega^2\cos 2\phi=C \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

For the sake of simplicity we will suppose that, initially, P coincides with A, and that its velocity is zero; hence, when $t=0$, we have $\phi=0$, and since, from (1),

$$\frac{d\theta}{dt}+\frac{d\phi}{dt}=\omega,$$

we have also initially $\frac{d\phi}{dt} = \omega$. Hence, from (2),

$$\omega^2 = C,$$

and therefore
$$2\frac{d\phi^2}{dt^2} - \omega^2 \cos 2\phi = \omega^2,$$

$$2\frac{d\phi^2}{dt^2} = \omega^2 (1 + \cos 2\phi) = 2\omega^2 \cos^2 \phi,$$

$$\frac{d\phi}{dt} = \omega \cos \phi,$$

$$\omega dt = \frac{d\phi}{\cos\phi} = \frac{\cos\phi d\phi}{1 - \sin^2\phi}.$$

Integrating,
$$\log\frac{1+\sin\phi}{1-\sin\phi} = 2\omega t,$$

no constant being added because $\phi = 0$ when $t = 0$.

From this equation we have

$$\frac{1+\sin\phi}{1-\sin\phi} = \epsilon^{2\omega t}, \quad \sin\phi = \frac{\epsilon^{\omega t} - \epsilon^{-\omega t}}{\epsilon^{\omega t} + \epsilon^{-\omega t}} = \frac{\epsilon^{\psi} - \epsilon^{-\psi}}{\epsilon^{\psi} + \epsilon^{-\psi}},$$

where ψ is equal to $\angle AOx$.

When $t = \infty$, $\sin\phi = 1$, and therefore $\phi = \frac{\pi}{2}$, a value towards which ϕ indefinitely tends as its limit. Thus it appears that after an infinite time the particle will arrive at the point O.

Again, since $r = 2a\cos\phi$, we may readily get

$$r = \frac{4a}{\epsilon^{\omega t} + \epsilon^{-\omega t}} = \frac{4a}{\epsilon^{\psi} + \epsilon^{-\psi}},$$

which gives the distance of the particle from O at any time during the motion.

From the above equations we may obtain for the pressure on the circle, corresponding to any position of the particle,

$$R = 2\omega^2 a \cos\phi\,(3\cos\phi - 2).$$

(4) Two particles P and Q (fig. 164) are connected together by an inflexible rod PQ without weight; P is capable of moving

along a smooth horizontal groove Ox, and Q may move any where upon a smooth horizontal plane passing through the groove: having given the initial circumstances of the particles, to determine their motions at any time after the commencement of the motion.

Let T be the tension of the rod at any time t; θ the inclination of the rod to the line xO; let $ON=x'$, $QN=y'$, where O is an assigned point in Ox, and QN at right angles to ON; $OP=x$; $\omega=$ the initial angular velocity of Q about P, $\beta=$ the initial velocity of P, $\alpha=$ the initial value of θ; let m, m', be the masses of P, Q; a the length of the rod.

For the motion of P there is

$$m\frac{d^2x}{dt^2}=-T\cos\theta \text{.....................} (1);$$

and, for the motion of Q,

$$m'\frac{d^2x'}{dt^2}=T\cos\theta \text{.........................} (2),$$

$$m'\frac{d^2y'}{dt^2}=-T\sin\theta \text{.......................} (3).$$

Adding together the equations (1) and (2),

$$m\frac{d^2x}{dt^2}+m'\frac{d^2x'}{dt^2}=0 \text{.....................} (4).$$

Multiplying (1) by $\sin\theta$, (3) by $\cos\theta$, and subtracting the latter of the resulting equations from the former,

$$m\sin\theta\frac{d^2x}{dt^2}-m'\cos\theta\frac{d^2y'}{dt^2}=0 \text{..............} (5).$$

Again, it is evident that

$$x'=x-a\cos\theta \text{........................} (6),$$

$$y'=a\sin\theta \text{..........................} (7).$$

From (4) and (6) we have

$$(m+m')\frac{d^2x}{dt^2}-m'a\frac{d^2}{dt^2}\cos\theta=0,$$

and, from (5) and (7),

$$m \sin\theta \frac{d^2x}{dt^2} - m'a\cos\theta \frac{d^2}{dt^2}\sin\theta = 0:$$

eliminating $\frac{d^2x}{dt^2}$ between the last two equations,

$$(m+m')\cos\theta \frac{d^2}{dt^2}\sin\theta - m\sin\theta\frac{d^2}{dt^2}\cos\theta = 0;$$

multiplying by $2\frac{d\theta}{dt}$, and integrating,

$$(m+m')\left(\frac{d}{dt}\sin\theta\right)^2 + m\left(\frac{d}{dt}\cos\theta\right)^2 = C,$$

$$(m+m'\cos^2\theta)\frac{d\theta^2}{dt^2} = C;$$

but, initially, $\theta = \alpha$, $\frac{d\theta}{dt} = \omega$; hence

$$(m+m'\cos^2\alpha)\,\omega^2 = C,$$

and therefore $$\frac{d\theta^2}{dt^2} = \omega^2\frac{m+m'\cos^2\alpha}{m+m'\cos^2\theta} \quad\text{.................. (8).}$$

Again, integrating (4), we get

$$m\frac{dx}{dt} + m'\frac{dx'}{dt} = C,$$

and therefore, by (6),

$$(m+m')\frac{dx}{dt} + m'a\sin\theta\frac{d\theta}{dt} = C;$$

but, at the commencement of the motion, $\frac{dx}{dt} = \beta$, $\theta = \alpha$, $\frac{d\theta}{dt} = \omega$;

hence $$(m+m')\beta + m'a\omega\sin\alpha = C,$$

and therefore $\frac{dx}{dt} = \beta + \frac{m'a\omega\sin\alpha}{m+m'} - \frac{m'a\sin\theta}{m+m'}\frac{d\theta}{dt}$;

whence, by (8),

$$\frac{dx}{dt} = \beta + \frac{m'a\omega\sin\alpha}{m+m'} - \frac{m'a\omega\sin\theta}{m+m'}\left(\frac{m+m'\cos^2\alpha}{m+m'\cos^2\theta}\right)^{\frac{1}{2}} \quad\text{.........(9).}$$

The equations (8) and (9) give us the velocity of P along Ox, and the angular velocity of Q about P, for any assignable inclination of the rod to the line Ox. If between these two equations we eliminate dt, we shall obtain a differential equation to the path of Q in x and θ.

From (8) we have

$$t = \frac{\omega}{(m + m' \cos^2 \alpha)^{\frac{1}{2}}} \int_\alpha^\theta (m + m' \cos^2 \theta)^{\frac{1}{2}} d\theta,$$

an elliptic transcendent for the determination of t for any value of θ.

Clairaut; *Mém. de l'Acad. des Sciences de Paris*, 1736, p. 10.

(5) A particle is attached to one extremity of a string, which is completely coiled round the circumference of a circular lamina, the other extremity of the string being fixed to the lamina: every particle of the lamina repels the particle with a force varying inversely as the distance: to find the velocity of the particle at any time after its departure from the circumference of the lamina.

Let a denote the radius of the lamina, r the distance of the particle from its centre at any time, and f the initial repulsive force experienced by the particle. Then, as may be ascertained by the performance of the appropriate integrations, the repulsive force on the particle at any time from the centre of the lamina will be $\frac{af}{r}$. Hence the particle may be considered as moving along a curve, which is the locus of the free extremity of the string, under the action of a central repulsive force $\frac{af}{r}$; and therefore, by the formula (D) of Section (I),

$$v^2 = C + 2\int \frac{af}{r} dr$$

$$= C + af \log r^2 = C + af \log (\rho^2 + a^2),$$

if ρ = the length of the string set free.

But, initially, $v=0$, $\rho=0$; hence

$$0=C+af\log a^2,$$

and therefore
$$v^2=af\log\frac{\rho^2+a^2}{a^2}.$$

Let θ denote the angle subtending the arc of the circumference of the lamina from which the string has been uncoiled; then $\rho=a\theta$, and we have

$$v^2=af\log(1+\theta^2).$$

(6) A particle is placed on a smooth inclined plane and is kept at a constant height by means of a horizontal motion given to the plane: to determine the space traversed by the particle at the end of any time.

If α be the inclination of the plane to the horizon, the space traversed by the particle at the end of any time t is equal to $\frac{1}{2}gt^2\tan\alpha$.

(7) A particle is placed within a smooth spherical surface at the lowest point: if the spherical surface be made to move parallel to a horizontal straight line so as to have at any time the velocity which gravity would generate in that time in a falling body, to find the greatest altitude to which the particle will rise.

The particle will just rise to the altitude of the centre of the sphere.

Griffin; *Solutions of the Examples on the Motion of a Rigid Body*, p. 85.

(8) A particle moves in a smooth straight tube, one point of which is fixed, and which revolves in a horizontal plane with a uniform angular velocity: if O be the immoveable point of the tube, and OP be drawn parallel and proportional to the pressure of the tube on the particle, to find the equation to the locus of P.

If θ be the inclination of OP to any given position of the tube,

$$OP=\alpha e^{\theta}+\beta e^{-\theta},$$

where α and β are constants.

(9) Two particles, connected together by a rigid rod without weight, are projected along a smooth horizontal plane: to determine their motion.

Let the plane of co-ordinates coincide with the plane of the motion. Let m, n, be the resolved parts of the initial velocity of the centre of gravity of the two particles parallel to the axes of x, y, and let a, b, be its initial co-ordinates. Let ω be the initial angular velocity of the rod, θ its inclination to the axis of x at the end of the time t, and ϵ at the beginning of the motion. Then the position of the centre of gravity is given at any time t by the equations

$$x = mt + a, \qquad y = nt + b;$$

and the inclination of the rod to the axis of x, by the equation

$$\theta = \omega t + \epsilon.$$

Clairaut; *Mémoires de l'Académie des Sciences de Paris*, 1736, p. 7. Euler; *Act. Acad. Petrop.* 1780, P. 1; *Opuscula, De motu corporum flexibilium*, Tom. III. p. 91.

(10) A spherical particle moves within a smooth tube, which revolves about one extremity with a uniform angular velocity in a vertical plane, the capacity of the tube being just sufficiently great for the reception of the particle: to determine the motion of the particle.

Let Ox, (fig. 161), which is horizontal, be the initial position of the tube, and P the position of the particle in the tube after a time t. Let ω denote the angular velocity of the tube about O, θ the inclination of OP to Ox, and let $OP = r$. Then, supposing the initial velocity of the particle to be zero, and that $r = a$ initially, the value of r at any time t is given by the equation

$$r = \frac{g}{2\omega^2} \sin \omega t + \frac{2a\omega^2 - g}{4\omega^2} \epsilon^{\omega t} + \frac{2a\omega^2 + g}{4\omega^2} \epsilon^{-\omega t},$$

and the polar equation to the path of the particle will result

from the substitution of θ for ωt in this equation. When t becomes very great, the polar equation becomes

$$r = \frac{2a\omega^2 - g}{4\omega^2} \epsilon^\theta,$$

which is the equation to an equiangular spiral.

The solution of this problem was attempted by M. Le Barbier, in the *Annales de Gergonne*, Tom. XIX. p. 285, who omitted to take into consideration the centrifugal force, an oversight which entirely vitiated his results. The correct solution was given in Tom. XX. by Ampère.

(11) A smooth wire in the form of a circle is made to revolve about a vertical diameter with uniform angular velocity: a small ring, capable of sliding upon the wire, would remain at rest relatively to the wire at a point of which the radius is inclined at an angle α to the vertical: to find the length of an isochronous simple pendulum for oscillations of the ring when slightly displaced from its position of relative rest.

If a be the radius of the circle, the length of the pendulum is equal to $a \cot \alpha \operatorname{cosec} \alpha$.

(12) A tube in the form of a cardioid, the axis of which is equal to $2a$, rotates with a uniform angular velocity $\left(\frac{g}{a}\right)^{\frac{1}{2}}$ about its axis, which is vertical, the cusp being at the top. A particle is projected within the tube at its lowest point with a velocity $(3ga)^{\frac{1}{2}}$: to find the greatest altitude to which the particle will ascend.

The particle when at its greatest altitude is in a horizontal line with the cusp.

(13) A particle falls from rest towards a fixed centre of force, which attracts directly as the distance: to find the equation to the path of the particle, supposing it to be included in a thin smooth rectilinear tube, which passes through the centre of force and revolves in one plane with a uniform angular velocity.

Let $\mu =$ the absolute force, $\omega =$ the angular velocity of the tube; and let a, the initial distance of the particle from the centre of force, be taken as the prime radius vector: then the equation to the path will be

$$r = \tfrac{1}{2} a \left\{e^{(\omega^2-\mu)^{\frac{1}{2}}\frac{\theta}{\omega}} + \epsilon^{-(\omega^2-\mu)^{\frac{1}{2}}\frac{\theta}{\omega}}\right\},$$

or $$r = a \cos \left\{(\mu - \omega^2)^{\frac{1}{2}} \frac{\theta}{\omega}\right\},$$

according as μ is less or greater than ω^2.

If $\mu = \omega^2$, the path becomes a circle.

(14) A particle P (fig. 165) is fixed to one end of a rigid rod PQ, which lies upon a smooth horizontal plane, and is so fine that its mass may be neglected. The end Q is constrained to move with a uniform velocity in the circumference of a circle ABQ; to find the velocity of the increase of the angle PQR, O being the centre of the circle, and OQR a straight line.

If $PQ = h$, $OQ = a$, $\angle PQR = \psi$ at any time t, $\alpha =$ the initial value of ψ, $\omega =$ the angular velocity of OQ, $\beta =$ the initial value of $\frac{d\psi}{dt}$, then

$$h\left(\frac{d\psi^2}{dt^2} - \beta^2\right) = 2a\omega^2 (\cos \psi - \cos \alpha).$$

Clairaut; *Mém. de l'Acad. des Sciences de Paris*, 1736, p. 14.

(15) QBA (fig. 166) is a circle on a horizontal plane, and QP a string touching it at the point Q; P is a particle attached to the end of the string. Supposing the particle P to be projected at right angles to QP with a given velocity so as to cause QP to be gradually wrapped about the circumference QBA; to find the velocity of the particle at any time during the motion, and the time which will elapse before the particle reaches the circumference.

Let β be the velocity of projection, v the velocity at any time during the motion, b the length of the string PQ, a the radius of the circle, T the time required. Then

$$v = \beta, \quad T = \frac{b^2}{2a\beta}.$$

(16) A circular horizontal lamina of matter ABC, (fig. 167), every particle of which attracts with a force varying inversely as the distance, is made to revolve with a uniform angular velocity round an axis through its centre O at right angles to its plane, the motion taking place in the direction of the arrows: to find the equation to the groove Aa which must be carved in the circular lamina in order that it may be described freely by a particle subject to the attraction of the lamina; the initial position of the particle being a point A in the circumference of the circle, and its initial velocity being zero.

Let P be any point in the groove; let $OP = r$, $OA = a$, $\angle POA = \theta$, $\omega =$ the angular velocity of the lamina about O, and $f =$ the attraction of the lamina on a particle in its circumference. Then the equation to the groove Aa will be

$$r = a \cos\left\{\left(\frac{f}{\omega^2 a}\right)^{\frac{1}{2}} \theta\right\}.$$

(17) Two small equal bodies A, B, connected together by a rigid line, are placed in a narrow rectilinear tube, in which they can move without friction; the tube is then made to revolve with a uniform angular velocity round a vertical axis which passes through a point C of the tube, this point C lying initially between A and B at a distance a from A and b from B: to find the time of A's arriving at C, and the tension of the rigid line at any time, a being considered less than b.

If ω denote the angular velocity of the tube, m the mass of each particle, t the required time, and T the tension; then

$$t = \frac{1}{\omega} \log \frac{b^{\frac{1}{2}} + a^{\frac{1}{2}}}{b^{\frac{1}{2}} - a^{\frac{1}{2}}}, \quad T = \tfrac{1}{2} m\omega^2 (a + b).$$

(18) A particle is drawn up an indefinitely thin cycloidal tube, the axis of the cycloid being vertical, by means of an equal particle, to which the former particle is attached by a thread passing over a pully at the highest point of the arc: to find the time of ascending to the highest point.

If T represent the required time, and t the time of a semi-oscillation in the cycloid,

$$T = 2^{\frac{1}{2}}t.$$

(19) A particle having been placed at a point in a straight line in a horizontal plane of indefinite extent, round which line as an axis the plane is then made to revolve with a uniform angular velocity: to find what time will elapse before the particle leaves the plane.

If ω be the angular velocity and t the required time, then

$$4 \cos \omega t = \epsilon^{\omega t} + \epsilon^{-\omega t}.$$

This problem was proposed in the *Lady's Diary*, for the year 1778, by John Landen, by whom a solution was given, which is singularly defective, not only in consequence of his neglecting the consideration of centrifugal force, but also from his erroneously supposing the horizontal velocity of the particle to be equal to its velocity along the plane, multiplied by the cosine of the plane's inclination to the horizon. See *Diarian Repository*, p. 512, where a correct solution is given by the Editors of the *Repository*, together with Landen's.

SECT. 7. *Constrained Motion of a Particle in Resisting Media.*

(1) A particle descends down a straight line AB, (fig. 168), inclined at an angle α to the vertical, in a medium of uniform density, in which the resistance varies as the velocity: to determine the velocity and the space at the end of any time.

Let P be the position of the particle at the end of any time t, v its velocity; let $AP = x$, and $k =$ the resistance for a unit of velocity. Then, since the resolved part of the force of gravity along AB is at every point $g \cos \alpha$, we have for the motion of P,

$$\frac{dv}{dt} = g \cos \alpha - kv,$$

$$\frac{dv}{g \cos \alpha - kv} = dt.$$

Integrating, we have

$$C - \frac{1}{k} \log (g \cos \alpha - kv) = t;$$

but $v = 0$ when $t = 0$, and therefore

$$C - \frac{1}{k} \log (g \cos \alpha) = 0:$$

hence
$$\frac{1}{k} \log \frac{g \cos \alpha - kv}{g \cos \alpha} = -t,$$

$$\frac{g \cos \alpha - kv}{g \cos \alpha} = \epsilon^{-kt},$$

$$v = \frac{g \cos \alpha}{k} (1 - \epsilon^{-kt}),$$

which gives the velocity for any value of t.

Again, since $dx = vdt$, we have

$$x = \frac{g \cos \alpha}{k} \int (1 - \epsilon^{-kt})\, dt$$

$$= C + \frac{g \cos \alpha}{k^2} (kt + \epsilon^{-kt}):$$

but, A being considered the initial position of the particle,

$$0 = C + \frac{g \cos \alpha}{k^2};$$

hence
$$x = \frac{g \cos \alpha}{k^2} (kt - 1 + \epsilon^{-kt}),$$

which gives the position of the particle at any time.

Euler; *Mechan.* Tom. II. p. 244.

(2) A particle descends from rest by the action of gravity from a point E, (fig. 169), down the arc EA of a cycloid BAB', of which the axis AC is vertical; the motion takes place in a medium of uniform density, where the resistance is equal to the sum of two quantities, one of which is constant and the other proportional to the square of the velocity: to find the velocity of the particle when it arrives at the point A, and

to determine at what point in its descent its velocity is a maximum.

Let $AM = x$, $AP = s$, $AE = c$, $v =$ the velocity at P; then, h and k being constant quantities, the equation of motion along the curve will be

$$v\frac{dv}{ds} = -g\frac{dx}{ds} + h + \frac{v^2}{k}:$$

hence $$d \,.\, v^2 - \frac{2ds}{k}v^2 = -2g\,dx + 2h\,ds:$$

but, by the nature of the cycloid, if $\frac{1}{4}a$ be the radius of the generating circle, $dx = \frac{s}{a}ds$: hence

$$d \,.\, v^2 - \frac{2ds}{k}v^2 = -\frac{2g}{a}s\,ds + 2h\,ds.$$

Multiplying both sides of the equation by $\epsilon^{-\frac{2s}{k}}$, we have

$$d\left(v^2\epsilon^{-\frac{2s}{k}}\right) = 2h\epsilon^{-\frac{2s}{k}}ds - \frac{2g}{a}\epsilon^{-\frac{2s}{k}}s\,ds.$$

Integrating, we have

$$v^2\epsilon^{-\frac{2s}{k}} = C - hk\epsilon^{-\frac{2s}{k}} - \frac{2g}{a}\int\epsilon^{-\frac{2s}{k}}s\,ds:$$

but $$\int\epsilon^{-\frac{2s}{k}}s\,ds = -\tfrac{1}{2}k\epsilon^{-\frac{2s}{k}}s + \tfrac{1}{2}k\int\epsilon^{-\frac{2s}{k}}ds$$

$$= -\tfrac{1}{2}k\epsilon^{-\frac{2s}{k}}s - \tfrac{1}{4}k^2\epsilon^{-\frac{2s}{k}}:$$

hence $$v^2\epsilon^{-\frac{2s}{k}} = C - hk\epsilon^{-\frac{2s}{k}} + \frac{1}{a}\left(gks + \tfrac{1}{2}gk^2\right)\epsilon^{-\frac{2s}{k}}$$

$$= C + \frac{1}{a}\left(gks + \tfrac{1}{2}gk^2 - ahk\right)\epsilon^{-\frac{2s}{k}}.$$

But, initially, $v = 0$, $s = c$: hence

$$0 = C + \frac{1}{a}\left(gkc + \tfrac{1}{2}gk^2 - ahk\right)\epsilon^{-\frac{2c}{k}}.$$

Let v_1 be the value of v when $s=0$; then

$$v_1^2 = C + \frac{1}{a}(\tfrac{1}{2}gk^2 - ahk);$$

and therefore $$v_1^2 = \frac{k}{a}(\tfrac{1}{2}gk - ah) - \frac{k}{a}(gc + \tfrac{1}{2}gk - ah)\,\epsilon^{-\frac{2c}{k}}.$$

Again, $$v^2\epsilon^{-\frac{2s}{k}} = \frac{k}{a}(gs + \tfrac{1}{2}gk - ah)\,\epsilon^{-\frac{2s}{k}} - \frac{k}{a}(gc + \tfrac{1}{2}gk - ah)\,\epsilon^{-\frac{2c}{k}},$$

$$v^2 = \frac{k}{a}(gs + \tfrac{1}{2}gk - ah) - \frac{k}{a}(gc + \tfrac{1}{2}gk - ah)\,\epsilon^{\frac{2}{k}(s-c)}.$$

When v is a maximum,

$$0 = \frac{gk}{a} - \frac{2}{a}(gc + \tfrac{1}{2}gk - ah)\,\epsilon^{\frac{2}{k}(s-c)},$$

$$\epsilon^{\frac{2}{k}(s-c)} = \frac{\tfrac{1}{2}gk}{gc + \tfrac{1}{2}gk - ah},$$

$$s = c + \frac{k}{2}\log\frac{\tfrac{1}{2}gk}{gc + \tfrac{1}{2}gk - ah} = c - \frac{k}{2}\log\frac{gc + \tfrac{1}{2}gk - ah}{\tfrac{1}{2}gk},$$

which gives the position of the particle when the velocity is a maximum. Euler; *Mechan.* Tom. II. p. 292.

(3) From a given point O, (fig. 170), an infinite number of straight lines OP are drawn in a vertical plane: to determine the nature of the curve APD, such that a particle descending down any line OP may always acquire the same velocity on arriving at P, the medium in which the motion takes place being uniform, and its resistance varying as any power of the velocity.

Let β be the velocity at P, v at any point p in OP; let $OP=r$, $OP=z$; $\angle POx = \theta$, Ox being vertical; draw PM horizontally, and let $OM=x$; then, k being the resistance for a unit of velocity, and m the index of its power,

$$v\frac{dv}{dz} = g\cos\theta - kv^m,$$

$$dz = \frac{vdv}{g\cos\theta - kv^m}.$$

Integrating, we have

$$r = \int_0^\beta \frac{vdv}{g\cos\theta - kv^m} = \int_0^\beta \frac{\beta d\beta}{g\cos\theta - k\beta^m}.$$

But $x = r\cos\theta$; hence

$$x = \int_0^\beta \frac{\beta d\beta}{g - k\beta^m \sec\theta} \quad \text{...(1)}$$

$$= \frac{\frac{1}{2}\beta^2}{g - k\beta^m\sec\theta} - \tfrac{1}{2}mk\sec\theta\int_0^\beta \frac{\beta^{m+1}d\beta}{(g - k\beta^m\sec\theta)^2} \quad \text{............(2).}$$

But, β being a constant quantity while x and θ vary, we have, from (1),

$$dx = kd(\sec\theta)\,.\int_0^\beta \frac{\beta^{m+1}d\beta}{(g - k\beta^m\sec\theta)^2};$$

and therefore, by (2),

$$2x = \frac{\beta^2}{g - k\beta^m\sec\theta} - \frac{m\sec\theta\,dx}{d\sec\theta};$$

or, putting $\frac{r}{x}$ for $\sec\theta$,

$$m\frac{r}{x}dx + 2xd\left(\frac{r}{x}\right) = \frac{\beta^2 d\left(\frac{r}{x}\right)}{g - k\frac{r}{x}\beta^m},$$

and therefore $\quad (m-2)\,rdx + 2x\,dr = \beta^2\dfrac{xdr - rdx}{gx - k\beta^m r};$

which is the differential equation to the curve in x and r.

Euler; *Mechan.* Tom. II. p. 246.

(4) To find the tautochrone in a medium the resistance of which varies as the square of the velocity, the particle being acted on by gravity.

Let O (fig. 171) be the point to which the particle is always to descend in the same time, AO being the tautochrone. Take Oy, a horizontal line, as the axis of y, Ox, a vertical line, as the axis of x. Let $OM = x$, $OP = s$; $v =$ the velocity of the particle at P, and $k =$ the resistance of the medium for a unit of velocity.

The equation for the motion along the curve will be

$$vdv = -gdx + kv^2ds:$$

multiplying by $2\epsilon^{-2ks}$, we have

$$d(v^2\epsilon^{-2ks}) = -2g\epsilon^{-2ks}\,dx.$$

Integrating, we obtain

$$v^2\epsilon^{-2ks} = C - 2g\int \epsilon^{-2ks}\,dx.$$

Suppose the velocity of the particle on its arrival at O to be that due to an altitude h in vacuum; then

$$2gh = C - 2g\int^0 \epsilon^{-2ks}\,dx;$$

hence $$v^2\epsilon^{-2ks} = 2g\{h - \int_0^x \epsilon^{-2ks}\,dx\}\text{ ....(1)},$$

and therefore, v being equal to $-\frac{ds}{dt}$ at any time t,

$$dt = -\frac{1}{(2g)^{\frac{1}{2}}}\frac{\epsilon^{-ks}\,ds}{(h-u)^{\frac{1}{2}}},$$

where $$u = \int_0^x \epsilon^{-2ks}\,dx\text{ ....(2)}.$$

Now, s being some function of x and therefore of u, we may assume $\epsilon^{-ks}\,ds = \phi(u)\,du$, and thus

$$dt = -\frac{1}{(2g)^{\frac{1}{2}}}\frac{\phi(u)\,du}{(h-u)^{\frac{1}{2}}},\quad t = -\frac{1}{(2g)^{\frac{1}{2}}}\int\frac{\phi(u)\,du}{(h-u)^{\frac{1}{2}}}.$$

But $t=0$ when $v=0$, and therefore, by (1) and (2), when $u=h$; and it will denote the whole time of descent to O from the beginning of the motion when $x=0$, and therefore, by (2), when $u=0$; hence the whole time of the descent is equal to

$$\frac{1}{(2g)^{\frac{1}{2}}}\int_0^h \frac{\phi(u)\,du}{(h-u)^{\frac{1}{2}}}\text{ ....(3)},$$

a result which, on the condition of tautochronism, must evidently be independent of h. From this it is plain that

$$\int \frac{\phi(u)\,du}{(h-u)^{\frac{1}{2}}}$$

must be of no dimensions in u and h together, and that consequently its differential $\dfrac{\phi(u)\,du}{(h-u)^{\frac{1}{2}}}$ must be of no dimensions in u, h, du; hence, since $\phi(u)$ evidently does not involve h, we must have

$$\phi(u) = \frac{\alpha}{u^{\frac{1}{2}}},$$

where α is some constant quantity. Hence, putting for $\phi(u)$ its value,

$$\epsilon^{-ks}\,ds = \alpha\frac{du}{u^{\frac{1}{2}}}.$$

Integrating, we have

$$C - \frac{1}{k}\epsilon^{-ks} = 2\alpha u^{\frac{1}{2}}.$$

But, by (2), when s and therefore x is equal to zero, $u=0$;

hence $$C - \frac{1}{k} = 0;$$

hence $$\frac{1}{k}(1-\epsilon^{-ks}) = 2\alpha u^{\frac{1}{2}}, \quad \frac{1}{k^2}(1-\epsilon^{-ks})^2 = 4\alpha^2 u,$$

$$\frac{1}{k}(1-\epsilon^{-ks})\,\epsilon^{-ks} = 2\alpha^2\frac{du}{ds} = 2\alpha^2\epsilon^{-2ks}\frac{dx}{ds}, \text{ by (2)};$$

and therefore the equation to the tautochrone will be

$$2\alpha^2 k\frac{dx}{ds} = \epsilon^{ks} - 1.$$

Since $\phi(u) = \dfrac{\alpha}{u^{\frac{1}{2}}}$, we have, from (3), if τ denote the whole time of descent,

$$\tau = \frac{\pi\alpha}{(2g)^{\frac{1}{2}}}, \quad \alpha^2 = \frac{2g\tau^2}{\pi^2};$$

and therefore the equation to the tautochrone for the time τ will be

$$4gk\tau^2 \frac{dx}{ds} = \pi^2 (\epsilon^{ks} - 1) \quad\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Euler; *Comment. Petrop.* 1729; *Mechan.* Tom. II. p. 392. John Bernoulli; *Mém. de l'Acad. des Sciences de Paris*, 1730; *Opera*, Tom. III. p. 173. See also the *Cambridge Mathematical Journal*, Vol. II. p. 153, where Mr Robert Leslie Ellis has reduced the solution of this problem to that of the Tautochrone in vacuum.

(5) A particle of mass m falls down the arc of a smooth cycloid, the axis of which is vertical and vertex upwards, in a medium the resistance of which on the particle is $\frac{mv^2}{2c}$, c being the length of the arc of the cycloid from the vertex to the initial position of the particle: to find the time of falling to the cusp.

If l be the length of the cycloidal arc between the vertex and the cusp, the required time is equal to

$$\left\{\frac{2l}{cg} \cdot (l - c)\right\}^{\frac{1}{2}}.$$

(6) A particle oscillates in an inverted cycloid, of which the axis is vertical, in a uniform medium where the resistance varies as the velocity; having given the first arc of descent, to find the whole space described by the particle before the motion ceases.

Let c denote the first arc of descent, k the resistance when the velocity is unity, a the radius of the generating circle; then the whole space will be equal to

$$c\,\frac{\epsilon^{\phi} + 1}{\epsilon^{\phi} - 1}, \quad \text{where } \phi = \frac{k\pi}{\left(\frac{g}{a} - k^2\right)^{\frac{1}{2}}}.$$

(7) An inelastic particle descends down the sides of a plane equilateral and equiangular polygon in a vertical plane, the

medium in which the motion takes place being uniform and its resistance varying as the square of the velocity; to determine the velocity of the particle when it has arrived at the end of any side of the polygon, the side down which the particle first descends being vertical.

Let $\pi - \alpha$ be the magnitude of each of the angles of the polygon, l the length of each of its sides, kv^2 the resistance for a velocity v; v_x the velocity at the end of the x^{th} side. Then

$$v_x = (\cos\alpha)^x \epsilon^{-klx} \left\{A^2 + \frac{g(\epsilon^{2kl} - 1)}{k(\cos\alpha)^2} \cdot \frac{M\cos x\alpha + N\sin x\alpha}{M^2 + N^2} \frac{\epsilon^{2klx}}{(\cos\alpha)^{2x}}\right\}^{\frac{1}{2}},$$

where $$M = \frac{\epsilon^{2kl} - \cos\alpha}{\cos\alpha}, \quad N = \frac{\sin\alpha}{(\cos\alpha)^2}\epsilon^{2kl};$$

and A is an arbitrary constant which will easily be determined if we know the value of v_x for any value of x.

Bordoni; *Memorie della Societa Italiana*, 1816, p. 173.

(8) From a given point O (fig. 170) an infinite number of straight lines OP are drawn in a vertical plane; to determine the nature of the curve APD, in order that a particle descending down any line OP may always acquire the same velocity on arriving at P; the motion taking place in a medium of uniform density where the resistance varies as the square of velocity.

Let OA be vertical and be represented by a; let $OP = r$, $OM = x$, $k =$ the resistance for a unit of velocity; then the equation to the curve will be

$$x = r\,\frac{\epsilon^{2kr}}{\epsilon^{2ka}}\,\frac{\epsilon^{2ka} - 1}{\epsilon^{2kr} - 1}.$$

Euler; *Mechan.* Tom. II. p. 251.

(9) From a given point O (fig. 172) an infinite number of straight lines OP are drawn in a vertical plane; to determine the nature of the curve OPA in order that a particle may descend through all its chords OP in the same time; the motion taking place in a medium of uniform density where the resistance varies as the square of the velocity.

Let OA be vertical and be equal to a; let $OP = r$, $\angle AOP = \theta$; then the polar equation to the curve OPA will be

$$(\cos\theta)^{\frac{1}{2}} = \frac{\log\{\epsilon^{kr} + (\epsilon^{2kr} - 1)^{\frac{1}{2}}\}}{\log\{\epsilon^{ka} + (\epsilon^{2ka} - 1)^{\frac{1}{2}}\}}.$$

Euler; *Mechan.* Tom. II. p. 256.

(10) A particle, acted on by gravity, is ascending the curve MNA, (fig. 173), in a medium where the resistance varies as the square of the velocity; to find the nature of the curve in order that the velocity of the particle at any point N may be the same as that which it would acquire by falling in the same medium down a vertical line IN, the length of which is equal to the arc AN of the curve measured from a fixed point A.

Draw through A a vertical line AB, and let fall NQ at right angles to AB; let $AQ = x$, $AN = s$, and $k =$ the resistance of the medium for a unit of velocity. Then the differential equation to the curve will be

$$k(x + s) = 1 - \epsilon^{-2kx}.$$

This problem was proposed to Clairaut on his journey to Lapland, by Klingstierna, Professor of Mathematics at Upsal, at which place he called on his way; Klingstierna's construction, together with his own solution, was published by Clairaut in the *Mémoires de l'Académie des Sciences de Paris*, 1740, p. 254.

SECTION 8. *Hodographs.*

(1) A particle descends from rest down a thin cycloidal tube, the axis of the cycloid being vertical and vertex the highest point: to investigate the form of the hodograph.

Let the axis of the cycloid be the axis of x, the tangent at its vertex being that of y. Let c be the initial vertical distance of the particle below the vertex. Let v be the velocity of the particle at any point of its descent, and ψ the inclination of the tangent to the curve at that point to the vertical. Then $v^2 = 2g(x - c)$.

Also, a being the radius of the generating circle and θ the eccentric angle, $x = a(1 - \cos\theta)$, $y = a(\theta + \sin\theta)$.

Hence $$\tan\psi = \frac{dy}{dx} = \frac{1 + \cos\theta}{\sin\theta},$$

$$\sin\theta \sin\psi = \cos\psi + \cos\theta\cos\psi,$$

$$\cos(\theta + \psi) = \cos(\pi - \psi),$$

$$\theta = \pi - 2\psi.$$

Hence $$v^2 = 2ga(1 - \cos\theta) - 2gc$$

$$= 2ga(1 + \cos 2\psi) - 2gc,$$

which is the equation to the hodograph.

COR. If the initial position of the particle coincide with the vertex of the cycloid, $c = 0$, and $v = 2ga\cos\psi$: the hodograph is then a circle.

(2) A particle moves on a curve in a vertical plane, the curve being such that the pressure on it is constant: to determine the form of the hodograph.

The hodograph is a conic section of which the directrix is horizontal.

(3) A straight rod, the ends of which are moveable along two perpendicular straight lines in one plane, revolves with a constant angular velocity: to find the hodographs of the paths of its points.

The required hodographs are ellipses enveloped by a hypocycloid.

CHAPTER V.

MOMENT OF INERTIA.

THE Moment of Inertia of a body, with regard to any axis, is the sum of all the products resulting from the multiplication of each element of the mass by the square of its distance from the axis. If M denote the whole mass of the body, the Moment of Inertia may be represented by the expression Mk^2, where k is a line called the Radius of Gyration. The term Moment of Inertia was first made use of by Euler. "Ratio hujus denominationis ex similitudine motus progressivi est desumpta: quemadmodum enim in motu progressivo, si à vi secundum suam directionem sollicitante acceleretur, est incrementum celeritatis ut vis sollicitans divisa per massam seu inertiam; ita in motu gyratorio, quoniam loco ipsius vis solicitantis ejus momentum considerari oportet, eam expressionem $\int r^2 dM$, quæ loco inertiæ in calculum ingreditur, *momentum inertiæ* appellemus, ut incrementum celeritatis angularis simili modo proportionale fiat momento vis sollicitantis diviso per momentum inertiæ[1]."

SECT. 1. *A Plane Curve about an Axis within its Plane.*

(1) To find the moment of inertia and radius of gyration of a circular arc about a radius through its vertex.

Let HAK (fig. 174) be the circular arc, A its vertex, C the centre of the circle. Take any point P in the arc; draw PM at right angles to the radius CA; join HK, intersecting CA in E. Join CH, CK, CP. Let $PM = y$, $CA = a$, arc $AP = s$, $HE = c = KE$, $\angle ACH = \alpha = \angle ACK$, $\angle PCA = \theta$. Then, the density of the arc and the indefinitely small area of the section of it

[1] Euler; *Theoria Motus Corporum Solidorum*, p. 167.

made by a plane through C, at right angles to its own plane, being represented respectively by ρ and κ, we shall have

$$\begin{aligned} Mk^2 &= \kappa\rho \int y^2 ds \\ &= \kappa\rho \int_{-\alpha}^{+\alpha} a^2 \sin^2\theta \,.\, a d\theta \\ &= \tfrac{1}{2}\kappa\rho a^3 \int_{-\alpha}^{+\alpha} (1 - \cos 2\theta)\, d\theta \\ &= \tfrac{1}{2}\kappa\rho a^3 (2\alpha - \sin 2\alpha): \end{aligned}$$

but $$M = \kappa\rho \,.\, 2a\alpha:$$

hence $$\begin{aligned} k^2 &= \tfrac{1}{2}a^2\left(1 - \frac{\sin 2\alpha}{2\alpha}\right) \\ &= \tfrac{1}{2}a^2 - \frac{a \sin\alpha \,.\, a\cos\alpha}{2\alpha} \\ &= \tfrac{1}{2}a^2 - \frac{c(a^2 - c^2)^{\frac{1}{2}}}{2\sin^{-1}\dfrac{c}{a}}. \end{aligned}$$

If the arc be a semicircle, $c = a$, and, if a circle, $c = 0$; in both cases $k^2 = \frac{1}{2}a^2$.

(2) To find the radius of gyration of a material straight line OB, (fig. 175), about an axis OA, to which it is inclined at a given angle, the density at any point of OB varying as some power of its distance from O.

Take any point P in OB; draw PM at right angles to OA; let $PM = y$, $OP = s$, $OB = l$, $\angle\, AOB = \alpha$, $\rho =$ the density at P: then $\rho = \mu s^n$, where μ is a constant quantity. Hence

$$Mk^2 = \kappa \int \rho y^2 ds = \kappa\mu \sin^2\alpha \int_0^l s^{n+2} ds = \frac{\kappa\mu \sin^2\alpha\, l^{n+3}}{n+3}.$$

Also, $$M = \kappa \int \rho ds = \kappa\mu \int_0^l s^n\, ds = \frac{\kappa\mu\, l^{n+1}}{n+1}.$$

Hence we get $$k^2 = \frac{n+1}{n+3}\, l^2 \sin^2\alpha.$$

If the density be invariable, $n = 0$, and $k^2 = \frac{1}{3}l^2 \sin^2\alpha$.

SECT. 2. *A Plane Curve about an Axis at Right Angles to its Plane.*

(1) To find the radius of gyration of a straight line AB, (fig. 176) about an axis through D at right angles to the plane ADB.

Let C be the middle point of the line; join CD. Let $AC=a$ $=BC$, $CD=b$; $k=$ the radius of gyration about the axis through D, and $k'=$ that about an axis parallel to this through C. Then

$$k^2 = k'^2 + b^2.$$

But, $2a\rho\kappa$ being the mass of AB,

$$2a\rho\kappa k'^2 = \rho\kappa \int_{-a}^{+a} s^2 ds = \frac{2\rho\kappa}{3} a^3,$$

$$k'^2 = \tfrac{1}{3}a^2.$$

Hence $$k^2 = \tfrac{1}{3}a^2 + b^2.$$

(2) To find the radius of gyration of a circular arc about an axis perpendicular to its plane through its centre of gravity.

Let k be the radius of gyration about the required axis, k' about an axis parallel to this through the centre of the circle, and h the distance between the centre of gravity of the arc and the centre of the circle. Then

$$k'^2 = k^2 + h^2.$$

But, r denoting the radius of the circle, c the chord, and a the length of the arc,

$$k'^2 = r^2, \qquad h^2 = \frac{c^2 r^2}{a^2}.$$

Hence, for the required radius of gyration,

$$k^2 = \frac{r^2}{a^2}(a^2 - c^2).$$

(3) To find the radius of gyration of a circular arc about an axis perpendicular to its plane through its vertex.

If r = the radius of the circle, a = the length and c = the chord of the arc,

$$k^2 = 2r^2\left(1 - \frac{c}{a}\right).$$

(4) If the density of a straight rod AB vary as the n^{th} power of the distance from one end A, and k, k', be the radii of gyration of the rod round axes at right angles to its length through A and B respectively; to compare the values of k and k', and to ascertain the value of n so that k may be equal to $6k'$.

$$\frac{k^2}{k'^2} = \frac{(n+1)(n+2)}{2}; \qquad n = 7, \text{ or } = -10.$$

(5) To find the shape of a uniform wire, lying in one plane, such that the moment of inertia of any portion about an axis perpendicular to its plane may vary as the difference of the distances of its extremities from the axis.

The equation to the form of the wire, a and α being constants, and the point where the axis intersects the plane being the origin of polar co-ordinates, is

$$2(\alpha - \theta) = \frac{(a^4 - r^4)^{\frac{1}{2}}}{r^2} + \sin^{-1}\frac{r^2}{a^2}.$$

SECT. 3. *A Plane Area about an Axis within or parallel to its Plane.*

(1) To find the radius of gyration of an elliptic area, of uniform thickness and density, about its principal axes.

Let ρ represent the uniform density of the area, and τ its indefinitely small thickness; then, x, y, denoting the co-ordinates of any point of the curve referred to the axes of the ellipse as axes of co-ordinates, we have for the moment of inertia, about the major axis of a quadrant of the ellipse,

$$Mk^2 = \rho\tau \int_0^b y^2 \,.\, x dy$$

$$= \frac{\rho\tau a}{b} \int_0^b y^2 (b^2 - y^2)^{\frac{1}{2}} dy :$$

but $$\int_0^b y^2 (b^2 - y^2)^{\frac{1}{2}} dy = \frac{1}{3} \int_0^b (b^2 - y^2)^{\frac{3}{2}} dy$$

$$= \frac{1}{4} b^2 \int_0^b (b^2 - y^2)^{\frac{1}{2}} dy$$

$$= \frac{1}{16} \pi b^4.$$

Hence the moment of inertia of the whole ellipse will be equal to

$$4Mk^2 = \frac{1}{4} \pi\rho\tau a b^3 ;$$

but $$4M = \pi\rho\tau a b ;$$

hence $$k^2 = \frac{1}{4} b^2.$$

If k' denote the radius of gyration about the minor axis, we shall have, by similar reasoning,

$$k'^2 = \frac{1}{4} a^2.$$

(2) To find the radius of gyration of a circular area about a straight line parallel to its plane, at a distance c from its centre.

If a be the radius of the circle, and k the required radius of gyration,

$$k^2 = \frac{1}{4} a^2 + c^2.$$

(3) To find the radius of gyration of an isosceles triangle about a perpendicular let fall from its vertex upon its base.

If $2b =$ the length of the base,

$$k^2 = \frac{1}{6} b^2.$$

(4) To find the radius of gyration of a lamina, bounded by the lemniscate $r^2 = a^2 \cos 2\theta$, about the axis of the curve.

If k be the radius of gyration,

$$k^2 = \frac{a^2}{48}(3\pi - 8).$$

(5) To find the radius of gyration of the area of the lemniscate $r^2 = a^2 \cos 2\theta$ about a tangent at its node.

The radius of gyration is equal to $\frac{1}{4}\pi^{\frac{1}{2}}a$.

(6) To find the moment of inertia of a triangle ABC about an axis drawn, in the plane of the triangle, through the angular point A.

If m be the mass of the triangle, and b, c, the lengths of the perpendiculars from B, C, upon the axis, the required moment of inertia is equal to

$$\tfrac{1}{6}m(b^2 + bc + c^2).$$

Messenger of Mathematics, Vol. IV. p. 115.

SECT. 4. *A Plane Area about a Perpendicular Axis.*

(1) To find the radius of gyration of a triangular lamina ABC, (fig. 177), about an axis through A at right angles to its plane.

Take two points P, p, indefinitely near to each other in the side AB, and draw PM, pm, parallel to BC. Take P', p', in PM, pm, and construct the indefinitely small parallelogram $P'p'$, two of the sides of which are parallel to AC. Let $AM = x$, $PM = y$, $P'M = y'$, $Am = x + dx$, $p'm = y' + dy'$, $\angle ACB = C$; let a, b, c, be the three sides of the triangle.

Then, Mk^2 denoting the moment of inertia about A, we have, τ denoting the indefinitely small thickness, and ρ the density of the lamina,

$$Mk^2 = \int_0^b \int_0^y (x^2 + y'^2 - 2xy' \cos C)\, \rho\tau \sin C\, dx\, dy'$$

$$= \rho\tau \sin C \int_0^b (x^2 y + \tfrac{1}{3} y^3 - xy^2 \cos C)\, dx$$

$$= \rho\tau \sin C \int_0^b \left(\frac{a}{b} x^3 + \tfrac{1}{3} \frac{a^3}{b^3} x^3 - \frac{a^2}{b^2} x^3 \cos C\right) dx$$

$$= \tfrac{1}{12} \cdot \tfrac{1}{2} \rho\tau\, ab \sin C\, (6b^2 + 2a^2 - 6ab \cos C)$$

$$= \tfrac{1}{12} M \{6b^2 + 2a^2 - 3(a^2 + b^2 - c^2)\},$$

and therefore

$$k^2 = \tfrac{1}{12} (3b^2 + 3c^2 - a^2).$$

(2) To find the radius of gyration of a triangular lamina ABC about a perpendicular through its centre of gravity G.

Let AG, BG, CG, be represented by α, β, γ; and BC, CA, AB, by a, b, c. Then, M denoting the mass of the whole triangle ABC, $\tfrac{1}{3} M$ will be the mass of each of the triangles BGC, CGA, AGB. Hence, by the preceding problem, the moment of inertia of these three triangles respectively about the axis through G will be

$$\tfrac{1}{36} M (3\beta^2 + 3\gamma^2 - a^2),$$
$$\tfrac{1}{36} M (3\gamma^2 + 3\alpha^2 - b^2),$$
$$\tfrac{1}{36} M (3\alpha^2 + 3\beta^2 - c^2);$$

and therefore the moment of inertia of the whole triangle about G will be equal to

$$\tfrac{1}{36} M \{6(\alpha^2 + \beta^2 + \gamma^2) - (a^2 + b^2 + c^2)\};$$

or, by a property of the centre of gravity of a triangle, to

$$\tfrac{1}{36} M \{2(a^2 + b^2 + c^2) - (a^2 + b^2 + c^2)\}$$
$$= \tfrac{1}{36} M (a^2 + b^2 + c^2).$$

Hence the square of the radius of gyration will be equal to

$$\tfrac{1}{36} (a^2 + b^2 + c^2).$$

Euler; *Theoria Motus Corporum Solidorum*, cap. VI.
Prob. 32. Cor. 1.

(3) To find the radius of gyration of an elliptic area about a perpendicular axis through its centre.

If M be the mass of the area, the moment of inertia about the two axes of the ellipse will be

$$\tfrac{1}{4}Mb^2, \qquad \tfrac{1}{4}Ma^2.$$

But the moment of inertia of a plane area, about any perpendicular axis, is equal to the sum of the moments of inertia about any two lines, at right angles to each other in the plane area, passing through the point at which the axis meets the area. Hence, in the present problem, the moment of inertia about the proposed axis is equal to

$$\tfrac{1}{4}M(a^2+b^2),$$

and the square of the radius of gyration $=\tfrac{1}{4}(a^2+b^2)$.

(4) To find the radius of gyration of a circular annulus about a perpendicular axis through the centre of the circle.

Let r be the distance of any point of the annular area from the centre of the circle, θ the angular co-ordinate, ρ the density, and τ the indefinitely small thickness of the area; then, a, b, being the radii of the two concentric circles,

$$Mk^2 = \int_0^{2\pi}\int_a^b r^2 . \rho\tau r d\theta dr$$

$$= \tfrac{1}{4}\rho\tau\int_0^{2\pi}(b^4-a^4)\,d\theta = \tfrac{1}{2}\pi\rho\tau(b^4-a^4).$$

But $$M = \int_0^{2\pi}\int_a^b \rho\tau r d\theta dr = \tfrac{1}{2}\rho\tau\int_0^{2\pi}(b^2-a^2)\,d\theta$$

$$= \pi\rho\tau(b^2-a^2);$$

hence $$k^2 = \tfrac{1}{2}(a^2+b^2).$$

(5) To determine the form of a uniform plane lamina, and the position of an axis at right angles to it, in order that the moment of inertia of the lamina about the axis may be a minimum, the mass, density, and thickness of the lamina being known.

Let mk^2 denote its moment of inertia, m denoting its mass: then, ρ being its density, and τ its thickness, x and y being polar co-ordinates,

$$mk^2 = \rho\tau \int_0^y \int_0^{2\pi} y\, dx\, dy \,.\, y^2 = \tfrac{1}{4}\rho\tau \int_0^{2\pi} y^4\, dx,$$

$$m = \rho\tau \int_0^y \int_0^{2\pi} y\, dx\, dy \quad = \tfrac{1}{2}\rho\tau \int_0^{2\pi} y^2\, dx.$$

Put $u = \int y^4\, dx, \qquad v = \int y^2\, dx$:

then, a being some constant,

$$u - av = \int (y^4 - ay^2)\, dx,$$

$$V = y^4 - ay^2.$$

Hence, by the formula of the Calculus of Variations,

$$N - \frac{d\,(P)}{dx} + \frac{d^2\,(Q)}{dx^2} - \&c. = 0,$$

we have, since P, Q, R, &c., are all zero,

$$4y^3 - 2ay = 0, \qquad y^2 = \tfrac{1}{2}\, a,$$

which shews that the disk is circular, its centre being in the axis of rotation.

Also $\quad m = \tfrac{1}{2}\rho\tau \int_0^{2\pi} y^2\, dx = \tfrac{1}{4}\rho\tau a \int_0^{2\pi} dx = \tfrac{1}{2}\pi\rho\tau a,$

$$a = \frac{2m}{\pi\rho\tau},$$

and therefore y, the radius, is equal to $\left(\frac{m}{\pi\rho\tau}\right)^{\frac{1}{2}}$.

(6) To find the radius of gyration of a parallelogram about an axis perpendicular to it through its centre of gravity.

If $2a$, $2b$, be the lengths of two adjoining sides of the parallelogram, then, whatever be the angle of their inclination,

$$k^2 = \tfrac{1}{3}\,(a^2 + b^2).$$

Euler; *Theoria Motus Corp. Solid.* Cap. VI. Prob. 35.

(7) To find the radius of gyration of a regular polygon about an axis perpendicular to it through its centre. If n be the number of sides, and c the length of each,

$$k^2 = \tfrac{1}{12} c^2 \frac{2 + \cos\dfrac{2\pi}{n}}{1 - \cos\dfrac{2\pi}{n}}.$$

(8) To find the radius of gyration of a portion of a parabola, bounded by a double ordinate to the axis, about a perpendicular line through its vertex.

If x, y, represent the extreme co-ordinates of the portion,

$$k^2 = \tfrac{3}{7}x^2 + \tfrac{1}{5}y^2.$$

SECT. 5. *Plane Area about an Oblique Axis.*

Having given the greatest of the moments of inertia of any plane figure about the three principal axes, which have the same origin, to find the moment of inertia about an axis passing through the same origin and equally inclined to the three principal axes.

Let A, B, C, be the moments of inertia about the three principal axes: one of these will evidently be at right angles to the plane area; we will suppose C to correspond to this. Then, μ being the moment of inertia about the other axis, and α, β, γ, its inclinations to the principal axes,

$$\begin{aligned}\mu &= A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma \\ &= (A + B + C)\cos^2\alpha,\end{aligned}$$

since $\alpha = \beta = \gamma$, by the supposition.

But we know that $A + B = C$: hence

$$\mu = 2C\cos^2\alpha,$$

C being evidently greater than either A or B.

But, since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$,

and $\alpha = \beta = \gamma$, we see that $\cos^2\alpha = \tfrac{1}{3}$: hence,

$$\mu = \tfrac{2}{3}C.$$

SECT. 6. *Symmetrical Solid about its Axis.*

(1) To find the radius of gyration of a homogeneous sphere about a diameter.

Let x, $x + dx$, be the distances of the circular faces of a thin circular slice of the sphere, at right angles to the diameter, from the centre, and let y be the radius of the section. Then, ρ denoting the density of the sphere, the moment of inertia of this slice about the diameter will be equal to

$$\tfrac{1}{2}\pi\rho y^4 dx;$$

and therefore the moment of inertia of the whole sphere, a being its radius, will be equal to

$$\tfrac{1}{2}\pi\rho\int_{-a}^{+a} y^4 dx = \tfrac{1}{2}\pi\rho\int_{-a}^{+a}(a^2 - x^2)^2 dx = \tfrac{8}{15}\pi\rho a^5.$$

But the mass of the sphere is equal to $\tfrac{4}{3}\pi\rho a^3$; hence

$$k^2 = \tfrac{2}{5}a^2.$$

Euler; *Theoria Motus Corporum Solidorum*, p. 198.

(2) To find the radius of gyration of a right cone about its axis.

If a denote the radius of the base of the cone,

$$k^2 = \tfrac{3}{10}a^2.$$

Euler; *Ib.* p. 197.

(3) To find the radius of gyration of a hollow sphere about a diameter.

If a, b, be the external and internal radii,

$$k^2 = \tfrac{2}{5}\frac{a^5 - b^5}{a^3 - b^3}.$$

Euler; *Ib.* p. 203.

(4) To find the radius of gyration of a solid cylinder about its axis.

If a denote the radius of the cylinder,

$$k^2 = \tfrac{1}{2}a^2.$$

Euler; *Ib.* p. 200.

(5) To find the moment of inertia of a sphere about a diameter, the density varying as the n^{th} power of the distance from the centre.

If $\mu =$ the density at a unit of distance from the centre, and $a =$ the radius, the moment of inertia is equal to

$$\frac{8\pi\mu}{3(n+5)} \cdot a^{n+5}.$$

(6) To find the radii of gyration of an ellipsoid about its axes.

If h, k, l, be the radii of gyration about the axes $2a$, $2b$, $2c$, respectively,

$$h^2 = \frac{b^2+c^2}{5}, \quad k^2 = \frac{c^2+a^2}{5}, \quad l^2 = \frac{a^2+b^2}{5}.$$

Poisson; *Traité de Mécanique*, Tom. 2, p. 47.

(7) If the density at any point of a right circular cone be proportional to the shortest distance of the point from the conical surface, to find the radius of gyration about the axis.

If a be the radius of the base of the cone, and k the radius of gyration,

$$k^2 = \frac{a^2}{5}.$$

(8) If a plane closed figure symmetrical on both sides of a line AA', or the annulus intercepted between two non-intersecting plane closed figures symmetrical on both sides of AA', revolve round any line BB', parallel to AA' and lying in the plane of the figure or annulus, but not intersecting it, prove that the moment of inertia of the generated solid, with respect to BB', is equal to $m(h^2+3k^2)$; where m is the mass of the solid, h the radius of the cylinder generated by the revolution of AA', and k the radius of gyration of the generating area with respect to AA'.

Townsend: *Quarterly Journal of Mathematics*, Vol. x. p. 203.

SECT. 7. *Moment of Inertia of a Solid not Symmetrical with respect to the Axis of Gyration.*

(1) To find the radius of gyration of a solid cylinder about an axis perpendicular to its own through its middle point.

Let x be the distance of any thin circular slice of the cylinder from the middle point of its axis; dx the thickness of the slice; ρ the density of the cylinder, b its radius, and $2a$ its length. Then, the moment of inertia of the slice about any diameter being equal to

$$\tfrac{1}{4}\pi\rho b^4 dx,$$

its moment of inertia about the axis of gyration in the present problem will be equal to

$$\pi\rho b^2 dx\,.\,(x^2 + \tfrac{1}{4}b^2).$$

Hence, Mk^2 denoting the moment of inertia of the whole cylinder about the proposed axis, we have

$$\begin{aligned} Mk^2 &= \pi\rho b^2 \int_{-a}^{+a} (x^2 + \tfrac{1}{4}b^2)\,dx \\ &= \pi\rho b^2 (\tfrac{2}{3}a^3 + \tfrac{1}{2}ab^2) \\ &= 2\,\pi\rho ab^2 (\tfrac{1}{3}a^2 + \tfrac{1}{4}b^2); \end{aligned}$$

and therefore, M being equal to $2\pi\rho ab^2$, we have

$$k^2 = \tfrac{1}{3}a^2 + \tfrac{1}{4}b^2.$$

Euler; *Theoria Motus Corporum Solidorum*, p. 196.

(2) To determine the moment of inertia of an ellipsoid about the diagonal of the inscribed parallelepiped of maximum volume.

If α, β, γ, be the angles made by the diagonal with the axes of x, y, z,

$$\cos^2\alpha = \frac{x^2}{r^2}, \quad \cos^2\beta = \frac{y^2}{r^2}, \quad \cos^2\gamma = \frac{z^2}{r^2},$$

where $$r^2 = x^2 + y^2 + z^2,$$

x, y, z, being the co-ordinates of an extremity of the diagonal.

Also, M being the mass of the ellipsoid, the moments of inertia about the axes of x, y, z, are

$$\frac{1}{5}M(b^2+c^2), \qquad \frac{1}{5}M(c^2+a^2), \qquad \frac{1}{5}M(a^2+b^2).$$

Hence the required moment of inertia is equal to

$$\frac{1}{5}M\left\{(b^2+c^2)\frac{x^2}{r^2}+(c^2+a^2)\frac{y^2}{r^2}+(a^2+b^2)\frac{z^2}{r^2}\right\}.$$

But, when the parallelepiped is a maximum, we may easily ascertain that

$$x=\frac{a}{\sqrt{3}}, \quad y=\frac{b}{\sqrt{3}}, \quad z=\frac{c}{\sqrt{3}}:$$

hence the moment of inertia becomes

$$\frac{2}{5}M.\frac{b^2c^2+c^2a^2+a^2b^2}{a^2+b^2+c^2}.$$

(3) **The vertex of a cone is at the centre of a sphere and the base of the cone is an area on the sphere environed by a circle: to find the moment of inertia of the cone about a straight line, through its vertex, perpendicular to its axis.**

Let Oz (fig. 178) be the axis of the cone; Ox, Oy, two lines through the vertex O at right angles to Oz; PN a perpendicular from P, any point in the cone, upon the plane xOy: join ON, and draw NM at right angles to Oy: join PO, PM. Let m = an element of the cone at the point P, Mk^2 = the required moment of inertia about Oy.

Let a = the radius of the sphere, $PO=r$, 2β = the vertical angle of the cone, $\angle POz=\theta$, $\angle NOx=\phi$.

Then $$Mk^2=\Sigma(m.PM^2).$$

But $$m=rd\theta\, dr\, r\sin\theta\, d\phi,$$

and $$PM^2=r^2\cos^2\theta+r^2\sin^2\theta\cos^2\phi:$$

hence
$$Mk^2 = \int_0^a \int_0^{2\pi} \int_0^{\beta} r^4 (\sin\theta \cos^2\theta + \sin^3\theta \cos^2\phi)\, d\theta\, d\phi\, dr$$
$$= \tfrac{1}{5} a^5 \int_0^{2\pi} \int_0^{\beta} (\sin\theta \cos^2\theta + \sin^3\theta \cos^2\phi)\, d\theta\, d\phi$$
$$= \tfrac{1}{5} \pi a^5 \int_0^{\beta} (2\sin\theta \cos^2\theta + \sin^3\theta)\, d\theta$$
$$= \tfrac{1}{5} \pi a^5 \, .\, {}_0^{\beta}\!\left(-\tfrac{1}{3}\cos^3\theta - \cos\theta\right)$$
$$= \tfrac{1}{15} \pi a^5 (4 - 3\cos\beta - \cos^3\beta).$$

If $\beta = \pi$, or the cone become a sphere,

$$Mk^2 = \tfrac{8}{15}\pi a^5; \quad \text{but } M = \tfrac{4}{3}\pi a^3: \quad \text{hence } k^2 = \tfrac{2}{5}a^2.$$

(4) To find the radius of gyration of a right cone about an axis through its vertex at right angles to its geometrical axis.

If a = the altitude of the cone, and c = the radius of the base,

$$k^2 = \tfrac{3}{20}(4a^2 + c^2).$$

(5) To find the radius of gyration of a right cone about an axis at right angles to the axis of the cone and passing through its centre of gravity.

If a be the altitude of the cone, and c the radius of its base; then

$$k^2 = \tfrac{3}{80}(a^2 + 4c^2).$$

Euler; *Ib.* p. 197.

(6) To find the radius of gyration of a circular right cone about a generating line.

If a be the altitude of the cone and c the radius of its base,

$$k^2 = \frac{3c^2}{20} \cdot \frac{6a^2 + c^2}{a^2 + c^2}.$$

Griffin: *Solutions of the Examples on the motion of a Rigid Body*, p. 9.

(7) To find the radius of gyration of a double convex lens about its axis, and about a diameter to the circle in which its

two spherical surfaces intersect; the two surfaces having equal radii.

If $a =$ the semi-axis of the lens, $b =$ the radius of the circular intersection of the two surfaces; $k =$ the radius of gyration of the lens about its axis, and k' about a diameter of the circle; we shall have

$$k^2 = \tfrac{1}{10}\frac{a^4 + 5a^2b^2 + 10b^4}{a^2 + 3b^2}, \quad k'^2 = \tfrac{1}{20}\frac{7a^4 + 15a^2b^2 + 10b^4}{a^2 + 3b^2}.$$

Euler; *Ib.* p. 201.

SECT. 8. *Principal Axes of a Plane Lamina at any proposed point in the Lamina.*

One of the axes is at right angles to the lamina. Let x, y, be the co-ordinates of any infinitesimal element m of the lamina, referred to axes in the plane of the lamina which pass through the point. Then, θ denoting the inclination of either of the other two principal axes at the point to the axis of x,

$$\tan 2\theta = \frac{2\Sigma\,(mxy)}{\Sigma m\,(x^2 - y^2)}.$$

(1) To find the principal axes of the area of a lamina, in the form of a semi-loop of a lemniscate, at the node.

We have

$$\tan 2\theta = \frac{2\Sigma\,(mxy)}{\Sigma m\,(x^2 - y^2)}$$

$$= 2\,\frac{\iint r^3 dr\,.\,\sin\theta\cos\theta\,d\theta}{\iint r^3 dr\,.\,\cos 2\theta\,d\theta},$$

$$= \frac{\int_0^{\frac{\pi}{4}} \cos^2 2\theta \sin 2\theta\,d\theta}{\int_0^{\frac{\pi}{4}} (1 - \sin^2 2\theta)\cos 2\theta\,d\theta} = \tfrac{1}{2}.$$

Thus the two principal axes, which lie in the plane of the lamina, are inclined to the axis of the lemniscate at angles

$$\frac{1}{2}\tan^{-1}\frac{1}{2},\quad \frac{1}{2}\left(\pi+\tan^{-1}\frac{1}{2}\right).$$

(2) To find the principal axes at an end of a wire in the form of a semicircle.

One of the axes is at right angles to the plane of the wire, and the inclinations of the other two to the chord of the semicircle are given by the equation

$$\tan 2\theta = \frac{4}{\pi}.$$

(3) To find the principal axes of a right-angled triangle at the right angle.

One of the axes is perpendicular to its plane and the other two are in its plane and are inclined to its sides at angles equal to $\frac{1}{2}\tan^{-1}\left(\frac{\tan 2\alpha}{\cancel{4}\ 7}\right)$, where α is one of the acute angles of the triangle.

Griffin; *Solutions of the examples on the motion of a Rigid Body*, p. 8.

(4) A parabolic area is included between the curve, the axis, and the semi-latus rectum: to find the positions of the principal axes of the area at the vertex of the parabola.

If θ be the inclination of either principal axis, in the plane of the area, to the axis of the parabola,

$$\tan 2\theta = \frac{35}{13}.$$

(5) To find the principal axes at a point in the circumference of an elliptic lamina.

One of the axes is at right angles to the lamina and, θ denoting the inclination of either of the other two to the major axis,

$$\tan 2\theta = \frac{8xy}{a^2 - b^2 + 4(x^2 - y^2)},$$

where x and y are the co-ordinates of the point referred to the axes of the ellipse as axes of co-ordinates.

Griffin; *Solutions of the Examples on the motion of a Rigid Body*, p. 8.

CHAPTER VI.

D'ALEMBERT'S PRINCIPLE.

A GENERAL method for the determination of the motion of a material system, acted on by any forces, was laid down by D'Alembert in his *Traité de Dynamique*, published in the year 1743[1], from which we have extracted the following passage in exposition of the Principle[2].

"Problême Général.

"Soit donné un systême de corps disposés les uns par rapport aux autres d'une manière quelconque; et supposons qu'on imprime à chacun de ces corps un mouvement particulier, qu'il ne puisse suivre à cause de l'action des autres corps; trouver le mouvement que chaque corps doit prendre.

"Solution.

"Soient A, B, C, &c. les corps qui composent le systême, et supposons qu'on leur ait imprimé les mouvemens a, b, c, etc. qu'ils soient forcés, à cause de leur action mutuelle, de changer dans les mouvemens a, b, c, etc. Il est clair qu'on peut regarder le mouvement a imprimé au corps A comme composé du mouvement a, qu'il a pris, et d'un autre mouvement α; qu'on peut de même regarder les mouvemens b, c, etc. comme composés des mouvemens b, β; c, κ; etc. d'où il s'ensuit que le mouvement des corps A, B, C, etc. entr'eux auroit été le même, si au lieu de leur donner les impulsions a, b, c, etc. on leur eût donné à-la-fois les doubles impulsions a, α; b, β; c, κ, etc. Or par la supposition, les corps A, B, C, etc. ont pris d'eux-mêmes les mouvemens a, b, c, etc. donc les mouvemens α, β, κ, etc. doivent être

[1] See also his *Recherches sur la Précession des Equinoxes*, p. 35, published in 1749.

[2] D'Alembert's Principle was first enunciated by him in a memoir which he read before the Academy of Sciences at the end of the year 1742.

tels qu'ils ne derangent rien dans les mouvemens a, b, c, etc. c'est-à-dire que, si les corps n'avoient reçu que les mouvemens α, β, κ, etc. ces mouvemens auroient dû se détruire mutuellement, et le systême demeurer en repos.

"De là résulte le principe suivant, pour trouver le mouvement de plusieurs corps qui agissent les uns sur les autres. *Décomposez les mouvemens a, b, c, etc. imprimés à chaque corps, chacun en deux autres* a, α; b, β; c, κ; *etc. qui soient tels, que si l'on n'eût imprimé aux corps que les mouvemens* a, b, c, *etc. ils eussent pu conserver ces mouvemens sans se nuire réciproquement; et que si on ne leur eût imprimé que les mouvemens* α, β, κ, *etc. le systême fût demeuré en repos; il est clair que* a, b, c, *etc. seront les mouvemens que ces corps prendront en vertu de leur action. Ce qu'il falloit trouver.*"

The idea of the general method developed by D'Alembert for the determination of the motion of material systems, had occurred somewhat earlier to Fontaine, as we are informed in the *Table des Mémoires*, prefixed to his *Traité de Calcul Différentiel et Integral*[1], having been communicated by him to the Academy of Sciences in the year 1739, and subsequently to several mathematicians. His views, however, on this subject were not made public till long after the appearance of the *Traité de Dynamique;* and in all probability D'Alembert, who did not become a member of the Academy before the year 1741, was not aware of Fontaine's generalization. D'Alembert, however, was the first to shew the wonderful fertility of the Principle by applying it to the solution of a great variety of difficult problems, among which may be mentioned that of the Precession of the Equinoxes, which had been inadequately attempted by Newton, and of which D'Alembert was the first to obtain a complete solution.

The earliest step towards the discovery of D'Alembert's Principle is to be met with in a memoir by James Bernoulli in the *Acta Eruditorum*, 1686, Jul. p. 356, entitled "Narratio Controversiæ inter Dn. Hugenium et Abbatem Catelanum agitatæ de Centro Oscillationis quæ loco animadversionis esse poterit in

[1] *Mémoires de l'Académie des Sciences de Paris*, 1770.

Responsionem Dn. Catelani, num. 27. Ephem. Gallic. anni 1684, insertam." Let m, m', denote two equal bodies attached to an inflexible straight line which is capable of motion in a vertical plane about one extremity which is fixed; let r, r', denote the distances of m, m', respectively, from the fixed extremity; v, v', their velocities for any position of the inflexible line in its descent from an assigned position; u, u', the velocities which they would have acquired by descending down the same arcs unconnectedly. Then, in consequence of the connection of the bodies, a velocity $u - v$ will be lost by m and a velocity $v' - u'$ gained by m' in their descent. Bernoulli proposes it to the consideration of mathematicians whether, according to the statical relation of two forces in equilibrium on a lever, the proportion $u - v : v' - u' :: r' : r$ be an accurate expression of the circumstances of the motion. This idea of Bernoulli's, although not free from error, contains however the first germ of the Principle of reducing the determination of the motions of material systems to the solution of statical problems. L'Hôpital, in a letter addressed to Huyghens[1], correctly observed that instead of considering the velocities acquired in a finite time, he should have considered the infinitesimal velocities acquired in an instant of time, and have compared them with those which gravity tends to impress upon the bodies during the same instant. He takes a complex pendulum, consisting of any two bodies attached to an inflexible straight line, and considers equilibrium to subsist between the quantities of motion lost and gained by these bodies in any instant of time, that is, between the differences of the quantities of motion which the bodies really acquire in this instant, and those which gravity tends to impress on them. He applies this Principle, which agrees with the general Principle of D'Alembert, to the determination of the Centre of Oscillation of a pendulum consisting of two bodies attached to an inflexible straight line oscillating about one extremity. He then extends his theory to a greater number of bodies in a straight line, and determines their Centre of Oscillation on the supposition, the truth of which is not however sufficiently obvious without demonstration, that

[1] *Histoire des Ouvrages des Sçavans*, 1690, *Juin*, p. 440.

any two of them may be collected at their particular Centre of Oscillation. On the publication of L'Hôpital's letter, James Bernoulli[1] reverted to the subject of the Centre of Oscillation, and at length succeeded in obtaining a direct and rigorous solution of the problem in the case where all the bodies are in one line, by the application of the principle laid down by L'Hôpital. Bernoulli[2] afterwards extended his method to the general case of the oscillations of bodies of any figure.

An ingenious investigation of the Centre of Oscillation, a problem from the beginning intimately connected with the development of D'Alembert's Principle, was shortly afterwards given by Brook Taylor[3] and John Bernoulli[4], between whom arose an angry controversy respecting priority of discovery[5]; the method given by these mathematicians, although depending upon the statical principles of the lever, did not however involve, in an explicit form, L'Hôpital's Principle of Equilibrium. Finally, Hermann[6] determined the Centre of Oscillation by the principle of the statical equivalence of the *solicitations of gravity*, and the *vicarious solicitations* applied in opposite directions, or, as it is expressed by modern mathematicians, by the equilibrium subsisting between the impressed forces of gravity and the effective forces applied in opposite directions; a method of investigation virtually coincident with that given by James Bernoulli. The idea of L'Hôpital became still more general in the hands of Euler[7], in a memoir on the determination of the oscillations of flexible strings printed in the year 1740. From the above historical sketch it will be easily seen that in the enunciation of a general Principle of Motion, Fontaine and D'Alembert had little more to do than to express in general language what had been distinctly conceived in the prosecution of particular re-

[1] *Acta Erudit. Lips.* 1691, *Jul.* p. 317, *Opera*, Tom. I. p. 460.

[2] *Mémoires de l'Académie des Sciences de Paris*, 1703, 1704.

[3] *Philosophical Transactions*, 1714, *May. Methodus Incrementorum.*

[4] *Acta Erudit. Lips.* 1714, *Jun.* p. 257; *Mém. Acad. Par.* 1714, p. 208, *Opera*, Tom. II. p. 168.

[5] *Act. Erudit. Lips.* 1716, 1718, 1719, 1721, 1722.

[6] *Phoronomia;* Lib. I. cap. 5.

[7] *Comment. Petrop.* Tom. VII.

searches by L'Hôpital, James and John Bernoulli, Brook Taylor, Hermann, and Euler. For additional information on the historical development of D'Alembert's Principle, the reader is referred to Lagrange's *Mécanique Analytique*, Seconde Partie, Section 1; Montucla's *Histoire des Mathématiques*, part. V. liv. 3, part. IV. liv. 7; and Whewell's *History of the Inductive Sciences*, Vol. II.

In modern treatises on Mechanics, D'Alembert's Principle is expressed under one or other of the following forms:

(1) When any material system is in motion under the action of any forces, the moving forces lost by the different molecules of the system must be in equlibrium.

(2) If the effective moving forces of the several particles of a system be applied to them in directions opposite to those in which they act; they will, conjointly with the impressed moving forces, constitute a system of forces statically disposed.

The former of these enunciations it will be seen is substantially the same as that given by D'Alembert, while the latter is a generalization of the idea developed by Hermann in his investigations on the particular problem of the Centre of Oscillation.

SECT. 1. *Motion of a single Particle*[1].

The object of this section is to apply D'Alembert's Principle to the exemplification of a general method for the determination of the motion of a particle within tubes and between contiguous surfaces, of which either the position, or the form, or both, are made to vary according to any assigned law whatever, the particle being acted on by given forces. Several of the problems of this section have been solved by particular methods in Chapter IV.

I. We will commence with the consideration of the motion of a particle along a tube, and, for the sake of perfect generality, we will suppose the tube to be one of double curvature. The tube is considered in all cases to be indefinitely narrow and

[1] The substance of this Section was published in the *Cambridge Mathematical Journal*, Vol. III. p. 49.

perfectly smooth, and every section at right angles to its axis to be circular.

Let the particle be referred to three fixed rectangular axes, and let x, y, z, be its co-ordinates at any time t; let x, y, z, become $x+\delta x$, $y+\delta y$, $z+\delta z$, when t becomes $t+\delta t$; δt, and consequently δx, δy, δz, being considered to be indefinitely small. Then the effective accelerating forces on the particle parallel to the three fixed axes will be, at the time t,

$$\frac{\delta^2 x}{\delta t^2}, \quad \frac{\delta^2 y}{\delta t^2}, \quad \frac{\delta^2 z}{\delta t^2}.$$

Also, let X, Y, Z, represent the impressed accelerating forces on the particle resolved parallel to the axes of x, y, z; and let $x+dx$, $y+dy$, $z+dz$, be the co-ordinates of a point in the tube very near to the point x, y, z, which the particle occupies at the time t. Then, observing that the action of the tube on the particle is always at right angles to its axis at every point and therefore, at the time t, to the line joining the two points x, y, z, and $x+dx$, $y+dy$, $z+dz$, we have, by D'Alembert's Principle, combined with the Principle of Virtual Velocities,

$$\left(\frac{\delta^2 x}{\delta t^2} - X\right) dx + \left(\frac{\delta^2 y}{\delta t^2} - Y\right) dy + \left(\frac{\delta^2 z}{\delta t^2} - Z\right) dz = 0 \ldots\ldots\text{(A)}.$$

Again, since the form and position of the tube are supposed to vary according to some assigned law, it is clear that, when t is known, the equations to the tube must be known; hence it is evident that, in addition to the equation (A), we shall have, from the particular conditions of each individual problem, a number of equations equivalent to two of the form

$$\phi(x, y, z, t) = 0, \qquad \chi(x, y, z, t) = 0 \ldots\ldots\ldots\text{(B)};$$

where ϕ and χ are symbols of functionality depending upon the law of the variations of the form and position of the tube.

The three equations (A) and (B) involve the four quantities x, y, z, t, and therefore, in any particular case, if the difficulty of the analytical processes be not insuperable, we may ascertain x, y, z, each of them in terms of t; in which consists the complete solution of the problem.

If the tube remain during the whole of the motion within one

plane, then, the plane of x, y, being so chosen as to coincide with this plane, the three equations (A) and (B) will evidently be reduced to the two

$$\left(\frac{\delta^2 x}{\delta t^2} - X\right) dx + \left(\frac{\delta^2 y}{\delta t^2} - Y\right) dy = 0 \dots\dots\dots\dots\dots (C),$$

$$\phi(x, y, t) = 0 \dots\dots\dots\dots\dots\dots\dots\dots (D).$$

We proceed to illustrate the general formulæ of the motion by the discussion of a few problems.

(1) A rectilinear tube revolves with a uniform angular velocity about one extremity in a horizontal plane: to find the motion of a particle within the tube.

Let ω be the constant angular velocity; r the distance of the particle at any time t from the fixed extremity of the tube; then, the plane of x, y, being taken horizontal, and the origin of co-ordinates at the fixed extremity of the tube, we shall have, supposing the tube initially to coincide with the axis of x,

$$x = r \cos \omega t \dots\dots\dots\dots\dots\dots\dots\dots (1),$$

$$y = r \sin \omega t \dots\dots\dots\dots\dots\dots\dots\dots (2).$$

From (1) we have

$$dx = dr \cos \omega t,$$

and, from (2),

$$dy = dr \sin \omega t.$$

Again, from (1) we have

$$\frac{\delta x}{\delta t} = \frac{\delta r}{\delta t} \cos \omega t - \omega r \sin \omega t,$$

$$\frac{\delta^2 x}{\delta t^2} = \frac{\delta^2 r}{\delta t^2} \cos \omega t - 2\omega \frac{\delta r}{\delta t} \sin \omega t - \omega^2 r \cos \omega t;$$

and, from (2),

$$\frac{\delta y}{\delta t} = \frac{\delta r}{\delta t} \sin \omega t + \omega r \cos \omega t,$$

$$\frac{\delta^2 y}{\delta t^2} = \frac{\delta^2 r}{\delta t^2} \sin \omega t + 2\omega \frac{\delta r}{\delta t} \cos \omega t - \omega^2 r \sin \omega t.$$

Substituting in the general formula (C) the values which we have obtained for dx, dy, $\frac{\delta^2 x}{\delta t^2}$, $\frac{\delta^2 y}{\delta t^2}$, we have, since $X = 0$, $Y = 0$,

$$\cos\omega t\left(\frac{\delta^2 r}{\delta t^2}\cos\omega t - 2\omega\frac{\delta r}{\delta t}\sin\omega t - \omega^2 r\cos\omega t\right)$$
$$+\sin\omega t\left(\frac{\delta^2 r}{\delta t^2}\sin\omega t + 2\omega\frac{\delta r}{\delta t}\cos\omega t - \omega^2 r\sin\omega t\right) = 0;$$

and therefore

$$\frac{\delta^2 r}{\delta t^2} - \omega^2 r = 0:$$

the integral of this equation is

$$r = C\epsilon^{\omega t} + C'\epsilon^{-\omega t}.$$

Let $r = a$ when $t = 0$; then

$$a = C + C':$$

also let $\frac{\delta r}{\delta t} = \beta$ when $t = 0$; then

$$\beta = C\omega - C'\omega:$$

from the two equations for the determination of C and C', we have

$$C = \frac{a\omega + \beta}{2\omega}, \qquad C' = \frac{a\omega - \beta}{2\omega}:$$

hence, for the motion of the particle along the tube,

$$2\omega r = (a\omega + \beta)\,\epsilon^{\omega t} + (a\omega - \beta)\,\epsilon^{-\omega t}.$$

This problem, which is the earliest problem of the motion of a particle subject to the constraint of a curve moving according to a prescribed law, is due to John Bernoulli[1]. A solution of this problem is given also by Clairaut[2], to whom it had probably been proposed by Bernoulli.

(2) Suppose the tube to revolve in a vertical instead of a horizontal plane, and let the time be reckoned from the moment of coincidence of the tube with the axis of x: then, the axes of x and y being supposed to be respectively horizontal and vertical, we have by the same process, since $X = 0$ and $Y = -g$,

$$\frac{\delta^2 r}{\delta t^2} - \omega^2 r = -g\sin\omega t.$$

[1] *Opera*, Tom. IV. p. 248.
[2] *Mémoires de l'Académie des Sciences de Paris*, 1742, p. 10.

The integral of this equation is

$$r = C\epsilon^{\omega t} + C'\epsilon^{-\omega t} + \frac{g}{2\omega^2}\sin\omega t;$$

and, if we determine the constants from the conditions that r, $\frac{\delta r}{\delta t}$, shall have initially values a, β, we shall get for the motion along the tube,

$$2\omega r = \left(a\omega + \beta - \frac{g}{2\omega}\right)\epsilon^{\omega t} + \left(a\omega - \beta + \frac{g}{2\omega}\right)\epsilon^{-\omega t} + \frac{g}{\omega}\sin\omega t.$$

This problem, of which an erroneous solution had been given by Barbier in the *Annales de Gergonne*, Tom. XIX., was correctly solved, in the following volume, by Ampère. In the *Cambridge Mathematical Journal*, Vol. III. p. 42, a solution is given by Professor Booth, who has discussed at length the more interesting cases of the motion.

(3) Suppose the tube to revolve in a horizontal plane about a fixed extremity with such an angular velocity, that the tangent of its angle of inclination to the axis of x is proportional to the time.

The equation to the tube at any time t will be

$$y = mtx \quad\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where m is some constant quantity: hence

$$dy = mt\,dx,$$

and therefore, from (C), since $X = 0$ and $Y = 0$,

$$\frac{\delta^2 x}{\delta t^2} + mt\frac{\delta^2 y}{\delta t^2} = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

But from (1) we have

$$\frac{\delta y}{\delta t} = mt\frac{\delta x}{\delta t} + mx,$$

$$\frac{\delta^2 y}{\delta t^2} = mt\frac{\delta^2 x}{\delta t^2} + 2m\frac{\delta x}{\delta t}:$$

hence, from (2),

$$(1 + m^2t^2)\frac{\delta^2 x}{\delta t^2} + 2m^2t\frac{\delta x}{\delta t} = 0,$$

$$\frac{\frac{\delta^2 x}{\delta t^2}}{\frac{\delta x}{\delta t}} + \frac{2m^2 t}{1 + m^2 t^2} = 0.$$

Integrating, we have

$$\log \frac{\delta x}{\delta t} + \log (1 + m^2 t^2) = \log C,$$

$$\frac{\delta x}{\delta t} (1 + m^2 t^2) = C.$$

Let β be the initial value of $\frac{\delta x}{\delta t}$, which will be the velocity of projection along the tube; then $C = \beta$, and therefore

$$\frac{\delta x}{\delta t} (1 + m^2 t^2) = \beta, \qquad \delta x = \frac{\beta \delta t}{1 + m^2 t^2}:$$

integrating, we get

$$x + C = \frac{\beta}{m} \tan^{-1} (mt).$$

Let $x = a$ when $t = 0$; then $a + C = 0$, and therefore

$$x = a + \frac{\beta}{m} \tan^{-1} (mt),$$

and consequently, from (1),

$$y = amt + \beta t \tan^{-1} (mt).$$

If θ be the inclination of the tube to the axis of x at any time, and r be the distance of the particle from the fixed extremity,

$$r = \frac{am + \beta\theta}{m \cos \theta}.$$

(4) A circular tube is constrained to move in a horizontal plane with a uniform angular velocity about a fixed point in its circumference: to determine the motion of a particle within the tube, which is placed initially at the extremity of the diameter passing through the fixed point.

Let the fixed point be taken as the origin of co-ordinates, and let the axis of x coincide with the initial position of the diameter through this point: let ω be the angular velocity of the

circle, a the radius: also let θ be the angle, at any time t, between the diameter through the particle and the diameter through the origin.

Then it will be easily seen that

$$x = a\cos\omega t + a\cos(\omega t - \theta) \quad\ldots\ldots\ldots\ldots(1),$$

$$y = a\sin\omega t + a\sin(\omega t - \theta) \quad\ldots\ldots\ldots\ldots(2).$$

From (1) we have

$$dx = ad\theta\sin(\omega t - \theta),$$

and, from (2),

$$dy = -ad\theta\cos(\omega t - \theta).$$

Hence, from (C), observing that $X = 0$ and $Y = 0$,

$$\sin(\omega t - \theta)\frac{\delta^2 x}{\delta t^2} - \cos(\omega t - \theta)\frac{\delta^2 y}{\delta t^2} = 0 \quad\ldots\ldots\ldots (3).$$

Again, from (1),

$$\frac{\delta x}{\delta t} = -a\omega\sin\omega t + a\left(\frac{\delta\theta}{\delta t} - \omega\right)\sin(\omega t - \theta),$$

$$\frac{\delta^2 x}{\delta t^2} = -a\omega^2\cos\omega t - a\left(\frac{\delta\theta}{\delta t} - \omega\right)^2\cos(\omega t - \theta) + a\frac{\delta^2\theta}{\delta t^2}\sin(\omega t - \theta);$$

and, from (2),

$$\frac{\delta y}{\delta t} = a\omega\cos\omega t - a\left(\frac{\delta\theta}{\delta t} - \omega\right)\cos(\omega t - \theta),$$

$$\frac{\delta^2 y}{\delta t^2} = -a\omega^2\sin\omega t - a\left(\frac{\delta\theta}{\delta t} - \omega\right)^2\sin(\omega t - \theta) - a\frac{\delta^2\theta}{\delta t^2}\cos(\omega t - \theta);$$

and therefore, by (3),

$$a\omega^2\{\sin\omega t\cos(\omega t - \theta) - \cos\omega t\sin(\omega t - \theta)\} + a\frac{\delta^2\theta}{\delta t^2} = 0,$$

$$\omega^2\sin\theta + \frac{\delta^2\theta}{\delta t^2} = 0:$$

multiplying by $2\frac{\delta\theta}{\delta t}$, and integrating,

$$\frac{\delta\theta^2}{\delta t^2} = C + 2\omega^2\cos\theta.$$

But, the absolute velocity of the particle being initially zero, it

is clear that 2ω will be the initial value of $\frac{\delta\theta}{\delta t}$; and therefore, θ being initially zero, we have

$$4\omega^2 = C + 2\omega^2, \qquad C = 2\omega^2,$$

and therefore

$$\frac{\delta\theta^2}{\delta t^2} = 2\omega^2(1+\cos\theta) = 4\omega^2\cos^2\frac{\theta}{2}, \quad \frac{\delta\theta}{\delta t} = 2\omega\cos\frac{\theta}{2},$$

$$\frac{\cos\frac{\theta}{2}\,\delta\theta}{\cos^2\frac{\theta}{2}} = 2\omega\delta t, \qquad \frac{\delta\sin\frac{\theta}{2}}{1-\sin^2\frac{\theta}{2}} = \omega\delta t.$$

Integrating, we have

$$\log\frac{1+\sin\frac{\theta}{2}}{1-\sin\frac{\theta}{2}} = 2\omega t + C:$$

but $\theta = 0$ when $t = 0$; hence $C = 0$, and we have

$$\frac{1+\sin\frac{\theta}{2}}{1-\sin\frac{\theta}{2}} = \epsilon^{2\omega t},$$

and therefore

$$\sin\frac{\theta}{2} = \frac{\epsilon^{\omega t} - \epsilon^{-\omega t}}{\epsilon^{\omega t} + \epsilon^{-\omega t}},$$

which determines the position of the particle within the tube at any time. When $t = \infty$, we have $\sin\frac{\theta}{2} = 1$, and therefore $\theta = \pi$, which shews that, after the lapse of an infinite time, the particle will arrive at the point of rotation.

(5) If we pursue the same course as in the solution of the problems (1), (2), (4), we may obtain a convenient formula for the following more general problem: A plane curvilinear tube of any invariable form whatever revolves in its own plane about a fixed point with a uniform angular velocity: to determine the motion of a particle, acted on by any forces, within the tube.

Let ω be the constant angular velocity of the tube about the fixed point; r the distance of the particle at any time from this point; ϕ the angle between the simultaneous directions of r and of a line joining an assigned point of the tube with the fixed point of rotation; ds an element of the length of the tube at the place of the particle, and S the accelerating force on the particle resolved along the element ds: then the equation for the motion of the particle will be

$$r^2 \frac{\delta\phi^2}{\delta t^2} + \frac{\delta r^2}{\delta t^2} - \omega^2 r^2 = 2\int S \frac{ds}{d\phi}\,\delta\phi :$$

but since, the form of the tube being invariable, $\delta\phi$, δr, may evidently be replaced by $d\phi$, dr, we have, putting, for the sake of uniformity of notation, dt in place of δt,

$$r^2 \frac{d\phi^2}{dt^2} + \frac{dr^2}{dt^2} - \omega^2 r^2 = 2\int S ds.$$

If ω be zero, the formula will become

$$r^2 \frac{d\phi^2}{dt^2} + \frac{dr^2}{dt^2} = 2\int S ds,$$

the well-known formula for the motion of a particle under the action of any forces within an immoveable plane tube.

(6) In the foregoing examples the position of the tube varies with the time; the form however remains invariable. We will now give an example in which the form changes with the time.

A particle is projected with a given velocity within a circular tube, the radius of which increases in proportion to the time while the centre remains stationary: to determine the motion of the particle, the tube being supposed to lie always in a horizontal plane.

The equation to the circle will be

$$x^2 + y^2 = a^2 (1 + \alpha t)^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where a and α are some constant quantities: hence

$$x\,dx + y\,dy = 0,$$

and therefore, by the general formula (C),

$$y\frac{\delta^2 x}{\delta t^2} - x\frac{\delta^2 y}{\delta t^2} = 0:$$

integrating, we have

$$y\frac{\delta x}{\delta t} - x\frac{\delta y}{\delta t} = C.$$

Let the axis of x be so chosen as to coincide with the initial distance of the particle from the centre, and let β be the initial velocity of the particle along the tube; then $C = -a\beta$, and therefore

$$x\frac{\delta y}{\delta t} - y\frac{\delta x}{\delta t} = a\beta \text{ ....} (2):$$

again, from (1), we have

$$x\frac{\delta x}{\delta t} + y\frac{\delta y}{\delta t} = a^2\alpha\,(1+\alpha t) \text{ ....} (3):$$

multiplying (2) by y and (3) by x, and subtracting the former result from the latter, we have

$$(x^2+y^2)\frac{\delta x}{\delta t} = a^2\alpha\,(1+\alpha t)\,x - a\beta y,$$

and therefore, by (1),

$$a\,(1+\alpha t)^2\frac{\delta x}{\delta t} = a\alpha\,(1+\alpha t)\,x - \beta\,\{a^2(1+\alpha t)^2 - x^2\}^{\frac{1}{2}}.$$

Put $1+\alpha t = \alpha\tau$; then

$$a\tau^2\frac{\delta x}{\delta\tau} = a\tau x - \frac{\beta}{\alpha^2}(a^2\alpha^2\tau^2 - x^2)^{\frac{1}{2}}:$$

again, put $x = m\tau$, and there is

$$a\tau^2\left(m + \tau\frac{\delta m}{\delta\tau}\right) = am\tau^2 - \frac{\beta\tau}{\alpha^2}(a^2\alpha^2 - m^2)^{\frac{1}{2}},$$

$$a\tau^3\frac{\delta m}{\delta\tau} = -\frac{\beta\tau}{\alpha^2}(a^2\alpha^2 - m^2)^{\frac{1}{2}}$$

$$-\frac{a\alpha^2\,\delta m}{(a^2\alpha^2 - m^2)^{\frac{1}{2}}} = \beta\frac{\delta\tau}{\tau^2}:$$

integrating, $$C + a\alpha^2\cos^{-1}\frac{m}{a\alpha} = -\frac{\beta}{\tau},$$

or, putting for m its value,

$$C + a\alpha^2 \cos^{-1} \frac{x}{a\alpha\tau} = -\frac{\beta}{\tau},$$

and, putting for τ its value $\frac{1}{\alpha}(1 + \alpha t)$,

$$C + a\alpha^2 \cos^{-1} \frac{x}{a(1+\alpha t)} = -\frac{\alpha\beta}{1+\alpha t}.$$

Now $x = a$ when $t = 0$; hence $C = -\alpha\beta$, and therefore

$$a\alpha^2 \cos^{-1} \frac{x}{a(1+\alpha t)} = \frac{\alpha^2\beta t}{1+\alpha t},$$

$$x = a(1+\alpha t) \cos \frac{\beta t}{a(1+\alpha t)},$$

and therefore, from (1),

$$y = a(1+\alpha t) \sin \frac{\beta t}{a(1+\alpha t)};$$

results which give the absolute position of the particle at any assigned time.

II. We proceed now to the consideration of the motion of a particle along a surface from which it is unable to detach itself, while the surface itself changes its position or its form, or both, according to any assigned law. To fix the ideas, we suppose the particle to move between two surfaces indefinitely close together, so as to be expressed by the same equation.

Let x, y, z, be the co-ordinates of the particle at any time t; and let $\delta x, \delta y, \delta z$, be the increments of x, y, z, in an indefinitely small time δt: also let dx, dy, dz, denote the increments of x, y, z, in passing from the point x, y, z, to any point near to it within the surface as it exists at the time t. Also let X, Y, Z, denote the resolved parts of the accelerating forces on the particle at the time t parallel to the axes of x, y, z: then, observing that the action of the surface on the particle is always in the direction of the normal at each point, we have, by D'Alembert's Principle combined with the Principle of Virtual Velocities,

$$\left(\frac{\delta^2 x}{\delta t^2} - X\right) dx + \left(\frac{\delta^2 y}{\delta t^2} - Y\right) dy + \left(\frac{\delta^2 z}{\delta t^2} - Z\right) dz = 0 \,\ldots\ldots\ldots (\text{A}').$$

Again, since the position and form of the surface vary according to an assigned law, its equation must evidently be known at any given time, and therefore we must have, from the nature of each particular problem, certain conditions between the quantities x, y, z, t, equivalent to a single equation

$$F = f(x, y, z, t) = 0 \text{ .... } (B').$$

Taking the total differential of (B′), we have

$$\frac{dF}{dx}dx + \frac{dF}{dy}dy + \frac{dF}{dz}dz = 0:$$

multiplying this equation by an arbitrary quantity λ and subtracting the resulting equation from (A′), we have

$$\left(\frac{\delta^2 x}{\delta t^2} - X - \lambda\frac{dF}{dx}\right)dx + \left(\frac{\delta^2 y}{\delta t^2} - Y - \lambda\frac{dF}{dy}\right)dy + \left(\frac{\delta^2 z}{\delta t^2} - Z - \lambda\frac{dF}{dz}\right)dz = 0.$$

Since λ is arbitrary, we may equate to zero the coefficients of any one of the three differentials dx, dy, dz: and, since the remaining two of the three differentials are independent of each other, their coefficients must be both zero: hence, eliminating λ between the three resulting equations, we have

$$\left(\frac{\delta^2 y}{\delta t^2} - Y\right)\frac{dF}{dz} = \left(\frac{\delta^2 z}{\delta t^2} - Z\right)\frac{dF}{dy},$$

$$\left(\frac{\delta^2 z}{\delta t^2} - Z\right)\frac{dF}{dx} = \left(\frac{\delta^2 x}{\delta t^2} - X\right)\frac{dF}{dz},$$

$$\left(\frac{\delta^2 x}{\delta t^2} - X\right)\frac{dF}{dy} = \left(\frac{\delta^2 y}{\delta t^2} - Y\right)\frac{dF}{dx}.$$

Any two of these three relations, together with the equation (B′), will give us three equations in x, y, z, t, whence x, y, z, are to be determined in terms of t.

The following example will serve to illustrate the use of these equations. We have taken a case where the form of the surface remains invariable, its position alone being liable to change. The analysis, however, in the solution of problems of the class which we are considering, receives its general character solely in consequence of the presence of t in the equation (B′), and therefore the example which we have chosen is sufficient for the general object we have in view.

A particle descends by the action of gravity down a plane which revolves uniformly about a vertical axis through which it passes: to determine the motion of the particle.

Let the plane of x, y, be horizontal, the axis of x coinciding with the initial intersection of the revolving plane with the horizontal plane through the origin, and let the axis of z be taken vertically downwards: then, ω denoting the angular velocity of the plane, its equation at any time t will be

$$F = y \cos \omega t - x \sin \omega t = 0 \text{ .... } (1);$$

whence $\quad \dfrac{dF}{dx} = -\sin \omega t, \quad \dfrac{dF}{dy} = \cos \omega t, \quad \dfrac{dF}{dz} = 0:$

also, $X = 0$, $Y = 0$, $Z = g$; and therefore, from either of the first two of the three general relations,

$$\frac{\delta^2 z}{\delta t^2} = g \text{ .... } (2),$$

and, from the third,

$$\frac{\delta^2 x}{\delta t^2} \cos \omega t + \frac{\delta^2 y}{\delta t^2} \sin \omega t = 0 \text{ .... } (3).$$

Let r denote the distance of the particle at any time from the axis of z; then

$$x = r \cos \omega t, \quad y = r \sin \omega t,$$

whence $\quad \dfrac{\delta x}{\delta t} = \dfrac{\delta r}{\delta t} \cos \omega t - \omega r \sin \omega t,$

$$\frac{\delta^2 x}{\delta t^2} = \frac{\delta^2 r}{\delta t^2} \cos \omega t - 2\omega \frac{\delta r}{\delta t} \sin \omega t - \omega^2 r \cos \omega t,$$

$$\frac{\delta y}{\delta t} = \frac{\delta r}{\delta t} \sin \omega t + \omega r \cos \omega t,$$

$$\frac{\delta^2 y}{\delta t^2} = \frac{\delta^2 r}{\delta t^2} \sin \omega t + 2\omega \frac{\delta r}{\delta t} \cos \omega t - \omega^2 r \sin \omega t;$$

and therefore, from (3),

$$\frac{\delta^2 r}{\delta t^2} - \omega^2 r = 0 \text{ .... } (4).$$

Let the initial values of z, $\frac{\delta z}{\delta t}$, be 0, β; and those of r, $\frac{\delta r}{\delta t}$, be a, α; then, from the equations (2) and (4), after executing obvious operations, we shall obtain

$$z = \tfrac{1}{2}gt^2 + \beta t,$$

$$2\omega r = (\omega a + \alpha)\,\epsilon^{\omega t} + (\omega a - \alpha)\,\epsilon^{-\omega t},$$

and $$\log \frac{(\omega^2 r^2 + \alpha^2 - \omega^2 a^2)^{\frac{1}{2}} + \omega r}{\alpha + \omega a} = \frac{\omega}{g}\{(2gz + \beta^2)^{\frac{1}{2}} - \beta\}:$$

the first two of these equations give the position of the particle on the revolving plane, and therefore, by virtue of the equation (1), the absolute position of the particle at any time; while the third is the equation to the path which the particle describes on the plane.

SECT. 2. *Systems of Particles.*

(1) Two heavy particles P, P' (fig. 179), are attached to a rigid imponderable rod APP', which is oscillating in a vertical plane about a fixed point at its extremity A: to determine the motion.

Let m, m', be the masses of the two particles; let $AP = a$, $AP' = a'$. Draw AB vertically downwards and let $\angle PAB = \theta$. Let ds, ds', denote the elements of the circular paths described by P, P', in a small time dt, estimated in a direction corresponding to an increase of θ. Then the effective moving forces of the two particles will be $m\frac{d^2 s}{dt^2}$, $m'\frac{d^2 s'}{dt^2}$, the moments of which about the point A will be $ma\frac{d^2 s}{dt^2}$, $m'a'\frac{d^2 s'}{dt^2}$. Also the moments of the impressed forces will be $-mag\sin\theta$, $-m'a'g\sin\theta$. Hence, for the equilibrium of the impressed forces, and the effective forces applied in directions opposite to their own, we have

$$ma\frac{d^2 s}{dt^2} + m'a'\frac{d^2 s'}{dt^2} + (ma + m'a')g\sin\theta = 0.$$

But $ds = ad\theta$, $ds' = a'd\theta$; hence

$$(ma^2 + m'a'^2)\frac{d^2\theta}{dt^2} + (ma + m'a')\,g\sin\theta = 0;$$

a result which shews that the rod will oscillate isochronously with a perfect pendulum of which the length is

$$\frac{ma^2 + m'a'^2}{ma + m'a'}.$$

(2) Two particles, attached to the extremities of a fine inextensible thread, are placed upon two inclined planes with a common summit: to determine the motion of the particles and the tension of the thread at any time.

Let m, m', be the masses of the particles; α, α', the inclinations of the planes to the horizon; x, x', the distances of the particles from the common summit of the planes at any time. Then the impressed accelerating forces on the particles m, m', estimated down the two planes, will be $g\sin\alpha$, $g\sin\alpha'$, respectively, and the effective accelerating forces, estimated in the same directions, will be $\dfrac{d^2x}{dt^2}$, $\dfrac{d^2x'}{dt^2}$. Hence, for the equilibrium of the impressed moving forces, and the effective moving forces applied in directions opposite to their own, we have

$$g(m\sin\alpha - m'\sin\alpha') = m\frac{d^2x}{dt^2} - m'\frac{d^2x'}{dt^2} \;\ldots\ldots\ldots\ldots(1).$$

But, if c denote the length of the thread,

$$x + x' = c, \qquad \frac{d^2x}{dt^2} + \frac{d^2x'}{dt^2} = 0:$$

hence, from (1),

$$(m + m')\frac{d^2x}{dt^2} = g(m\sin\alpha - m'\sin\alpha') \;\ldots\ldots\ldots\ldots\ldots\ldots(2);$$

which determines the common acceleration of the two particles estimated in accordance with an increase of x: should the expression for $\dfrac{d^2x}{dt^2}$ be a negative quantity, x will decrease and x' increase.

If T denote the tension of the thread, we shall have, for the equilibrium of the impressed moving forces T, $mg \sin \alpha$, exerted on the particle m, and the effective moving force $m \frac{d^2x}{dt^2}$ applied in a direction opposite to its own,

$$T = m\left(g \sin \alpha - \frac{d^2x}{dt^2}\right)$$

$$= \frac{mm'g}{m+m'}(\sin \alpha + \sin \alpha'), \text{ by } (2);$$

which gives the value of T, which is therefore of invariable magnitude during the whole motion.

Poisson; *Traité de Mécanique,* Tom. II. p. 12.

(3) One body draws up another on the wheel and axle: to determine the motion of the weights and the tension of the strings.

Let a, a', denote the radii of the wheel and axle; m, m', the masses of the bodies suspended from them; s the arc described, at the end of the time t, by a molecule μ of the mass of the wheel and axle, r the distance of the molecule from the axis of rotation; x, x', the vertical distances, below the horizontal plane through the axis, of the masses m, m'.

Then the moment of the impressed forces about the axis of rotation will be

$$mag - m'a'g:$$

and the moments of the effective forces, estimated in the same direction, will be

$$ma \frac{d^2x}{dt^2} - m'a' \frac{d^2x'}{dt^2} + \Sigma\left(\mu r \frac{d^2s}{dt^2}\right).$$

Hence, by D'Alembert's Principle,

$$ma \frac{d^2x}{dt^2} - m'a' \frac{d^2x'}{dt^2} + \Sigma\left(\mu r \frac{d^2s}{dt^2}\right) = mag - m'a'g \,\ldots\ldots\ldots (1).$$

Let θ represent the whole angle through which the wheel and axle have rotated at the end of the time t; then, b, b', denoting the initial values of x, x', it is clear that

$$x = b + a\theta, \quad x' = b' - a'\theta,$$

and therefore

$$\frac{d^2x}{dt^2} = a\frac{d^2\theta}{dt^2}, \qquad \frac{d^2x'}{dt^2} = -a'\frac{d^2\theta}{dt^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Also, it is manifest that $s = r\theta$, and therefore

$$\Sigma\left(\mu r\frac{d^2s}{dt^2}\right) = \Sigma\left(\mu r^2\frac{d^2\theta}{dt^2}\right) = \frac{d^2\theta}{dt^2}\Sigma(\mu r^2) = Mk^2\frac{d^2\theta}{dt^2} \quad\ldots\ldots(3),$$

where Mk^2 denotes the moment of inertia of the wheel and axle together about the axis of rotation.

From (1), (2), (3), we obtain

$$(ma^2 + m'a'^2 + Mk^2)\frac{d^2\theta}{dt^2} = mag - m'a'g,$$

and therefore, if the system be supposed to have no motion when $t = 0$,

$$\theta = \tfrac{1}{2}gt^2\frac{ma - m'a'}{ma^2 + m'a'^2 + Mk^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Let T denote the tension of the string supporting m; then

$$\begin{aligned} T &= m\left(g - \frac{d^2x}{dt^2}\right) \\ &= m\left(g - a\frac{d^2\theta}{dt^2}\right) \\ &= mg\left\{1 - \frac{a(ma - m'a')}{ma^2 + m'a'^2 + Mk^2}\right\} \\ &= mg\frac{m'a'(a + a') + Mk^2}{ma^2 + m'a'^2 + Mk^2}. \end{aligned}$$

Similarly, the tension of the other string being denoted by T',

$$T' = m'g\frac{ma(a' + a) + Mk^2}{m'a'^2 + ma^2 + Mk^2}.$$

(4) A thin uniform rod AB (fig. 180) slides down between the vertical and horizontal rods OBy, OAx, to which it is attached by small rings at A and B: to find the angular velocity of AB in any position.

Let X = the pressure of Oy on AB,

Y = ... Ox on AB.

Let m denote the mass of an elemental length ds of the rod at P: let $OM=x$, $PM=y$, $AB=a$, $\angle\, OAB=\theta$, $AP=s$.

The moving forces on m will be

the effective force $m\dfrac{d^2x}{dt^2}$, parallel to Ox,

the impressed force mg, parallel to yO,

the effective force $m\dfrac{d^2y}{dt^2}$, parallel to Oy.

Reverse the directions of $m\dfrac{d^2x}{dt^2}$ and $m\dfrac{d^2y}{dt^2}$, as indicated in the figure: and let the same thing be done in regard to all the molecules of the descending rod. Then the system of forces will satisfy the conditions of equilibrium.

Hence $$\Sigma\left(m\frac{d^2x}{dt^2}\right)=X \quad\ldots\ldots\ldots\ldots (1),$$

$$\Sigma m\left(\frac{d^2y}{dt^2}+g\right)=Y \quad\ldots\ldots\ldots\ldots (2),$$

$$\Sigma\left\{m\left(\frac{d^2y}{dt^2}+g\right)x-m\frac{d^2x}{dt^2}y\right\}+Xa\sin\theta-Ya\cos\theta=0 \ldots (3).$$

Let λ denote the mass of a unit of length of the rod: then $m=\lambda ds$. Also

$$x=(a-s)\cos\theta, \qquad y=s\sin\theta.$$

Hence $$\Sigma\left(m\frac{d^2x}{dt^2}\right)=\lambda\int_0^a (a-s)\,ds\,\frac{d^2\cos\theta}{dt^2}$$

$$=\tfrac{1}{2}\lambda a^2\frac{d^2\cos\theta}{dt^2},$$

$$\Sigma m\left(\frac{d^2y}{dt^2}+g\right)=\lambda\int_0^a\left(sds\frac{d^2\sin\theta}{dt^2}+gds\right)$$

$$=\lambda\left(\tfrac{1}{2}a^2\frac{d^2\sin\theta}{dt^2}+ga\right),$$

$$\Sigma m\left(x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right)=\frac{d}{dt}\Sigma m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)$$

$$=\lambda\frac{d^2\theta}{dt^2}\int_0^a s\,(a-s)\,ds$$

$$=\tfrac{1}{6}\lambda a^3\frac{d^2\theta}{dt^2},$$

$$\Sigma(mgx) = \lambda g \cos\theta \int_0^a (a-s)\,ds$$
$$= \tfrac{1}{2}\lambda g a^2 \cos\theta.$$

The equations (1), (2), (3), therefore become, if M denote the mass of the whole rod,

$$X = \tfrac{1}{2} Ma \, . \frac{d^2 \cos\theta}{dt^2} \text{.........................(4)},$$

$$Y = M\left(g + \tfrac{1}{2} a \frac{d^2 \sin\theta}{dt^2}\right) \text{..................(5)},$$

$$\tfrac{1}{3} Ma \frac{d^2\theta}{dt^2} + \tfrac{1}{2} Mg\cos\theta + X\sin\theta - Y\cos\theta = 0 \text{.....(6)}.$$

From (4), (5), (6), we shall get, after eliminating X and Y,

$$\frac{d^2\theta}{dt^2} = -\frac{3g}{2a}\cos\theta \text{.........................(7)}:$$

integrating, we have, α being the initial value of θ,

$$\frac{d\theta^2}{dt^2} = \frac{3g}{a}(\sin\alpha - \sin\theta) \text{.................(8)},$$

a result which determines the angular velocity of AB in any position.

COR. From (4), (5), (7), (8), we may easily ascertain that

$$X = \tfrac{3}{4} Mg\cos\theta\,(3\sin\theta - 2\sin\alpha),$$
$$Y = Mg - \tfrac{3}{4} Mg\,(1 + 2\sin\alpha\sin\theta + \sin^2\theta).$$

(5) A uniform heavy rod OA (fig. 181), which is at liberty to oscillate in a vertical plane about a horizontal axis through O, falls from a horizontal position: to determine the angle included between the direction of the rod and the direction of the pressure for any position of the rod.

Let Ox, Oy, be the axes of co-ordinates in the plane of oscillation, Ox being horizontal and Oy vertical; let Oz be at right angles to the plane xOy. Let U, V, represent the resolved parts of the reaction of the axis Oz upon the rod, estimated along xO, yO. Let ρ = the density of the rod, κ = the area of a section of it taken at right angles to its length; let P be any point in OA, draw PM at right angles to Ox; let

$$OM = x, \quad PM = y, \quad OP = r, \quad OA = a, \quad \angle AOx = \theta.$$

Then, by D'Alembert's Principle, resolving forces parallel to Ox,

$$U = -\int_0^a \left\{\kappa\rho\, dr \frac{d^2x}{dt^2}\right\} = -\kappa\rho \int_0^a \left(dr \frac{d^2x}{dt^2}\right) \text{ .... (1);}$$

resolving forces parallel to Oy,

$$V = -\kappa\rho \int_0^a \left\{dr \left(\frac{d^2y}{dt^2} - g\right)\right\} \text{..................(2);}$$

and, taking moments about the axis Oz,

$$\int_0^a \kappa\rho\, g dr \,.\, x = \int_0^a \left\{\kappa\rho\, dr \left(x \frac{d^2y}{dt^2} - y \frac{d^2x}{dt^2}\right)\right\},$$

$$g\int_0^a x dr = \int_0^a \left\{dr \left(x \frac{d^2y}{dt^2} - y \frac{d^2x}{dt^2}\right)\right\} \text{..................(3).}$$

But, from the geometry, we see that

$$x = r\cos\theta, \quad \frac{dx}{dt} = -r\sin\theta \frac{d\theta}{dt}, \quad \frac{d^2x}{dt^2} = -r\cos\theta \frac{d\theta^2}{dt^2} - r\sin\theta \frac{d^2\theta}{dt^2},$$

and similarly

$$\frac{d^2y}{dt^2} = -r\sin\theta \frac{d\theta^2}{dt^2} + r\cos\theta \frac{d^2\theta}{dt^2};$$

hence, from (1), we have

$$U = \kappa\rho \int_0^a r dr \left(\cos\theta \frac{d\theta^2}{dt^2} + \sin\theta \frac{d^2\theta}{dt^2}\right)$$

$$= \tfrac{1}{2}a^2\kappa\rho \left(\cos\theta \frac{d\theta^2}{dt^2} + \sin\theta \frac{d^2\theta}{dt^2}\right) \text{...............(4);}$$

and, from (2),

$$V = \kappa\rho\, ga + \tfrac{1}{2}a^2\kappa\rho \left(\sin\theta \frac{d\theta^2}{dt^2} - \cos\theta \frac{d^2\theta}{dt^2}\right) \text{............... (5).}$$

Again, from (3), substituting for x and y their values in terms of r and θ, we get

$$g\int_0^a \cos\theta\, r dr = \int_0^a r^2 dr \frac{d^2\theta}{dt^2},$$

and therefore

$$\tfrac{1}{2}ga^2\cos\theta = \tfrac{1}{3}a^3 \frac{d^2\theta}{dt^2}, \qquad \frac{d^2\theta}{dt^2} = \frac{3g}{2a}\cos\theta:$$

multiplying by $2\dfrac{d\theta}{dt}$, integrating, and bearing in mind that $\theta = 0$ when $\dfrac{d\theta}{dt} = 0$, we have

$$\frac{d\theta^2}{dt^2} = \frac{3g}{a}\sin\theta.$$

Hence, substituting for $\dfrac{d\theta}{dt}$ and $\dfrac{d^2\theta}{dt^2}$ their values in (4) and (5), we obtain

$$U = \tfrac{9}{4}\kappa\rho\, ag \sin\theta\cos\theta,$$
$$V = \tfrac{1}{4}\kappa\rho\, ag\,(10 - 9\cos^2\theta).$$

From these equations we get

$$U\cos\theta + V\sin\theta = \tfrac{5}{2}\kappa\rho ag\sin\theta,$$
$$V\cos\theta - U\sin\theta = \tfrac{1}{4}\kappa\rho ag\cos\theta.$$

But $U\cos\theta + V\sin\theta$ and $V\cos\theta - U\sin\theta$ are the expressions for the resolved parts of the reaction of the fixed axis, estimated along AO and at right angles to AO; hence, if ϕ denote the inclination of the resultant reaction to the line AO produced, or of the resultant pressure on the axis to the line OA, we shall have

$$\tan\phi = \frac{V\cos\theta - U\sin\theta}{U\cos\theta + V\sin\theta} = \tfrac{1}{10}\cot\theta,$$
$$\tan\theta\tan\phi = \tfrac{1}{10}.$$

(6) A small body is suspended from a fixed point by a string, and is attracted towards a point, the distance of which from it is large compared with the length of the string: if the time of a small oscillation is proportional to the distance at which the attracting point is removed, to determine the law of the attracting force.

The attracting force varies inversely as the square of the distance.

(7) To determine the motion of a chain on two inclined planes, the intersection of which is horizontal, the chain being in a plane perpendicular to the said intersection.

Let α, β be the inclinations of the planes to the horizon, l the length of the chain: then, x being the portion of the chain on the former plane at the end of the time t,

$$x = \frac{l \sin \beta}{\sin \alpha + \sin \beta} + Ae^{\lambda t} + Be^{-\lambda t},$$

where $$\lambda = \left(\frac{g}{l}\right)^{\frac{1}{2}} (\sin \alpha + \sin \beta)^{\frac{1}{2}};$$

A and B being constants which may be found when the initial values of x and $\frac{dx}{dt}$ are given.

Duhamel: *Cours de Mécanique, Deuxième Partie*, p. 82.

(8) A flexible chain of uniform thickness moves upon two inclined planes, placed back to back: to find its tension at any point: also to find the greatest tension at the common summit of the planes, and to determine whether it is greater or less than the tension at the same point when there is equilibrium.

Let l denote the whole length of the chain, m the mass of a unit of its length; α, β, the inclinations of CP, CQ, the two portions of the chain, to the horizon; let $CP = r$; $T =$ the tension at any point E in CP; $CE = x$. Then

$$T = \frac{mg}{l}(r - x)(l - r)(\sin \alpha + \sin \beta):$$

the greatest tension at the common summit is equal to

$$\tfrac{1}{4} mgl (\sin \alpha + \sin \beta),$$

a less quantity than when there is equilibrium, unless $\alpha = \beta$.

(9) A narrow smooth semicircular tube is fixed in a vertical plane, the vertex of the semicircle being its highest point; and a heavy flexible string, passing through it, hangs at rest: if the string be cut at one of the ends of the tube, to find the velocity which the longer portion of the string will have attained when it is just leaving the tube.

If a be the radius of the tube and l the length of the longer portion, the square of the required velocity is equal to

$$ag \,.\, \left\{2\pi - \frac{a}{l}(\pi^2 - 4)\right\}.$$

(10) Two particles, connected together by a rigid imponderable rod, are constrained to move along two grooves Ox, Oy, respectively, the former horizontal, the latter vertical: supposing the particles to be placed in any assigned position, to find the angular velocity of the rod in any position of its descent, and the pressures on the grooves.

Let θ denote the inclination of the rod to the horizon at any time, ω the corresponding angular velocity, α the initial value of θ, l the length of the rod; X, Y, the pressures on the grooves Oy, Ox, respectively; m, m', the masses of the particles in the horizontal and vertical grooves respectively: then

$$\omega = \left(\frac{2m'g}{l}\right)^{\frac{1}{2}} \cdot \left(\frac{\sin\alpha - \sin\theta}{m\sin^2\theta + m'\cos^2\theta}\right)^{\frac{1}{2}},$$

$$X = \frac{mm'g\cos\theta}{(m\sin^2\theta + m'\cos^2\theta)^2}\{\sin\theta(m\sin^2\theta + m'\cos^2\theta) - 2m'(\sin\alpha - \sin\theta)\},$$

$$Y = mg + \frac{mm'g\sin\theta}{(m\sin^2\theta + m'\cos^2\theta)^2}\{\sin\theta(m\sin^2\theta + m'\cos^2\theta) - 2m'(\sin\alpha - \sin\theta)\}.$$

CHAPTER VII.

MOTION OF RIGID BODIES ABOUT FIXED AXES.

SECT. 1. *Various Problems.*

LET F denote the resolved part of any one of a system of forces acting on a rigid body, at right angles to a fixed axis, r being the perpendicular distance between the fixed axis and the direction of F. Then Fr will be the moment of this force about the axis, and, if $\Sigma\,(Fr)$ denote the sum of the moments of all the forces affected by their appropriate signs, we shall have, for the determination of the motion of the body, the general formula

$$\frac{d\omega}{dt} = \frac{\Sigma\,(Fr)}{Mk^2},$$

where $\omega =$ the angular velocity of the body after a time t, and $Mk^2 =$ its moment of inertia about the fixed axis.

(1) A straight uniform rod, moveable about its upper end, hangs vertically: to find the least angular velocity with which it must begin to move in order that it may perform complete revolutions in a vertical plane.

Let OA (fig. 182) be the rod in any position; let $\theta =$ its inclination to the vertical line Ox at any time t. Let G be the centre of gravity: draw GH at right angles to Ox. Let $OA = a$, $m =$ the mass of the rod.

Then, for the motion,

$$mk^2 \frac{d^2\theta}{dt^2} = -\tfrac{1}{2} amg \sin\theta:$$

but $k^2 = \frac{1}{3}a^2$: hence

$$2a \frac{d^2\theta}{dt^2} = -3g \sin\theta,$$

$$a \frac{d\theta^2}{dt^2} = C + 3g \cos\theta:$$

let ω = the initial angular velocity of the rod: then

$$a\omega^2 = C + 3g,$$

and therefore

$$a\frac{d\theta^2}{dt^2} = a\omega^2 - 3g(1 - \cos\theta).$$

Again, the condition of the problem requires that $\frac{d\theta}{dt} = 0$ when $\theta = \pi$: hence

$$0 = a\omega^2 - 6g,$$

and therefore $$\omega = \left(\frac{6g}{a}\right)^{\frac{1}{2}}.$$

(2) A straight rod AB (fig. 183) is freely moveable about its lower end A, which is fixed, while the other end B is suspended by a fine string BC attached to a fixed point C: when the system is slightly displaced from its position of equilibrium, so as to keep the string at full stretch, to find the time of a small oscillation.

Let $AB = a$; join CA; let α = the inclination of CA to the horizon, $\angle BAC = \epsilon$, θ = the inclination of the plane BAC to the vertical plane through AC, mk^2 = the moment of inertia of AB about AC.

The component of the weight mg of the rod at right angles to AC is $mg\cos\alpha$, and the arm of the moment of this component about AC is $\frac{1}{2}a\sin\epsilon \,.\, \sin\theta$: hence, for the motion,

$$mk^2\frac{d^2\theta}{dt^2} = -mg\cos\alpha \,.\, \tfrac{1}{2}a\sin\theta \,.\, \sin\epsilon:$$

but $k^2 = \frac{1}{3}a^2\sin^2\epsilon$: hence, θ being small,

$$\frac{d^2\theta}{dt^2} + \frac{3g\cos\alpha}{2a\sin\epsilon}\cdot\theta = 0.$$

Hence the time of an oscillation is equal to $\pi\left(\frac{2a\sin\epsilon}{3g\cos\alpha}\right)^{\frac{1}{2}}$.

(3) Supposing the force which acts on the crank of a steam-engine to be vertical, and to vary as the sine of the angle through which the crank has revolved at any time from a ver-

tical position; to find the angular velocity of the crank in any position, the moment of the resistance being always equal to half the greatest moment of the force, and the moment of the weight of the crank being regarded as inconsiderable.

Let AO (fig. 184) be the crank, O being the fixed extremity; draw Ox vertical; let $\angle AOx = \theta$ at any time t; $F =$ the force acting at the extremity A; $OA = a$: assume $F = \mu \sin\theta$; let mk^2 denote the moment of inertia of the crank about O.

Then, the moment of the resistance about O being $\frac{1}{2}\mu a$, we have, for the motion of the crank,

$$mk^2 \frac{d^2\theta}{dt^2} = \mu \sin\theta \,.\, a \sin\theta - \tfrac{1}{2}\mu a$$

$$= -\tfrac{1}{2}\mu a \cos 2\theta:$$

multiplying by $2\dfrac{d\theta}{dt}$ and integrating, we obtain

$$mk^2 \frac{d\theta^2}{dt^2} = C - \tfrac{1}{2}\mu a \sin 2\theta:$$

let ω denote the angular velocity of the crank when $\theta = 0$; then

$$mk^2\omega^2 = C:$$

hence
$$\frac{d\theta^2}{dt^2} = \omega^2 - \frac{\mu a \sin 2\theta}{2mk^2},$$

which gives the angular velocity of the crank in any position: from this result we see that the angular velocity is always ω when the crank is in either a horizontal or a vertical position.

(4) The extremities of a uniform rod, moveable about its middle point, are connected with a fixed point by elastic strings, the natural length of each string being equal to the distance of the fixed point from the middle point of the rod: to find the period of the rod's oscillations, when it has been slightly displaced from its position of equilibrium in the plane through the rod and the fixed point.

Let b be the distance of the fixed point from the end of the rod in the position of equilibrium, c the distance between the fixed point and the middle point of the rod, m the mass of the

rod, and λ the modulus of elasticity of either string: then the time of oscillation is equal to $\frac{\pi b}{c}\left(\frac{mb}{6\lambda}\right)^{\frac{1}{2}}$.

SECT. 2. *Uniform Revolution.*

(1) An isosceles right-angled triangle ABC (fig. 185) is suspended at the right angle A, and its side AB is kept in a vertical position by a ring at B: an angular velocity ω being communicated to the triangle round AB, to determine the magnitude of ω in order that there may be no pressure at B.

Bisect BC in L, join AL, and take $AG = \frac{2}{3}AL$; then G will be the centre of gravity of the triangle; draw GH at right angles to AC. Take P any point in the area of the triangle, and draw PM at right angles to AB. Let $AM = x$, $PM = y$, $AC = a = AB$; m = the mass of a unit of area of the triangle.

Then

$$AH = AG\cos\frac{\pi}{4} = \frac{2}{3}AL\cos\frac{\pi}{4} = \frac{2}{3}a\left(\cos\frac{\pi}{4}\right)^2 = \frac{1}{3}a.$$

Also the area of the triangle is equal to $\frac{1}{2}a^2$, and therefore its mass to $\frac{1}{2}ma^2$: hence the moment of the triangle about an axis through A at right angles to its plane at any instant, in consequence of gravity, is

$$\tfrac{1}{2}ma^2g \, . \, \tfrac{1}{3}a = \tfrac{1}{6}ma^3g.$$

Again, the moment about the same axis due to centrifugal force is equal to

$$\iint m\omega^2 y\,dx\,dy \, . \, x = \tfrac{1}{2}m\omega^2\int y^2 x\,dx$$

$$= \tfrac{1}{2}m\omega^2\int_0^a (a-x)^2 x\,dx = \tfrac{1}{24}m\omega^2a^4.$$

Now, since there is no pressure on the ring at B, the moments of gravity and of centrifugal force about the axis through A must be equal; hence we have

$$\tfrac{1}{6}ma^3g = \tfrac{1}{24}m\omega^2a^4,$$

and therefore $$\omega^2 = \frac{4g}{a}, \quad \omega = 2\left(\frac{g}{a}\right)^{\frac{1}{2}}.$$

(2) A string lying in the form of a circle on a smooth table is revolving like a wheel: to find the tension of the string.

Let m = the mass of a unit of length of the string, mds = the mass of the element Pp (fig. 186): the moving force on the element due to rotation is equal to $mds \,.\, \omega^2 r$, ω being the angular velocity and r the radius.

Let t be the tension at P, the tension at p being accordingly $t + dt$. Resolving tangentially we have, $\angle\, POp$ being denoted by θ, for the equilibrium of Pp,

$$t = (t + dt) \cos \theta,$$

or, in the limit,

$$t = t + dt,$$

or $\qquad dt = 0, \quad t = \text{a constant quantity.}$

To find this constant value we have, resolving normally,

$$mds \,.\, \omega^2 r = (t + dt) \sin \theta = t\theta, \text{ in the limit:}$$

whence

$$t = mr^2\omega^2,$$

or the tension varies as the square of the angular velocity.

(3) Two equal uniform rods AB, AC (fig. 187) are connected at one extremity A by a fixed hinge, the other extremities being connected by a fine string BC: they are whirled round with a given angular velocity, so that the axis of the isosceles triangle formed by the string and rods is always vertical: to find the tension of the string.

Let $AB = 2a$, T = the tension of the string, W = the weight of either rod, ω = the angular velocity about the vertical axis AE of the triangle, $\angle\, BAE = \alpha$. Take P any point in AB; let $AP = r$.

Then, taking moments of the forces acting upon AB, about the point A, we have

$$T \,.\, 2a \cos \alpha = Wa \sin \alpha + \int_0^{2a} \omega^2 r \sin \alpha \,.\, \frac{Wdr}{2ag} \,.\, r \cos \alpha$$

$$= Wa \sin \alpha + \frac{W\omega^2 \sin \alpha \cos \alpha}{2ag} \cdot \frac{8}{3} a^3$$

$$= Wa \sin \alpha \left(1 + \frac{4a\omega^2 \cos \alpha}{3g}\right),$$

$$T = \tfrac{1}{2} W \tan \alpha \,.\left(1 + \frac{4a\omega^2 \cos \alpha}{3g}\right).$$

(4) One end A of a rod AB is attached to a hinge A in a vertical axis, and the other end B is connected with a weight P by means of a fine string passing through a small hole in the axis at a distance, above A, equal to AB: supposing the rod to revolve about the axis, to determine its angular velocity in order that it may be inclined during the whole motion at an angle $\frac{\pi}{4}$ to the vertical line drawn downwards from A.

Let $2a$ denote the length of the rod, and W its weight: then, ω representing the angular velocity,

$$\omega^2 = \frac{3g}{2a\sqrt{2}} \left\{1 - \frac{P}{W} \sqrt{(4 - \sqrt{8})}\right\}.$$

(5) A rod revolves freely about a fixed point at its upper end so as to be always inclined to the vertical at a given angle: to find its angular velocity about the vertical through its higher end, and the direction of the pressure on the fixed point.

Let $2a$ be the length of the rod and α the given angle: then the angular velocity will be equal to

$$\left(\frac{3g}{4a \cos \alpha}\right)^{\frac{1}{2}};$$

and, ψ being the inclination of the direction of the pressure to the vertical,

$$\tan \psi = \frac{3}{4} \tan \alpha.$$

Griffin: *Solutions of the Examples on the motion of a rigid body*, p. 35.

(6) An elastic string, the weight of a unit of length of which in its natural state is w, is placed within a circular tube of radius a, which revolves uniformly about a vertical diameter: the modulus of elasticity is wa and the angular velocity $\left(\frac{g}{a}\right)^{\frac{1}{2}}$:

if the string occupy the upper half of the tube, to find the natural length of the string.

The natural length of the string is equal to the length of the diameter of the tube.

(7) A square lamina revolves about an edge, which is fixed in a vertical position: to ascertain the angular velocity in order that the resultant pressure on the axis of revolution may pass through the highest point.

If $2a$ be the length of an edge, the required angular velocity is equal to $\left(\frac{g}{a}\right)^{\frac{1}{2}}$.

(8) A carriage moves on a railroad with a given velocity round a curve of given radius: to find the amount by which the outer rail must be elevated above the inner one in order that the carriage may not be overturned towards the outside.

We will suppose the radii of the circles described by the molecules of the carriage to be the same, as will be approximately the case in railroads.

Let $2b$ = the breadth of the road between the rails, a = the distance of the centre of gravity of the carriage from the road, r = the radius of the curve, v = the velocity of each molecule of the carriage, and θ = the inclination of the road to the horizon: then

$$\tan\theta = \frac{av^2 - bgr}{bv^2 + agr}.$$

(9) A thin book lies on one of the faces of a desk: to find the greatest angular velocity round a vertical axis which can be given to the desk without throwing off the book.

Let α = the inclination of the desk to the horizon, a = the length of the book; and, the book being supposed to be placed symmetrically on one face of the desk, let c = the distance of its lower edge from the axis of revolution, ω = the required angular

velocity. Then, the book being supposed to be moveable about its lower edge, which is kept at rest by the ledge of the desk,

$$\omega^2 = \frac{3g \cot \alpha}{3c - 2a \cos \alpha}.$$

(10) A circular disc is capable of motion about a horizontal tangent, which rotates with a uniform angular velocity about a vertical axis through the point of junction, which is fixed: to find the angular velocity of the tangent in order that the inclination of the disc to the horizon may have a given constant value.

Let a be the radius of the disc and α the inclination of the disc to the horizon: then the required angular velocity is equal to

$$\left(\frac{4g}{5a \sin \alpha}\right)^{\frac{1}{2}}.$$

(11) A Ring, surrounding a Planet, revolves uniformly about a diameter passing through the common centre of the Ring and the Planet: to determine the form of the Ring in order that the tangential stress may be the same at all points.

If $\omega =$ the angular velocity, $\mu =$ the attraction of the planet at a unit of distance, $2a =$ the diameter of revolution; then, the prime radius vector being supposed to be coincident with the diameter of revolution, the equation to the ring will be

$$\frac{1}{a} - \frac{1}{r} = \frac{\omega^2}{2\mu} r^2 \sin^2 \theta.$$

SECT. 3. *Centre of Oscillation.*

Conceive a body of any figure, acted on by gravity, to be oscillating about a fixed horizontal axis AB (fig. 188); let G be the centre of gravity of the body; draw GO at right angles to AB. Produce OG to a point C such that

$$OC = \frac{h^2 + k^2}{h},$$

where $h = OG$ and $k =$ the radius of gyration of the body about an axis through G parallel to AB; then, if the whole mass of the body be collected at the point C, the period of its oscillations about AB will be the same as before. The point C is called the Centre of Oscillation or of Agitation.

The theory of the Centre of Oscillation of bodies originated in questions addressed, about the year 1646, by Mersenne to the mathematicians of his day, who were called upon by him to exert their ingenuity to discover the time of oscillation of bodies moveable about horizontal axes. It is rather singular that all those who first attempted the solution of this celebrated problem, among whom Mersenne[1] himself is to be numbered, together with Descartes[2], Roberval[3], Wallis[4], and Fabri[5], tacitly supposed the Centre of Oscillation to be coincident with the Centre of Percussion; a supposition which, although true, is by no means obvious without a rigorous demonstration. On the strength of this assumption, however, the Centre of Oscillation was correctly determined in the case of certain figures. Descartes gave a true solution of the case where a plane area oscillates *in planum*, but failed in the case of solid bodies and of plane areas oscillating *in latus*. Roberval assigned correctly the position of the Centre of Oscillation, not only of plane areas oscillating *in planum*, but also in certain instances of oscillation *in latus*, while together with Descartes he failed to give a correct solution of the problem in the case of solid figures. The labours of Huyghens, who in his earlier efforts to obtain a solution of Mersenne's problem had been utterly baffled, were at length crowned with success, and accordingly in the fourth part of his *Horologium Oscillatorium*, which appeared in the year 1673, was given the first rigorous and general investigation of the Centre of Oscillation. The two following axioms constitute the basis of his researches: first, that the centre of gravity of a system

[1] *Mersenni Reflexiones Physico-Mathematicæ*, Cap. XI. et XII.

[2] *Lettres de Descartes*, Tom. III. p. 487, &c.

[3] *Lettres de Descartes*, ib.

[4] *Mechanica, sive De Motu.*

[5] *Tract. de Motu, Append. Physico-Math. De Centro Percussionis.*

of heavy bodies cannot of itself rise to an altitude greater than that from which it has fallen, whatever change be made in the mutual disposition of the bodies; and, secondly, that a compound pendulum will always ascend to the same height as that from which it has descended freely. Some years after the publication of the *Horologium Oscillatorium*, the truth of these fundamental axioms, which although true, it must be admitted, are not sufficiently elementary, was called in question by the Abbé Catelan[1], who substituted certain frail theories of his own in place of the valuable researches of Huyghens. The attention of the mathematicians of the day having been more closely directed to the subject by the controversy which arose between Huyghens and Catelan, the views of Huyghens received ample corroboration from the more elementary investigations of L'Hôpital, James Bernoulli, and other mathematicians. For information respecting the subsequent history of Mersenne's problem, the reader is referred to the Chapter on D'Alembert's Principle.

(1) To find at what point of the rod of a perfect pendulum must be fixed a given weight of indefinitely small volume, so as to have the greatest effect in accelerating the pendulum.

Let m be the mass of the bob of the perfect pendulum, and a its length; m' the mass of the given weight, and a' the distance of its point of attachment from the centre of suspension; l the distance between the centre of suspension and the centre of oscillation of the complex pendulum. Then we shall have, m and m' being both of indefinitely small volume,

$$l = \frac{ma^2 + m'a'^2}{ma + m'a'}.$$

Now the shorter the rod of a perfect pendulum, the shorter will be the time of its oscillations: hence l must be a minimum: differentiating then with respect to a' we get

$$\frac{dl}{da'} = \frac{2m'a'(ma + m'a') - m'(ma^2 + m'a'^2)}{(ma + m'a')^2} = 0:$$

[1] *Journal des Sçavans*, 1682 et 1684.

hence
$$m'a'^2 + 2maa' = ma^2,$$
$$m'^2a'^2 + 2mam'a' + m^2a^2 = (m^2 + mm')\,a^2,$$
$$m'a' + ma = (m^2 + mm')^{\frac{1}{2}}a,$$
$$a' = \frac{a}{m'}\left\{(m^2 + mm')^{\frac{1}{2}} - m\right\},$$
which determines the required point of attachment.

Lady's and Gentleman's Diary, 1742. *Diarian Repository*, p. 394. Euler; *De Motu Corp. Solid.*, Prob. 48. Cor. 1. p. 216.

(2) To compare the times in which a circular plate will vibrate round a horizontal tangent and round a horizontal axis, through the point of contact, at right angles to the tangent.

Let l, l' denote the lengths of the isochronous pendulums in the former and latter case respectively; a the radius of the plate; k, k', the radii of gyration about axes through the centre of the plate parallel in each case to the axis of oscillation. Then
$$l = \frac{a^2 + k^2}{a}, \qquad l' = \frac{a^2 + k'^2}{a}.$$

Let A denote the area of the plate; r the distance of a point within it from its centre, and θ the inclination of this distance to the horizon when the plate is hanging at rest. Then
$$Ak^2 = \iint r\,d\theta\,dr\,.\,r^2\sin^2\theta, \text{ between the proper limits,}$$
$$= \int_0^{2\pi}\int_0^a r^3\sin^2\theta\,d\theta\,dr = \tfrac{1}{4}a^4\int_0^{2\pi}\sin^2\theta\,d\theta = \tfrac{1}{4}\pi a^4.$$
Also
$$Ak'^2 = \iint r\,d\theta\,dr\,.\,r^2 = \int_0^{2\pi}\int_0^a r^3\,d\theta\,dr = \tfrac{1}{4}a^4\int_0^{2\pi}d\theta = \tfrac{1}{2}\pi a^4.$$
But $A = \pi a^2$; hence $k^2 = \frac{1}{4}a^2$ and $k'^2 = \frac{1}{2}a^2$; and therefore
$$l = a + \tfrac{1}{4}a = \tfrac{5}{4}a, \qquad l' = a + \tfrac{1}{2}a = \tfrac{6}{4}a.$$
Hence, if t, t' denote the times of vibration,
$$\frac{t}{t'} = \left(\frac{l}{l'}\right)^{\frac{1}{2}} = \left(\frac{5}{6}\right)^{\frac{1}{2}}.$$

(3) To find the length of a simple pendulum oscillating in the same time as the arc of a given circle, the axis of oscillation passing horizontally through the middle point of the arc at right angles to its plane.

Let C, (fig. 189), be the centre of the circle, A the middle point of the arc, P any point in the arc. Draw PM at right angles to AC: let $AM = x$, $AC = a$, $\delta m =$ the mass of an element of the arc at P, $l =$ the length of the required pendulum. Then

$$l = \frac{\Sigma (r^2 \delta m)}{\Sigma (x \delta m)} = \frac{\Sigma (2ax\delta m)}{\Sigma (x\delta m)} = 2a,$$

a result which shews that the length of the simple pendulum depends only upon the radius of the circle, and not upon the length of the arc. *Lady's Diary*, 1841.

(4) If l and h be the distances of the centres of oscillation and gravity of a mercurial pendulum, of which the weight is m, from the axis of suspension, and h' be the distance of the centre of gravity of a small quantity of mercury μ by the addition of which the pendulum is made to vibrate seconds exactly, to determine the approximate ratio of μ to m, L being the length of the seconds pendulum, and r the radius of the cylinder containing the mercury.

The moment of inertia of the mercury μ, which may be regarded approximately as a circular lamina of fluid, about any diameter, and therefore about a diameter parallel to the axis from which the pendulum is suspended, will be $\frac{1}{4}\mu r^2$, and therefore its moment of inertia about the axis of suspension will be

$$\mu (h'^2 + \tfrac{1}{4} r^2).$$

Also, the radius of gyration of the mercury m about a line through its centre of gravity parallel to the axis of suspension being k, the moment of inertia about the axis of suspension will be $m (h^2 + k^2)$. Hence, by the formula for the Centre of Oscillation, we have approximately

$$(\mu h' + mh) L = \mu (h'^2 + \tfrac{1}{4} r^2) + m (h^2 + k^2).$$

But also we shall have

$$hl = h^2 + k^2;$$

hence $$(\mu h' + mh) L = \mu (h'^2 + \tfrac{1}{4}r^2) + mhl,$$

$$\frac{\mu}{m} \{h' (L - h') - \tfrac{1}{4}r^2\} = h (l - L),$$

$$\frac{\mu}{m} = \frac{4h (l - L)}{4h' (L - h') - r^2}.$$

(5) A bent lever, the lengths of the arms of which are a and b, and the angle between them θ, makes small oscillations in its own plane about the fulcrum: to find the length of the isochronous simple pendulum.

$$\text{The required length} = \frac{\tfrac{2}{3} (a^3 + b^3)}{(a^4 + 2a^2b^2 \cos \theta + b^4)^{\frac{1}{2}}}.$$

(6) A bent lever, the arms of which are of equal weight, and which are inclined to each other at right angles, makes small oscillations in its own plane about its fulcrum: to find the length of the isochronous simple pendulum.

The length of the required pendulum is equal to two-thirds of the diameter of a circle of which the arms of the lever are chords.

(7) To ascertain at what point in its length a uniform straight rod of small thickness must be suspended in order that it may oscillate isochronously with a given simple pendulum.

Let $2a$ = the length of the rod, l = the length of the given pendulum, h = the distance of the required point of suspension from the rod's centre of gravity. Then

$$h = \tfrac{1}{2} l \pm (\tfrac{1}{4} l^2 - \tfrac{1}{3} a^2)^{\frac{1}{2}},$$

which shews that $\dfrac{2a}{\sqrt{3}}$ is the least admissible value of l.

(8) A heavy circular arc, of which the radius is a, and which subtends an angle 2α at the centre of the circle, oscillates, in a

vertical plane, between two inclined planes: to find the length of the isochronous simple pendulum.

The required length is equal to $\dfrac{a\alpha}{\sin \alpha}$.

(9) To investigate the form of an isosceles triangle, the oscillations of which may have the same amplitude and period round an axis, perpendicular to its plane, through its vertex, and round an axis, parallel to the former, through the middle point of its base.

The vertical angle of the triangle must be a right angle.

(10) A square oscillates about a horizontal axis perpendicular to its plane: to find where the axis must pierce the square in order that the time of oscillation may be a minimum.

If $c =$ the length of a side of the square, the locus of the required point is a circle described about the centre of the square with a radius $\dfrac{c}{\sqrt{6}}$.

(11) A square lamina oscillates flat-ways about a horizontal axis passing through one of its angular points: to find the length of the isochronous simple pendulum.

The required length $= \frac{7}{12} \times$ diagonal.

(12) A sector of a circle oscillates round a horizontal axis at right angles to its plane through the centre of the circle: to find the angle of the sector when the length of the isochronous simple pendulum is equal to one half the length of the arc.

If $\phi =$ the angle of the sector,

$$\cos\phi = -\tfrac{1}{8}.$$

(13) A uniform rod of given length is bent into the form of a cycloid, and oscillates about a horizontal line joining its extremities: to find the length of the isochronous simple pendulum.

If a be the length of the rod, the length of the isochronous pendulum will be $\frac{1}{2}a$.

(14) A pendulum consists of an indefinitely thin rigid rod OA, and a globe of which the centre is A: to determine the point A', in the line OA, at which the centre of another globe must be fixed in order that the oscillations of the system of the two globes may be executed in the smallest time possible.

Let $OA = a$, $OA' = a'$; also let r, r', be the radii, and m, m', the masses of the globes A, A'. Then

$$a' = \frac{1}{m'}\left\{m(m+m')a^2 + \tfrac{2}{5}m'(mr^2 + m'r'^2)\right\}^{\frac{1}{2}} - \frac{ma}{m'}.$$

Euler; *Theoria Motus Corporum Solidorum*, p. 215.

CHAPTER VIII.

MOTION OF RIGID BODIES. SMOOTH SURFACES.

IF a body be in motion about a Principal Axis[1], and be acted on by forces which do not tend to perturb the direction of this axis; then, the motion of the centre of gravity of the body remaining the same as if all the forces were impressed on the mass condensed at this point, the Principal Axis will always remain parallel to itself as an axis of permanent rotation, and the angular acceleration about this axis will be the same as if it were a fixed axis. The discovery of the existence of three principal axes in every body as axes of permanent rotation is due to Professor Segner of Gottingen, by whom it was communicated to the world in a memoir entitled *Specimen Theoriæ Turbinum*, published at Halle in the year 1755. For the complete development of the theory of rotation about permanent axes, the student is referred to Euler's *Theoria Motus Corporum Solidorum*, cap. VIII., a work of the greatest value for those who wish to acquire profound views on the subject of the motion of rigid bodies.

If a body be revolving at any instant of time about an axis which is not a principal one, this axis will not be one of permanent rotation; the body will revolve successively about a series of instantaneous axes, the positions of which both in relation to the body and to absolute space are different. The solution of the great physical problem of the Precession of the Equinoxes, published by D'Alembert[2] in the year 1749, unfolded a complete method for the investigation of the general problem of

[1] The following is the definition of Principal Axes given by Euler, *Theoria Motus Corporum Solidorum*, p. 175: "Axes principales cujusque corporis sunt tres illi axes per ejus centrum inertiæ transeuntes, quorum respectu momenta inertiæ sunt vel maxima vel minima."

[2] *Recherches sur la Précession des Équinoxes*, 1749.

rotation. In the following year was published by Euler[1] a memoir entitled *Découverte d'un nouveau principe de Mécanique,* the object of which was to investigate general formulæ for the motion of a body under the most general circumstances of motion and force. The equations, however, expressing under the most simple form the general conditions of rotation, were first given by Euler[2] in the year 1758, who availed himself of the principles of simplification afforded by the recent discoveries of Segner[3] respecting the existence of the three Principal Axes of material bodies. The consideration of the general problem of rotation was resumed by D'Alembert, and presented under its most general aspect in the first volume of his *Opuscules Mathématiques,* published in 1761, where he expresses disapprobation of the title prefixed by Euler to his memoir of 1749, in consideration of his own investigations on the Precession of the Equinoxes. The subject of rotation was thoroughly investigated and exemplified by Euler in his *Theoria Motus Corporum Solidorum et Rigidorum,* which appeared in the year 1767. The same subject was afterwards investigated by Lagrange[4] on more general principles of analysis. In the year 1777 appeared a memoir entitled, *A new Theory of the Rotatory Motion of Bodies affected by Forces disturbing such motion,* by Landen[5], a celebrated English mathematician, in which he expresses himself dissatisfied with the conclusions of the great continental philosophers on the subject of rotation. The subject was again resumed by Landen[6] a few years afterwards, when he developes more fully his own views, and persists in his opposition to the doctrines of his predecessors. There is a memoir by Wildbore in the *Philosophical Transactions* for the year 1790, in which the subject is investigated under a new light: the conclusions of the author are un-

[1] *Mémoires de l'Académie des Sciences de Berlin,* 1750.

[2] *Ibid.* 1758.

[3] *Specimen Theoriæ Turbinum,* 1755.

[4] *Mémoires de l'Académie des Sciences de Berlin,* 1753; *Mécanique Analytique,* Seconde Partie, Section IX.

[5] *Philosophical Transactions,* 1777.

[6] *Ibid.* 1785.

favourable to the cause of Landen, whose views are in fact now generally exploded. For further information on the history of the theory of rotation and Landen's controversy, the student is referred to a memoir by Mr Whewell, in the second volume of the *Cambridge Philosophical Transactions*, 1827. The investigation of Euler's general equations of rotatory motion has been effected with great elegance and simplicity by Mr O'Brien, in the fifth chapter of his *Mathematical Tracts*, Part I.

SECT. 1. *Single Body. Axis of Rotation not Rotating.*

(1) A rod PQ (fig. 190), of uniform thickness and density, having been placed in a given position, such that one end is in contact with a smooth horizontal plane OA and the other with a smooth vertical plane OB, descends in a vertical plane AOB by the action of gravity: to determine where the rod will detach itself from the vertical plane.

Let $PG = a = GQ$, G being the centre of gravity of the rod; let GH be vertical and equal to y at any time t of the motion; $OH = x$, $\angle QPO = \phi$; $k =$ the radius of gyration of the rod about G; $R =$ the reaction of the vertical plane, which will be horizontal, and $S =$ that of the horizontal plane, which will be vertical; $m =$ the mass of the rod.

Then, for the motion of the rod, we have, resolving forces horizontally,

$$m\frac{d^2x}{dt^2} = R \quad \text{......................} \quad (1);$$

resolving vertically,

$$m\frac{d^2y}{dt^2} = S - mg \quad \text{.................} \quad (2);$$

and taking moments about G,

$$mk^2\frac{d^2\phi}{dt^2} = Ra\sin\phi - Sa\cos\phi \text{.................} \quad (3).$$

Eliminating R and S between the three equations (1), (2), (3), we have

$$k^2 \frac{d^2\phi}{dt^2} = a \sin\phi \frac{d^2x}{dt^2} - a\cos\phi \frac{d^2y}{dt^2} - ag\cos\phi \text{......(4)}.$$

Now, from the geometry, it is clear that

$$x = a\cos\phi, \quad y = a\sin\phi,$$

and therefore

$$\frac{dx}{dt} = -a\sin\phi \frac{d\phi}{dt}, \quad \frac{d^2x}{dt^2} = -a\cos\phi \frac{d\phi^2}{dt^2} - a\sin\phi \frac{d^2\phi}{dt^2}, \text{...(5)},$$

and

$$\frac{dy}{dt} = a\cos\phi \frac{d\phi}{dt}, \quad \frac{d^2y}{dt^2} = -a\sin\phi \frac{d\phi^2}{dt^2} + a\cos\phi \frac{d^2\phi}{dt^2}:$$

hence we have

$$a\sin\phi \frac{d^2x}{dt^2} - a\cos\phi \frac{d^2y}{dt^2} = -a^2 \frac{d^2\phi}{dt^2};$$

and therefore, from (4),

$$(a^2 + k^2) \frac{d^2\phi}{dt^2} = -ag\cos\phi \text{..................(6)}:$$

multiplying by $2\frac{d\phi}{dt}$, and integrating, we have

$$(a^2 + k^2) \frac{d\phi^2}{dt^2} = C - 2ag\sin\phi:$$

but, if α be the initial value of ϕ, we have, since $\frac{d\phi}{dt} = 0$ initially,

$$0 = C - 2ag\sin\alpha;$$

and therefore $$(a^2 + k^2) \frac{d\phi^2}{dt^2} = 2ag(\sin\alpha - \sin\phi) \text{..........(7)}.$$

Now, at the instant when the rod detaches itself from the vertical plane, $R = 0$; hence, by (1) and the value of $\frac{d^2x}{dt^2}$ in (5),

$$\cos\phi \frac{d\phi^2}{dt^2} + \sin\phi \frac{d^2\phi}{dt^2} = 0;$$

and therefore, by (6) and (7),

$$2ag(\sin\alpha - \sin\phi)\cos\phi = ag\cos\phi\sin\phi;$$

whence, since ϕ cannot be equal to $\frac{1}{2}\pi$, we have

$$2\sin\alpha - 2\sin\phi = \sin\phi, \quad \sin\phi = \tfrac{2}{3}\sin\alpha;$$

which gives the position of the rod at the moment of its separation from the vertical plane.

This problem was proposed by Weston, a disciple of Landen's, in the *Lady's and Gentleman's Diary* for the year 1757; and solved by Peter Walton, a contributor to the *Diary*. See *Diarian Repository*, p. 467.

(2) A uniform rod of given length hangs horizontally by two equal vertical strings attached to its ends: if it be twisted horizontally through a very small angle, so that its centre of gravity remains in the same vertical line, to find the time of an oscillation, the inertia of the strings being neglected.

Let P, Q, (fig. 191), be the points from which the strings PA, QB, are suspended, AB being the position in which the rod will rest; let ab be the position of the rod at any instant after disturbance; G the centre of gravity of AB, and therefore approximately of ab. Let $AG = a = BG$, $AP = b = BQ$, $\angle AGa = \theta$, $m =$ the mass of the rod.

Then, for small oscillations, the tension of each string may be considered equal to $\frac{1}{2}mg$ and Aa equal to $a\theta$. Also the resolved part of the tension of aP along aA will be nearly equal to

$$\tfrac{1}{2}mg\frac{a\theta}{b} = \frac{mag\theta}{2b},$$

and its moment about G will be nearly equal to

$$\frac{ma^2g\theta}{2b}:$$

similarly for the tension of the string bQ: hence for the angular motion of ab about G, taking into account the tensions of both the strings,

$$mk^2\frac{d^2\theta}{dt^2} = -\frac{ma^2g\theta}{b}:$$

but $k^2 = \frac{1}{3}a^2$: hence

$$\frac{d^2\theta}{dt^2} = -\frac{3g}{b}\theta:$$

hence the time of an oscillation will be equal to $\left(\frac{b}{3g}\right)^{\frac{1}{2}}\pi$;

which is that of a simple pendulum, of which the length is $\frac{1}{3}b$, and is independent of the length of the rod.

Lady's and Gentleman's Diary, 1842, p. 51.

(3) A heterogeneous sphere is placed upon a perfectly smooth horizontal plane, its centre of gravity being slightly distant from the vertical through its geometrical centre: to find the time of the small oscillation of the centre of gravity about the geometrical centre.

Let ABS (fig. 192) be a vertical section of the sphere, passing through C its geometrical centre and G its centre of gravity. Draw GQ horizontal, intersecting the vertical line SCK, through the point of contact of the sphere and the horizontal plane, in the point Q; draw GM vertical to cut the horizontal plane in M; and let CGA be the radius through G. Let $\angle AGM = \phi = \angle ACS$; $CG = c$, $MG = y$, $m =$ the mass of the sphere, $k =$ the radius of gyration about G, $R =$ the reaction of the plane at S upon the sphere, which will exert itself vertically.

Then for the motion of the sphere we have, resolving forces vertically,

$$m\frac{d^2y}{dt^2} = R - mg \text{.....................} (1),$$

and, taking moments about G,

$$mk^2\frac{d^2\phi}{dt^2} = -Rc\sin\phi \text{................} (2):$$

but $y = a - c\cos\phi$, and therefore, from (1),

$$R = mg - mc\frac{d^2\cos\phi}{dt^2}:$$

hence from (2) we have

$$mk^2\frac{d^2\phi}{dt^2} = -mcg\sin\phi + mc^2\sin\phi\,\frac{d^2\cos\phi}{dt^2},$$

and therefore

$$(k^2 + c^2 \sin^2 \phi) \frac{d^2\phi}{dt^2} + c^2 \sin \phi \cos \phi \frac{d\phi^2}{dt^2} = -cg \sin \phi :$$

multiplying this equation by $2\frac{d\phi}{dt}$, and integrating, we get

$$(k^2 + c^2 \sin^2 \phi) \frac{d\phi^2}{dt^2} = 2cg \cos \phi + C:$$

suppose α to be the initial value of ϕ; then, $\frac{d\phi}{dt}$ being supposed to be initially zero, we have

$$0 = 2cg \cos \alpha + C,$$

and therefore

$$(k^2 + c^2 \sin^2\phi) \frac{d\phi^2}{dt^2} = 2cg (\cos \phi - \cos \alpha) \ldots\ldots\ldots (3),$$

whence, $\frac{dt}{d\phi}$ being considered negative because as t increases $d\phi$ is negative from the beginning to the end of every complete oscillation,

$$\frac{dt}{d\phi} = -\frac{1}{(2cg)^{\frac{1}{2}}} \frac{(k^2 + c^2 \sin^2 \phi)^{\frac{1}{2}}}{(\cos\phi - \cos\alpha)^{\frac{1}{2}}}:$$

now from (3) we see that when $\frac{d\phi}{dt} = 0$, $\cos \phi = \cos \alpha$, and therefore $\phi = \pm \alpha$, the positive value of ϕ corresponding to the beginning and the negative value to the end of a complete oscillation; hence, if T denote the time of a complete oscillation,

$$T = -\frac{1}{(2cg)^{\frac{1}{2}}} \int_{+\alpha}^{-\alpha} d\phi \frac{(k^2 + c^2 \sin^2\phi)^{\frac{1}{2}}}{(\cos\phi - \cos\alpha)^{\frac{1}{2}}}$$

$$= \frac{1}{(2cg)^{\frac{1}{2}}} \int_{-\alpha}^{+\alpha} d\phi \frac{(k^2 + c^2 \sin^2\phi)^{\frac{1}{2}}}{(\cos\phi - \cos\alpha)^{\frac{1}{2}}}.$$

Assume

$$\sin\frac{\phi}{2} = s, \quad \sin\frac{\alpha}{2} = b;$$

then $$\tfrac{1}{2}\cos\frac{\phi}{2}\,d\phi = ds, \qquad d\phi = \frac{2ds}{(1-s^2)^{\frac{1}{2}}},$$

$$\cos\phi - \cos\alpha = 2\,(b^2 - s^2), \quad \sin^2\phi = 4s^2\,(1-s^2):$$

hence we have

$$T = \frac{1}{(cg)^{\frac{1}{2}}}\int_{-b}^{+b} ds\,\frac{(1-s^2)^{-\frac{1}{2}}\,\{k^2 + 4c^2s^2\,(1-s^2)\}^{\frac{1}{2}}}{(b^2-s^2)^{\frac{1}{2}}},$$

or, neglecting as inconsiderable powers of the small quantity s beyond the second,

$$= \frac{k}{(cg)^{\frac{1}{2}}}\int_{-b}^{+b} ds\,\frac{1 + \frac{1}{2}s^2 + \frac{2c^2}{k^2}s^2}{(b^2-s^2)^{\frac{1}{2}}}$$

$$= \frac{k}{(cg)^{\frac{1}{2}}}\int_{-b}^{+b} ds\left\{\frac{1}{(b^2-s^2)^{\frac{1}{2}}} + \frac{4c^2+k^2}{2k^2}\,\frac{s^2}{(b^2-s^2)^{\frac{1}{2}}}\right\}:$$

but $$\int_{-b}^{+b}\frac{ds}{(b^2-s^2)^{\frac{1}{2}}} = \pi, \qquad \int_{-b}^{+b}\frac{s^2ds}{(b^2-s^2)^{\frac{1}{2}}} = \tfrac{1}{2}\pi b^2:$$

hence we have, for the time of a complete oscillation,

$$T = \frac{\pi k}{(cg)^{\frac{1}{2}}}\left(1 + \frac{4c^2+k^2}{4k^2}\,b^2\right) = \frac{\pi k}{(cg)^{\frac{1}{2}}}\left(1 + \frac{4c^2+k^2}{4k^2}\sin^2\frac{\alpha}{2}\right).$$

Euler; *Nova Acta Acad. Petrop.* 1783; p. 119.

(4) One end A of a beam AB (fig. 193) is placed upon a smooth inclined plane EF: to find the motion of the beam and its pressure on the plane at any time.

Let G be the position of the centre of gravity of the beam at any time t from the commencement of the motion, R = the reaction of the plane upon the extremity A, $\angle BAF = \phi$; let E be the initial position of A, and β the initial value of ϕ; m = the mass of the beam, k = its radius of gyration about G, $EA = z$, $EH = x$, $GH = y$, α = the inclination of FE to the horizon.

Then for the motion of the beam we have, resolving forces parallel to the plane,

$$m\frac{d^2x}{dt^2} = mg\sin\alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

resolving forces at right angles to the plane,

$$m\frac{d^2y}{dt^2} = R - mg\cos\alpha \quad\text{..................(2)};$$

and, taking moments about G,

$$mk^2\frac{d^2\phi}{dt^2} = -Ra\cos\phi \quad\text{..................(3)}.$$

From (1) we get

$$\frac{dx}{dt} = gt\sin\alpha + C:$$

but $\frac{dx}{dt} = 0$ when $t = 0$; and therefore $C = 0$; hence

$$\frac{dx}{dt} = gt\sin\alpha:$$

integrating, and observing that $x = a\cos\beta$ when $t = 0$, we have

$$x = \tfrac{1}{2}gt^2\sin\alpha + a\cos\beta \quad\text{.....................(4)},$$

which gives the position of the point H at any assigned time from the commencement of the motion.

Again, from (2) and (3), by the elimination of R,

$$a\cos\phi\frac{d^2y}{dt^2} = -k^2\frac{d^2\phi}{dt^2} - ag\cos\alpha\cos\phi:$$

but, by the geometry, we see that $y = a\sin\phi$: hence

$$a^2\cos\phi\frac{d^2\sin\phi}{dt^2} = -k^2\frac{d^2\phi}{dt^2} - ag\cos\alpha\cos\phi;$$

and therefore, multiplying by $2\frac{d\phi}{dt}$, and integrating,

$$a^2\left(\frac{d\sin\phi}{dt}\right)^2 = C - k^2\frac{d\phi^2}{dt^2} - 2ag\cos\alpha\sin\phi:$$

but, initially, $\frac{d\phi}{dt} = 0$ and $\phi = \beta$: hence there is

$$0 = C - 2ag\cos\alpha\sin\beta,$$

and therefore

$$(a^2\cos^2\phi + k^2)\frac{d\phi^2}{dt^2} = 2ag\cos\alpha(\sin\beta - \sin\phi) \quad\text{.......(5)},$$

which gives the angular velocity of the beam for every position which it can assume during its descent.

From the geometry it is evident that

$$z = x - a\cos\phi$$
$$= \tfrac{1}{2}gt^2 \sin\alpha + a(\cos\beta - \cos\phi), \text{ by (4)},$$

and $$y = a\sin\phi:$$

if therefore from (5) we could obtain ϕ in terms of t, we might determine the values of y and z at any time from the beginning of the motion.

Again, for the pressure on the plane at any time, we have, from (3),

$$R = -\frac{mk^2}{a\cos\phi}\frac{d^2\phi}{dt^2}:$$

but from (5), $\dfrac{d\phi^2}{dt^2} = 2ag\cos\alpha\dfrac{\sin\beta - \sin\phi}{a^2\cos^2\phi + k^2}$,

and therefore, differentiating with respect to t, and dividing by $2\dfrac{d\phi}{dt}\cos\phi$,

$$\frac{1}{\cos\phi}\frac{d^2\phi}{dt^2} = \frac{2a^2g\cos\alpha\sin\phi(\sin\beta - \sin\phi)}{(a^2\cos^2\phi + k^2)^2} - \frac{ga\cos\alpha}{a^2\cos^2\phi + k^2}:$$

hence $$R = \frac{mk^2g\cos\alpha}{a^2\cos^2\phi + k^2} - \frac{2ma^2k^2g\cos\alpha\sin\phi(\sin\beta - \sin\phi)}{(a^2\cos^2\phi + k^2)^2}$$

$$= \frac{mk^2g\cos\alpha}{(a^2\cos^2\phi + k^2)^2}\{k^2 + a^2(1 + \sin^2\phi - 2\sin\beta\sin\phi)\},$$

which gives the pressure on the plane for any of the successive positions of the beam.

Fuss; *Nova Acta Petrop.* 1795; p. 70.

(5) A cylinder *KLM* (fig. 194) is placed with its axis horizontal upon a smooth inclined plane; a string *EPMKL*, one end *E* of which is attached to a fixed point at a distance *EA* from the plane equal to the radius of the cylinder, having been wound about the cylinder in a vertical plane through the centre of gravity *O* of the cylinder at right angles to its axis: to find the tension of the string and the velocity of decrease of its angle of inclination to the plane corresponding to any position of the cylinder in its descent; the length of the free string being initially equal to zero.

Let M be the point of contact of the section KLM of the cylinder, about which the string is wound, with the inclined plane; and P the point at which the free string EP touches the cylinder. Produce EP to meet the inclined plane at S; join OP, OM; at any time t from the commencement of the motion let $AM = x$, $T =$ the tension of the string, $\angle ESA = \theta = \angle POM$, $\phi =$ the whole angle through which the cylinder has revolved about its centre of gravity; also, let $m =$ the mass of the cylinder, $k =$ its radius of gyration about its axis, $\alpha =$ the inclination of the plane to the horizon, and $AE = a = MO$.

Then for the motion of the cylinder we have, resolving forces parallel to the plane,

$$m\frac{d^2x}{dt^2} = mg\sin\alpha - T\cos\theta \;\ldots\ldots\ldots\ldots (1);$$

and taking moments about O, the centre of gravity,

$$mk^2\frac{d^2\phi}{dt^2} = Ta \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2):$$

by the elimination of T between these two equations, we get

$$a\frac{d^2x}{dt^2} = ag\sin\alpha - k^2\cos\theta\frac{d^2\phi}{dt^2} \;\ldots\ldots\ldots\ldots (3).$$

Take along EA, produced if necessary, Ep equal to EP: then, if the cylinder were made to roll from E to p, and then Ep were made to revolve about E into the position EP, the cylinder would clearly on the whole have revolved about its centre of gravity through the very angle which actually belongs to its real motion in setting free the length EP of the string. Now, in the first stage of the hypothetical motion, the cylinder would obviously move through an angle equal to $\frac{Ep}{a}$ or $\frac{EP}{a}$, which is equal to $\cot\theta$; and, in the second stage, through an angle $pES = \frac{1}{2}\pi - \theta$, in an opposite direction. Hence clearly we have

$$\phi = \cot\theta - (\tfrac{1}{2}\pi - \theta) = \cot\theta + \theta - \tfrac{1}{2}\pi \;\ldots\ldots\ldots\ldots (4).$$

Also, from the geometry, it is obvious that

$$x = \frac{a}{\sin\theta} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

From (4) and (5) we have

$$d\phi = -\frac{d\theta}{\sin^2\theta} + d\theta = -\frac{\cos^2\theta}{\sin^2\theta}\, d\theta \text{ .... (6)},$$

$$dx = -\frac{a\cos\theta}{\sin^2\theta}\, d\theta \text{ .... (7)}.$$

Multiplying (3) by $2\frac{dx}{dt}$, we get

$$2a\frac{dx}{dt}\frac{d^2x}{dt^2} = 2ag\sin\alpha\frac{dx}{dt} - 2k^2\frac{dx}{dt}\cos\theta\frac{d^2\phi}{dt^2}:$$

but from (6) and (7) it is clear that

$$\cos\theta\frac{dx}{dt} = a\frac{d\phi}{dt} \text{ .... (8)}:$$

hence we obtain

$$2\frac{dx}{dt}\frac{d^2x}{dt^2} = 2g\sin\alpha\frac{dx}{dt} - 2k^2\frac{d\phi}{dt}\frac{d^2\phi}{dt^2}:$$

integrating, and adding the arbitrary constant C,

$$\frac{dx^2}{dt^2} + k^2\frac{d\phi^2}{dt^2} = 2gx\sin\alpha + C:$$

but, initially, $\frac{dx}{dt} = 0$, $\frac{d\phi}{dt} = 0$, $x = a$: hence

$$0 = 2ga\sin\alpha + C,$$

and therefore $\quad \dfrac{dx^2}{dt^2} + k^2\dfrac{d\phi^2}{dt^2} = 2g\sin\alpha\,(x - a):$

substituting in this equation the values of x, $d\phi$, dx, given in (5), (6), (7), we have

$$\left(\frac{a^2\cos^2\theta}{\sin^4\theta} + \frac{k^2\cos^4\theta}{\sin^4\theta}\right)\frac{d\theta^2}{dt^2} = 2g\sin\alpha\left(\frac{a}{\sin\theta} - a\right),$$

and therefore $\quad \dfrac{d\theta^2}{dt^2} = \dfrac{2ga\sin\alpha\,(1-\sin\theta)\sin^3\theta}{\cos^2\theta\,(a^2 + k^2\cos^2\theta)}$ ....(9);

which gives the angular velocity of the string about E in terms of its inclination to the plane, or for any position of the cylinder.

Again, from (8), we have

$$a\frac{d^2\phi}{dt^2} = \cos\theta\frac{d^2x}{dt^2} - \sin\theta\frac{dx}{dt}\frac{d\theta}{dt},$$

and therefore $$\frac{d^2x}{dt^2}=\frac{a}{\cos\theta}\frac{d^2\phi}{dt^2}+\tan\theta\frac{dx}{dt}\frac{d\theta}{dt}$$
$$=\frac{a}{\cos\theta}\frac{d^2\phi}{dt^2}-\frac{a}{\sin\theta}\frac{d\theta^2}{dt^2}, \text{ by (7):}$$

substituting this value of $\frac{d^2x}{dt^2}$ in (3), we obtain

$$\left(\frac{a}{\cos\theta}+\frac{k^2}{a}\cos\theta\right)\frac{d^2\phi}{dt^2}=g\sin\alpha+\frac{a}{\sin\theta}\frac{d\theta^2}{dt^2},$$

and therefore, by (9),

$$\frac{a^2+k^2\cos^2\theta}{a\cos\theta}\frac{d^2\phi}{dt^2}=g\sin\alpha+\frac{2ga^2\sin\alpha\sin^2\theta\,(1-\sin\theta)}{\cos^2\theta\,(a^2+k^2\cos^2\theta)}$$
$$=g\sin\alpha\frac{a^2(1+\sin^2\theta-2\sin^3\theta)+k^2\cos^4\theta}{\cos^2\theta\,(a^2+k^2\cos^2\theta)}:$$

hence, by (2), we have for the tension of the string for any position of the cylinder,

$$T=\frac{mk^2}{a}\frac{d^2\phi}{dt^2}=mk^2g\sin\alpha\frac{a^2(1+\sin^2\theta-2\sin^3\theta)+k^2\cos^4\theta}{\cos\theta\,(a^2+k^2\cos^2\theta)^2}.$$

Euler; *Nova Acta Acad. Petrop.* 1795; p. 64.

(6) A uniform heavy rod OA, (fig. 195), which is at liberty to oscillate in a vertical plane about a horizontal axis through O, falls from a horizontal position: to determine the angle included between the direction of the rod and the direction of the pressure upon the fixed axis, for any position of the rod.

From O draw Om at right angles to OA and to the fixed axis; and produce AO indefinitely to a point n. Let R, S, denote the resolved parts of the reaction of the fixed axis along Om, On, for any position of the rod. Draw Ox horizontal and at right angles to the fixed axis. Let $OA=a$; $m=$ the mass of the rod; $\angle AOx=\theta$, at any time t. Then, for the motion of the rod about its centre of gravity G, the moment of inertia about G being $\frac{1}{12}ma^2$,

$$\tfrac{1}{12}ma^2\frac{d^2\theta}{dt^2}=\tfrac{1}{2}aR,$$

$$ma\frac{d^2\theta}{dt^2}=6R\text{.......................(1).}$$

Also, for the motion about O, the moment of inertia about O being $\frac{1}{3}ma^2$,

$$\tfrac{1}{3}ma^2\frac{d^2\theta}{dt^2} = mg\,.\,\tfrac{1}{2}a\cos\theta,$$

$$2a\frac{d^2\theta}{dt^2} = 3g\cos\theta \text{.......................}(2).$$

Eliminating $\frac{d^2\theta}{dt^2}$ between (1) and (2), we get

$$R = \tfrac{1}{4}mg\cos\theta \text{.......................}(3).$$

Again, equating S to the resolved part of the weight along OA and the centrifugal force,

$$S = mg\sin\theta + \int_0^a \left(\frac{mdr}{a}r\frac{d\theta^2}{dt^2}\right)$$

$$= mg\sin\theta + \tfrac{1}{2}ma\frac{d\theta^2}{dt^2} \text{.....................}(4).$$

Again, multiplying (2) by $\frac{d\theta}{dt}$, and integrating,

$$a\frac{d\theta^2}{dt^2} = C + 3g\sin\theta:$$

but $\frac{d\theta}{dt} = 0$, when $\theta = 0$: hence $C = 0$, and therefore

$$a\frac{d\theta^2}{dt^2} = 3g\sin\theta:$$

hence, from (4), we get

$$S = mg\sin\theta + \tfrac{3}{2}mg\sin\theta = \tfrac{5}{2}mg\sin\theta \text{............}(5).$$

Let ϕ be the angle which the whole reaction of the fixed axis makes with the line On: then

$$\tan\phi = \frac{R}{S},$$

and therefore, by the equations (3) and (5),

$$\tan\theta\tan\phi = \tfrac{1}{10},$$

which gives the value of ϕ for any position of the rod: ϕ is evidently the angle between the direction of the whole pressure on the fixed axis and the length OA of the rod.

A solution of this problem was given in Chap. VI., by the direct application of D'Alembert's Principle.

(7) A uniform rod, acted on by gravity, is oscillating in a vertical plane about one extremity: to find the tendency of the vis inertia in any position to bend the rod at any point, and to ascertain the point at which this tendency is a maximum.

Let OA (fig. 196) be the position of the rod at any time t; Ox an indefinite horizontal line through O, the fixed extremity of the rod, in the vertical plane through OA. Take C any point in OA, P any point in CA. Let $OA = 2a$, $OC = c$, $OP = r$, $\angle AOx = \theta$, $k =$ the radius of gyration about O; $m =$ the mass of the rod.

Then the *force gained* by an element dr of the rod at the point P, resolved at right angles to OP, will be equal to

$$m\frac{dr}{2a}\left(r\frac{d^2\theta}{dt^2} - g\cos\theta\right);$$

and the moment of this about C will be equal to

$$\frac{m}{2a}\left(r\frac{d^2\theta}{dt^2} - g\cos\theta\right)(r-c)\,dr:$$

hence the whole moment to produce bending at C will be equal to

$$\frac{m}{2a}\int_c^{2a}\left(r\frac{d^2\theta}{dt^2} - g\cos\theta\right)(r-c)\,dr \;\ldots\ldots\ldots\ldots\; (1).$$

But, for the motion of the rod, we have

$$mk^2\frac{d^2\theta}{dt^2} = mga\cos\theta,$$

and therefore, $\frac{4}{3}a^2$ being the value of k^2,

$$\frac{d^2\theta}{dt^2} = \frac{3g}{4a}\cos\theta.$$

Hence the expression (1) becomes

$$\frac{mg\cos\theta}{8a^2}\int_c^{2a}(3r-4a)(r-c)\,dr$$

$$=\frac{mg\cos\theta}{8a^2}\int^{2a}\{3(r-c)-(4a-3c)\}(r-c)\,dr$$

$$=\frac{mg\cos\theta}{8a^3}\{(2a-c)^3-\tfrac{1}{2}(4a-3c)(2a-c)^2\}$$

$$=\frac{mg\cos\theta}{8a^3}(2a-c)^2\{2a-c-\tfrac{1}{2}(4a-3c)\}$$

$$=\frac{mg\cos\theta}{16a^3}c(2a-c)^2.$$

When this expression is a maximum, we have

$$(2a-c)^2-2c(2a-c)=0,$$

$$2a-3c=0,\quad c=\tfrac{2}{3}a,$$

or

$$OC=\tfrac{1}{3}OA.$$

The following is a different solution of the same problem.

Let X, Y, (fig. 197), be the transversal and longitudinal actions and reactions of any two portions OC, CA, of the rod; and let μ be the wrenching force at C estimated as tending to elevate OC.

Then, for the motion of OC, taking moments about O,

$$m\frac{c}{2a}\cdot\tfrac{1}{3}c^2\frac{d^2\theta}{dt^2}=m\frac{c}{2a}\cdot\tfrac{1}{2}gc\cos\theta-X.c-\mu,$$

and, for the motion of CA, taking moments about its centre of gravity,

$$m\frac{2a-c}{2a}\cdot\tfrac{1}{3}\left(\frac{2a-c}{2}\right)^2\cdot\frac{d^2\theta}{dt^2}=\mu-X\frac{2a-c}{2}$$

But, for the motion of the whole rod,

$$\frac{d^2\theta}{dt^2}=\frac{3g}{4a}\cos\theta:$$

hence the equations for the motion of the two pieces become

$$\mu+cX=\frac{mgc^2\cos\theta}{8a^2}(2a-c),$$

and

$$2\mu-(2a-c)X=\frac{mg(2a-c)^3\cos\theta}{32a^2}.$$

Eliminating X, we shall easily see that

$$\mu=\frac{mgc(2a-c)^2\cos\theta}{16a^2},$$

the same expression for the wrench as we obtained in the former solution.

(8) A uniform beam is supported symmetrically on two props: to find where they must be placed in order that, when one of them is removed, the instantaneous pressure on the other may be the same as the previous statical pressure.

Let A (fig. 198) be the position of the prop which is not removed; G the centre of gravity of the beam. Let $AG = h$, k = the radius of gyration of the beam about G, m = the mass of the beam, R = the reaction of the prop at A before and immediately after the removal of the other prop, f = the instantaneous angular acceleration.

Then, taking moments about A,

$$m(h^2 + k^2)f = mgh \;\ldots\ldots\ldots\ldots\ldots\ldots\; (1).$$

Again, taking moments about G,

$$mk^2f = Rh \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Also, for the equilibrium of the beam while supported by both props,

$$R = \tfrac{1}{2}mg \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

From (1), (2), (3), we see that $h = k$; and therefore $2k$ is the required distance between the two props.

(9) A hemisphere revolves about a fixed axis, which coincides with a diameter of its base and is inclined at a given angle to the vertical, from a position of instantaneous rest in which the plane containing the centre of gravity and fixed axis was perpendicular to the vertical plane through that axis: to find the whole pressure on the axis, when these two planes coincide.

Let ACB (fig. 199) be the axis of revolution, Ax a vertical line, G the centre of gravity of the hemisphere in any position during the motion, H the lowest position of G. Let $\angle BAx = \alpha$, a = the radius of the sphere, $CG = c$, $\angle GCH = \theta$, m = the mass of the hemisphere.

The whole pressure on the axis, when G is at H, will be equal to the sum of the pressure due to gravity, and the pressure due to centrifugal force.

The weight of the hemisphere may be resolved into $mg\cos\alpha$, parallel to BA, which produces no effect on the motion, and $mg\sin\alpha$, parallel to CH. The moment of the latter component about AB is equal to

$$mg\sin\alpha \,.\, c\sin\theta:$$

hence, mk^2 being the moment of inertia of the hemisphere about AB,

$$mk^2\frac{d^2\theta}{dt^2} = -mgc\sin\alpha\sin\theta,$$

and therefore, since $\dfrac{d\theta}{dt} = 0$ when $\theta = \frac{1}{2}\pi$,

$$\frac{d\theta^2}{dt^2} = \frac{2gc}{k^2}\sin\alpha\cos\theta,$$

or, since $c = \frac{3}{8}a$, and $k^2 = \frac{2}{5}a^2$,

$$\frac{d\theta^2}{dt^2} = \frac{15g}{8a}\sin\alpha\cos\theta.$$

Hence the pressure on the axis, arising from centrifugal force, is equal to

$$m\frac{d\theta^2}{dt^2}\,.\,CH = \frac{45}{64}mg\sin\alpha.$$

Again, the pressure arising from gravity is equivalent to $mg\sin\alpha$, at right angles to BA, and $mg\cos\alpha$, parallel to BA.

Hence the whole pressure exerted on the fixed axis at right angles to it is equal to

$$mg\sin\alpha\left(1 + \frac{45}{64}\right) = \frac{109}{64}mg\sin\alpha.$$

The resultant of the two pressures on the fixed axis is therefore equal to

$$mg\,.\left\{\cos^2\alpha + \left(\frac{109}{64}\right)^2\sin^2\alpha\right\}^{\frac{1}{2}}.$$

(10) A chain, ten yards long, consisting of indefinitely small equal links, being laid straight on a perfectly smooth horizontal plane, except one part, a yard in length, which hangs down,

through a hole in the plane, in a vertical tube: in what time will the chain entirely quit the plane?

The time required $= 2{\cdot}890663$ seconds nearly.

Lady's and Gentleman's Diary, 1758; *Diarian Repository*, p. 683.

(11) A cylinder descends down a plane, the inclination of which to the horizon is α, unwrapping a fine string fixed at the highest point of the plane: to find the angle through which the plane must be depressed in order that a sphere, descending under like circumstances, may experience the same acceleration.

The required angle of depression is equal to

$$\alpha - \sin^{-1}\left(\frac{14}{15}\sin\alpha\right).$$

(12) The lower end of a uniform rod, inclined at a given angle to the horizon, is placed upon a smooth horizontal plane: supposing a horizontal force to be continually applied at its lower end such as to cause the rod to descend in a vertical plane with a given uniform angular velocity, to find the velocity of the lower end of the rod in any position.

If $\omega =$ the angular velocity, $\theta =$ the inclination of the rod at any time to the horizon, and $\alpha =$ the initial value of θ; the velocity of the lower end will be equal to

$$\frac{g}{\omega}\log\left(\frac{\sin\alpha}{\sin\theta}\right).$$

If $\omega = 0$, the lower end will have traversed a space equal to $\frac{1}{2}gt^2\cot\alpha$ at the end of a time t.

(13) A homogeneous sphere is suspended by a fine wire attached to a fixed point at its upper extremity: the sphere is then turned round by the hand through n revolutions, and then let go: to determine the motion communicated to it by the untwisting of the wire, the elasticity of torsion being supposed proportional to the angle.

If θ be the trigonometrical angle through which, at the end of any time t, the sphere has been twisted from its position of rest; then, ρ denoting the density of the sphere and μ a constant, the whole motion is expressed by the equation

$$\theta = 2n\pi \cos\left\{\left(\frac{15\mu}{8\pi\rho a^5}\right)^{\frac{1}{2}} t\right\}.$$

(14) A uniform rod hangs by one end from a fixed point, the other end being close to the ground: an angular velocity is then communicated to the rod, and, when it has revolved through an angle of ninety degrees, the end by which it was hanging is loosed: to find the least initial angular velocity so that on falling to the ground it may pitch in an upright position.

If a be the length of the rod and ω the required angular velocity,

$$\omega^2 = \frac{g}{4a}\left(6 + \frac{\pi^2}{\pi + 2}\right).$$

(15) Small rings, attached to the angular points of a triangular lamina, are moveable on a smooth circular wire circumscribing the triangle: to find the time of a small oscillation when the wire is so held that the triangle is nearly in its position of stable equilibrium.

If h be the distance of the centre of the wire from the centre of gravity of the triangle, k the radius of gyration of the triangle about the centre of the wire, and α the inclination of the plane of the lamina to the vertical, the time of oscillation is equal to

$$\frac{\pi k}{(gh \cos\alpha)^{\frac{1}{2}}}.$$

(16) A uniform rod is supported by means of two fine strings of equal lengths, the lower ends of which are fastened to the ends of the rod and the upper ends to fixed points in the same horizontal line: to find the time of a small oscillation when the system is slightly displaced in the vertical plane in which it is situated, the strings not being slackened.

If $2a$ be the length of the rod, b that of each string, and α the inclination of the strings to the horizon in the position of equilibrium, the time of oscillation is equal to

$$\pi\left\{\frac{ab\sin\alpha}{3g}\cdot\frac{1+2\sin^2\alpha}{a+b\cos^2\alpha}\right\}^{\frac{1}{2}}.$$

(17) A thin uniform rod, one end of which is attached to a smooth hinge, is allowed to fall from a horizontal position: to find the vertical strain on the hinge when the horizontal strain on it is the greatest.

If W be the weight of the rod, the required vertical strain is equal to $\frac{11}{8}W$.

(18) A given square board, two edges of which are horizontal, is supported by two vertical strings attached to its higher edge at given points: supposing either of the two strings to be cut, to find the initial tension of the other.

If W be the weight of the board, a the length of an edge, and b the distance of the point of attachment of the uncut string from the middle of the higher edge, the required tension is equal to

$$\frac{a^2W}{a^2+6b^2}.$$

(19) An equilateral triangle is suspended from a point by three strings, each equal to one of the sides, attached to its angular points: if one of the strings be cut, to find the instantaneous change in the tensions of the other two.

Let T, T', be the tensions of either of the two strings before and just after the third string is cut: then

$$T' : T :: 36 : 43.$$

(20) A uniform circular table is supported by three equal and equidistant props placed at the circumference: if one prop be suddenly removed, to find the alteration in the pressure on each of the other props in the first instant.

The pressure on each of the remaining props is instantaneously diminished by one-twelfth of the weight of the table.

(21) An elliptic lamina is supported, with its plane vertical and transverse axis horizontal, by two weightless pins passing through its foci: if one of the pins be released, to determine the eccentricity of the ellipse in order that the pressure on the other may be initially unaltered.

The required value of the eccentricity is $\left(\frac{2}{5}\right)^{\frac{1}{2}}$.

(22) A uniform rod is suspended by two strings of equal lengths, attached to its extremities and to two fixed points in the same horizontal plane, the distance between which is equal to the length of the rod. An angular velocity of such magnitude is communicated to the rod about a vertical line through its centre that it just rises to the level of the fixed points: to find the tension of either string the instant after the communication of the angular velocity.

The instantaneous tension of either string is seven times as great as it was before motion commenced.

(23) A heavy rod is suspended from a fixed point by two inextensible strings without weight, the strings and the rod forming an equilateral triangle: supposing either of the strings to be cut, to determine the initial tension of the other.

If W be the weight of the rod, the required tension is equal to

$$\frac{\sqrt{12}}{13} \cdot W.$$

(24) A uniform sphere, moveable about a fixed point in its surface, rests against an inclined plane: supposing the diameter which passes through the fixed point to be horizontal, to determine whether, if the plane be suddenly removed, the pressure on the fixed point will be increased or diminished.

The pressure will be increased or diminished accordingly as the inclination of the plane is less or greater than $\tan^{-1} \frac{2}{7}$.

(25) A hemisphere oscillates about a horizontal axis which coincides with a diameter of the base: to compare the maxi-

mum pressure on the axis with the weight of the hemisphere, the base of the hemisphere at the commencement of the motion being inclined to the horizon at an angle of 60°.

The greatest pressure $= \frac{178}{128} \times$ weight of hemisphere.

(26) A cone, moveable about the horizontal diameter of its base, which is fixed, is supported, its axis being horizontal, by a vertical string fastened to its vertex: supposing the string to be cut, to compare the initial pressure on the fixed diameter with the pressure in the former case.

If α be the vertical angle of the cone, P the pressure on the horizontal diameter before the string is cut, and P' the pressure after it is cut, then

$$P' = P.\frac{5 - 3\cos\alpha}{5 - \cos\alpha}.$$

(27) An angular velocity having been impressed upon a heterogeneous sphere about an axis, perpendicular to the vertical plane which contains its centre of gravity G and its geometrical centre C, and passing through G (fig. 192), it is then placed upon a smooth horizontal plane: to determine the magnitude of the impressed angular velocity in order that G may rise to a point in the vertical line SCK through C, and there rest; the initial magnitude of the angle between CG and the vertical radius CS being given.

Let $CG = c$, $k =$ the radius of gyration about G, $\alpha =$ the initial value of the angle GCS, and $\omega =$ the required angular velocity; then ω will be determined by the equation

$$(k^2 + c^2 \sin^2 \alpha)\, \omega^2 = 2cg\,(1 + \cos\alpha).$$

Euler; *Nova Acta Acad. Petrop.* 1783; p. 119.

(28) A uniform rod, not acted on by any forces, is in motion, its ends being constrained to slide along two fixed rods at right angles to each other in one plane: to find the wrenching force at any point.

Let AB be the rod, C any point in it, O the intersection of the two fixed rods; let CH, CK, be perpendiculars from C

upon OA, OB, respectively; let m = the mass of AB. Then the angular velocity ω of AB will be invariable, and the wrenching force at C will be equal to

$$\tfrac{1}{2} m\omega^2 . CH . CK.$$

Mackenzie and Walton; *Solutions of the Cambridge Problems for* 1854.

SECTION 2. *Single Body. Axis of Rotation rotating.*

The problems in this section, the solutions of which are worked out in full, or which are proposed for the exercise of the student, require a knowledge of theorems the demonstrations of which are given in ordinary treatises on Rigid Dynamics. Every student is of course expected to be familiar with the notation and theorems in the excellent work by Mr Routh, *On the Dynamics of a System of Rigid Bodies.*

(1) A plane lamina, not acted on by any forces, of uniform density and thickness, the boundary of which is a curve represented in polar co-ordinates by the equation

$$r = a + b \sin^2 2\theta,$$

moves about its pole as a fixed point: to determine the nature of the cone described in space by its instantaneous axis.

The moments of inertia of the lamina about the prime radius vector and a line through the pole perpendicular, in the plane of the lamina, to this radius vector, which are principal axes at the pole, are equal: hence, by Euler's equations, we see that, ω_1, ω_2, ω_3, denoting the angular velocities about these two axes and the principal axis normal to the lamina at their intersection,

$$\frac{d\omega_1}{dt} = -\omega_2\omega_3, \quad \frac{d\omega_2}{dt} = \omega_3\omega_1, \quad \frac{d\omega_3}{dt} = 0:$$

hence $$\omega_1^2 + \omega_2^2 = \alpha^2, \quad \omega_3 = \gamma,$$

where α and β are constant quantities. Thus, ω denoting the angular velocity about the instantaneous axis,

$$\omega^2 = \omega_1^2 + \omega_2^2 + \omega_3^2 = \alpha^2 + \gamma^2.$$

Let r be the radius vector of Poinsot's momental ellipsoid, which coincides with the instantaneous axis, and p the perpendicular from the centre on the tangent plane to the ellipsoid at the extremity of this radius vector: then

$$\frac{p}{r}=\frac{T}{G\omega},$$

where, A, A, C, denoting the moments of inertia about the principal axes,

$$T=A(\omega_1^2+\omega_2^2)+C\omega_3^2=A(\alpha^2+\beta^2)+C\gamma^2,$$

and $$G=\{A^2(\omega_1^2+\omega_2^2)+C^2\omega_3^2\}^{\frac{1}{2}}=(A^2\alpha^2+C^2\gamma^2)^{\frac{1}{2}}.$$

Hence $$\frac{p}{r}=\text{a constant quantity}.$$

But the position of the perpendicular p is absolutely fixed in space, and the position of the radius vector r coincides with the instantaneous axis[1]: hence the instantaneous axis describes a right cone in space.

(2) A solid of revolution, not acted on by any forces, is revolving about a fixed point at its centre of gravity: supposing the instantaneous axis and the axis of figure to be equally inclined to the invariable line, to find the magnitude of this inclination.

Let C denote the moment of inertia about the axis of figure, and A that about a perpendicular line through the centre of gravity. Then, the principal axes at the fixed point being taken as axes of co-ordinates, the equations to the instantaneous axis are

$$\frac{x}{\omega_1}=\frac{y}{\omega_2}=\frac{z}{\omega_3},$$

and to the invariable line

$$\frac{x}{A\omega_1}=\frac{y}{A\omega_2}=\frac{z}{C\omega_3}.$$

Since the inclinations of the invariable line to the instantaneous axis and to the axis of figure are equal, we have, supposing

[1] Routh, *Dynamics of a System of Rigid Bodies*, Second Edition, p. 328.

the axis of figure, which is one of the principal axes, to be the axis of z,

$$\frac{A\omega_1^2 + A\omega_2^2 + C\omega_3^2}{(\omega_1^2 + \omega_2^2 + \omega_3^2)^{\frac{1}{2}} (A^2\omega_1^2 + A^2\omega_2^2 + C^2\omega_3^2)^{\frac{1}{2}}}$$

$$= \frac{C\omega_3}{(A^2\omega_1^2 + A^2\omega_2^2 + C^2\omega_3^2)^{\frac{1}{2}}},$$

$$A(\omega_1^2 + \omega_2^2) + C\omega_3^2 = C\omega_3(\omega_1^2 + \omega_2^2 + \omega_3^2)^{\frac{1}{2}},$$

$$A^2(\omega_1^2 + \omega_2^2)^2 + 2AC\omega_3^2(\omega_1^2 + \omega_2^2) = C^2\omega_3^2(\omega_1^2 + \omega_2^2),$$

$$A^2(\omega_1^2 + \omega_2^2) = C(C - 2A)\,\omega_3^2 \ldots\ldots\ldots\ldots\ldots (1).$$

If then θ denote the inclination of the invariable line to either axis,

$$\cos\theta = \frac{C\omega_3}{\{A^2(\omega_1^2 + \omega_2^2) + C^2\omega_3^2\}^{\frac{1}{2}}},$$

$$\cos 2\theta = \frac{C^2\omega_3^2 - A^2(\omega_1^2 + \omega_2^2)}{A^2(\omega_1^2 + \omega_2^2) + C^2\omega_3^2}$$

$$= \frac{A}{C - A}.$$

The relation (1) shews that the hypothesis is impossible if C be less than $2A$.

(3) Of a rigid plane lamina, not acted on by any forces, one point, about which it can turn freely, is fixed: an angular velocity is communicated to it about a line in its plane, the moment of inertia about which is given: to determine the ratio of the greatest to the least angular velocity of the lamina.

Let A, B, be the moments of inertia about the principal axes through the fixed point in the plane of the lamina, C the moment of inertia about the third principal axis. Then, adopting the ordinary notation,

$$\omega_1^2 + \omega_2^2 + \omega_3^2 = \omega^2,$$

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = T,$$

$$A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = G^2.$$

Let Q be the given moment of inertia, α being the inclination of the initial axis of rotation to the principal axis corresponding to the moment A: let ω' be the initial angular velocity. Then

$$\omega'^2 (A \cos^2 \alpha + B \sin^2 \alpha) = T,$$
$$\omega'^2 (A^2 \cos^2 \alpha + B^2 \sin^2 \alpha) = G^2,$$

whence
$$\frac{T}{G^2} = \frac{A + B \tan^2 \alpha}{A^2 + B^2 \tan^2 \alpha}:$$

but
$$Q = A \cos^2 \alpha + B \sin^2 \alpha,$$

and therefore
$$\tan^2 \alpha = \frac{A - Q}{Q - B}:$$

consequently
$$\frac{T}{G^2} = \frac{Q}{Q(A+B) - AB} \quad\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

But
$$\omega^2 \frac{d\omega^2}{dt^2} = (\lambda_1 - \omega^2)(\lambda_2 - \omega^2)(\lambda_3 - \omega^2) \quad\ldots\ldots\ldots\ldots (2),$$

where
$$\lambda_1 = \frac{T(B+C) - G^2}{BC},$$
$$\lambda_2 = \frac{T(C+A) - G^2}{CA},$$
$$\lambda_3 = \frac{T(A+B) - G^2}{AB}.$$

Also by (1), bearing in mind that $C = A + B$,

$$\frac{\lambda_1}{T} = \frac{Q(C-A) + AB}{BCQ} = \frac{Q+A}{CQ},$$
$$\frac{\lambda_2}{T} = \frac{Q(C-B) + AB}{CAQ} = \frac{Q+B}{CQ},$$
$$\frac{\lambda_3}{T} = \frac{1}{Q} = \frac{A+B}{CQ}.$$

Suppose that A is greater than B: then, by the relation $Q = A \cos^2 \alpha + B \sin^2 \alpha$, it is evident that Q is less than A and greater than B. Hence, by the formulæ for λ_1, λ_2, λ_3, it is evident that λ_1 is the greatest and that λ_3 is greater than λ_2. Also, since $C = A + B$, $\lambda_3 = \frac{T}{Q} = \omega'^2$. Hence, initially, ω^2 is greater than λ_2; and therefore, by the formula (2), since $\omega^2 \frac{d\omega^2}{dt^2}$

is essentially positive, ω^2 cannot be greater than λ_1 and cannot become less than λ_2: hence the ratio of the greatest to the least angular velocity of the lamina is $(A+Q)^{\frac{1}{2}}$ to $(A+B)^{\frac{1}{2}}$.

(4) The axis of a solid of revolution, the vertex of which is fixed, is set rotating about the vertical with a given angular velocity at a given inclination to the vertical, without any rotation of the solid about its axis of figure: to find the differential equation for the determination of the inclination of the axis of figure to the vertical at any future time.

Let α be the initial inclination of the axis of figure to the vertical, θ its inclination at the end of any time t: let A be the moment of inertia about a line, through the vertex O, at right angles to the axis of figure, and h the distance of the centre of gravity from the vertex. Let ω_3 be the angular velocity about the axis of figure OC, at any time, ω_1 and ω_2 the angular velocities at the same time about other two principal axes, OA, OB, fixed in relation to the solid, at the vertex. Let m be the mass of the solid, ω the given initial angular velocity of its axis. Let Z be vertically below O. The vertical force mg is equivalent to $mg\cos\theta$ along OC and $mg\sin\theta$ at right angles to OC, in the plane of ZC, the former component having no effect on the motion. The component $mg\sin\theta$ is equivalent to two forces, $mg\sin\theta\sin\phi$ in the plane of BOC and $mg\sin\theta\cos\phi$ in the plane of COA, the moments of these two forces about OA, OB, being respectively represented by

$$-mgh\sin\theta\sin\phi, \quad -mgh\sin\theta\cos\phi.$$

By the third of Euler's equations of motion we have $\frac{d\omega_3}{dt}=0$, and therefore ω_3 is constant: but it is initially zero: hence it is always zero. Thus the first two of Euler's equations are

$$A\frac{d\omega_1}{dt}=-mgh\sin\theta\sin\phi\quad\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$A\frac{d\omega_2}{dt}=-mgh\sin\theta\cos\phi\quad\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The relations[1] between ω_1, ω_2, ω_3, θ, ϕ, ψ; since ω_3 is zero, are

$$\omega_1 = \frac{d\theta}{dt}\sin\phi - \frac{d\psi}{dt}\sin\theta\cos\phi \quad\text{...............} (3),$$

$$\omega_2 = \frac{d\theta}{dt}\cos\phi + \frac{d\psi}{dt}\sin\theta\sin\phi \quad\text{...............} (4),$$

$$0 = \frac{d\psi}{dt}\cos\theta + \frac{d\phi}{dt} \quad\text{...............} (5).$$

From (3) and (5) we have

$$\omega_1\cos\theta = \frac{d}{dt}(\sin\theta\sin\phi) \quad\text{...............} (6);$$

and, from (4) and (5),

$$\omega_2\cos\theta = \frac{d}{dt}(\sin\theta\cos\phi) \quad\text{...............} (7).$$

By (1) and (6) we have

$$A\frac{d^2\omega_1}{dt^2} = -mgh\omega_1\cos\theta;$$

and, by (2) and (7),

$$A\frac{d^2\omega_2}{dt^2} = -mgh\omega_2\cos\theta;$$

and therefore

$$\omega_2\frac{d^2\omega_1}{dt^2} - \omega_1\frac{d^2\omega_2}{dt^2} = 0,$$

$$\omega_2\frac{d\omega_1}{dt} - \omega_1\frac{d\omega_2}{dt} = C \quad\text{...............} (8),$$

where C is constant.

From (1), (2), (6), (7), (8), we see that

$$\sin\theta\sin\phi\left(\frac{d\theta}{dt}\cos\theta\cos\phi - \frac{d\phi}{dt}\sin\theta\sin\phi\right)$$

$$-\sin\theta\cos\phi\left(\frac{d\theta}{dt}\cos\theta\sin\phi + \frac{d\phi}{dt}\sin\theta\cos\phi\right) = C'\cos\theta,$$

where C' is constant; and therefore

$$\frac{d\phi}{dt} = -\frac{C'\cos\theta}{\sin^2\theta} \quad\text{...............} (9).$$

[1] Routh: *Rigid Dynamics*, p. 168.

From (3), (4), (5), (9), we have

$$\omega_1 = \frac{d\theta}{dt}\sin\phi - \frac{C'\cos\phi}{\sin\theta},$$

and
$$\omega_2 = \frac{d\theta}{dt}\cos\phi + \frac{C'\sin\phi}{\sin\theta},$$

and therefore
$$\omega_1^2 + \omega_2^2 = \frac{d\theta^2}{dt^2} + \frac{C'^2}{\sin^2\theta} \quad\text{..................}\quad (10).$$

From (1) and (2) we have

$$\tfrac{1}{2}A\frac{d}{dt}(\omega_1^2+\omega_2^2) = -mgh\sin\theta\,(\omega_1\sin\phi + \omega_2\cos\phi)$$

$$= -mgh\sin\theta\frac{d\theta}{dt}, \text{ by (3) and (4);}$$

whence $A(\omega_1^2+\omega_2^2) = C'' + 2mgh\cos\theta,$

and therefore, by (10),

$$A\frac{d\theta^2}{dt^2} + \frac{AC'^2}{\sin^2\theta} = C'' + 2mgh\cos\theta.$$

But, initially, $\frac{d\theta}{dt}=0$, and $\theta=\alpha$: hence

$$\frac{AC'^2}{\sin^2\alpha} = C'' + 2mgh\cos\alpha;$$

and therefore

$$A\frac{d\theta^2}{dt^2} + AC'^2\left(\frac{1}{\sin^2\theta} - \frac{1}{\sin^2\alpha}\right) = 2mgh(\cos\theta - \cos\alpha):$$

but, initially, $\frac{d\psi}{dt}=\omega$, $\theta=\alpha$: hence, by (5), the initial value of $\frac{d\phi}{dt}$ is equal to $-\omega\cos\alpha$: and therefore, by (9),

$$\omega\cos\alpha = \frac{C'\cos\alpha}{\sin^2\alpha}, \quad C' = \omega\sin^2\alpha:$$

thus our equation becomes

$$A\frac{d\theta^2}{dt^2} + A\omega^2\sin^4\alpha\left(\frac{1}{\sin^2\theta} - \frac{1}{\sin^2\alpha}\right) = 2mgh(\cos\theta - \cos\alpha),$$

$$A\frac{d\theta^2}{dt^2} = 2mgh\,(\cos\theta - \cos\alpha) + \frac{A\omega^2 \sin^2\alpha}{\sin^2\theta}\cdot(\cos^2\alpha - \cos^2\theta)$$

$$= (\cos\theta - \cos\alpha)\,.\left\{2mgh - \frac{A\omega^2\sin^2\alpha}{\sin^2\theta}\cdot(\cos\theta + \cos\alpha)\right\}.$$

(5) A circular disc, revolving about an axis through its centre perpendicular to its plane, which is inclined at a given angle to the horizon, is placed upon a smooth plane: to determine the motion.

Let a be the radius of the disc, m its mass, A its moment of inertia about a diameter, R the reaction of the smooth plane; ω_1, ω_2, ω_3, the angular velocities at any time t about principal axes at the centre of the disc; γ the initial value of ω_3, the angular velocity about that principal axis which is perpendicular to the disc. Then, θ being the inclination of the disc at any time to the horizon,

$$R = m\left(g + a\frac{d^2\sin\theta}{dt^2}\right).$$

Now R is equivalent to $R\sin\theta$ along the plane of the disc, which produces no effect on the rotation, and $R\cos\theta$ at right angles to the disc: the latter component has moments represented by $-Ra\cos\theta\sin\phi$, $-Ra\cos\theta\cos\phi$, about the principal axes to which ω_1, ω_2, respectively, relate. Hence, by Euler's equations,

$$A\frac{d\omega_1}{dt} = -A\omega_2\omega_3 - ma\cos\theta\sin\phi\left(a\frac{d^2\sin\theta}{dt^2} + g\right)\text{......(1)},$$

$$A\frac{d\omega_2}{dt} = A\omega_3\omega_1 - ma\cos\theta\cos\phi\left(a\frac{d^2\sin\theta}{dt^2} + g\right)\text{.........(2)},$$

$$\omega_3 = \gamma\text{.................................(3)}.$$

Also
$$\frac{d\theta}{dt} = \omega_1\sin\phi + \omega_2\cos\phi\text{ ....(4)},$$

$$\sin\theta\frac{d\psi}{dt} = -\omega_1\cos\phi + \omega_2\sin\phi\text{..............(5)},$$

$$\gamma = \frac{d\psi}{dt}\cos\theta + \frac{d\phi}{dt}\text{......................(6)}.$$

From (1) and (2),

$$A\left(\cos\phi\frac{d\omega_1}{dt}-\sin\phi\frac{d\omega_2}{dt}\right)=-A\omega_3(\omega_2\cos\phi+\omega_1\sin\phi)$$

$$=-A\omega_3\frac{d\theta}{dt}, \text{ by (4)}:$$

hence

$$A\frac{d}{dt}(\omega_1\cos\phi-\omega_2\sin\phi)+A(\omega_1\sin\phi+\omega_2\cos\phi)\frac{d\phi}{dt}=-A\omega_3\frac{d\theta}{dt},$$

and therefore, by (4),

$$A\frac{d}{dt}(\omega_1\cos\phi-\omega_2\sin\phi)+A\frac{d\theta}{dt}\frac{d\phi}{dt}=-A\omega_3\frac{d\theta}{dt},$$

$$\frac{d}{dt}(\omega_1\cos\phi-\omega_2\sin\phi)+\left(\omega_3+\frac{d\phi}{dt}\right)\frac{d\theta}{dt}=0.$$

But, by (5) and (6),

$$\frac{d\phi}{dt}=\gamma-\frac{d\psi}{dt}\cos\theta=\gamma-\cot\theta(\omega_2\sin\phi-\omega_1\cos\phi):$$

hence, observing the relation (3),

$$\frac{d}{dt}(\omega_1\cos\phi-\omega_2\sin\phi)+\frac{d\theta}{dt}\{2\gamma-\cot\theta(\omega_2\sin\phi-\omega_1\cos\phi)\}=0,$$

$$\sin\theta d(\omega_1\cos\phi-\omega_2\sin\phi)$$
$$+d\theta\{2\gamma\sin\theta+\cos\theta(\omega_1\cos\phi-\omega_2\sin\phi)\}=0,$$

$$\sin\theta(\omega_1\cos\phi-\omega_2\sin\phi)-2\gamma\cos\theta=C.$$

But initially $\omega_1=0$, $\omega_2=0$: let $\theta=\epsilon$ initially: then

$$\sin\theta.(\omega_1\cos\phi-\omega_2\sin\phi)=2\gamma(\cos\theta-\cos\epsilon)\text{ ....(7).}$$

Again, from (1), (2), and (4),

$$A\left(\omega_1\frac{d\omega_1}{dt}+\omega_2\frac{d\omega_2}{dt}\right)=-ma\cos\theta\frac{d\theta}{dt}\left(a\frac{d^2\sin\theta}{dt^2}+g\right),$$

$$A(\omega_1^2+\omega_2^2)=\text{const.}-ma^2\left(\frac{d\sin\theta}{dt}\right)^2-2mga\sin\theta:$$

but $A=\dfrac{ma^2}{4}$: therefore

$$\omega_1^2+\omega_2^2=\text{const.}-4\cos^2\theta\frac{d\theta^2}{dt^2}-8\frac{g}{a}\sin\theta:$$

but, initially, $\omega_1 = 0$, $\omega_2 = 0$, $\theta = \epsilon$, $\frac{d\theta}{dt} = 0$: hence

$$0 = \text{const.} - 8\frac{g}{a}\sin\epsilon:$$

hence $$\omega_1^2 + \omega_2^2 + 4\cos^2\theta\frac{d\theta^2}{dt^2} = 8\frac{g}{a}(\sin\epsilon - \sin\theta) \quad\text{.........(8).}$$

From (4) and (5), squaring and adding,

$$\frac{d\theta^2}{dt^2} + \sin^2\theta\frac{d\psi^2}{dt^2} = \omega_1^2 + \omega_2^2,$$

and therefore, by (8),

$$\frac{d\theta^2}{dt^2} + \sin^2\theta \cdot \frac{d\psi^2}{dt^2} = \frac{8g}{a}(\sin\epsilon - \sin\theta) - 4\cos^2\theta\frac{d\theta^2}{dt^2} \quad\text{.........(9).}$$

From (5) and (7),

$$\sin^2\theta \cdot \frac{d\psi}{dt} = 2\gamma(\cos\epsilon - \cos\theta) \quad\text{..............(10).}$$

From (9) and (10),

$$\frac{d\theta^2}{dt^2}(1 + 4\cos^2\theta) = \frac{8g}{a}(\sin\epsilon - \sin\theta) - \frac{4\gamma^2}{\sin^2\theta}(\cos\epsilon - \cos\theta)^2 \quad\text{......(11).}$$

The equation (11) determines θ in terms of t: then (10) determines ψ in terms of t: thence (6) determines ϕ in terms of t.

COR. Let $a\gamma^2$ be very great compared with g: then, as is evident from equation (11), the quantity

$$\left(\frac{\cos\epsilon - \cos\theta}{\sin\theta}\right)^2,$$

being essentially positive, must be very small compared with $\sin\epsilon - \sin\theta$: that is

$$\frac{(\cos\epsilon - \cos\theta)^2}{\sin^2\theta \,.\, (\sin\epsilon - \sin\theta)}$$

is a small quantity. Since, however, the denominator cannot be large, the numerator must be small: hence $\theta = \eta + \epsilon$, where

η is a small quantity. The equation (11) becomes, going to the second order of small quantities,

$$\frac{d\eta^2}{dt^2}(1+4\cos^2\epsilon)=\frac{8g}{a}\cdot(\tfrac{1}{2}n^2\sin\epsilon-n\cos\epsilon)-4\gamma^2\eta^2,$$

whence $\dfrac{d^2\eta}{dt^2}(1+4\cos^2\epsilon)+\dfrac{4}{a}(a\gamma^2-g\sin\epsilon)\,\eta+\dfrac{4g}{a}\cos\epsilon=0,$

$$\frac{d^2}{dt^2}\left(n+\frac{g\cos\epsilon}{a\gamma^2-g\sin\epsilon}\right)+\frac{4}{a}\cdot\frac{a\gamma^2-g\sin\epsilon}{1+4\cos^2\epsilon}\left(\eta+\frac{g\cos\epsilon}{a\gamma^2-g\sin\epsilon}\right)=0,$$

whence, λ and μ being constants,

$$\eta+\frac{g\cos\epsilon}{a\gamma^2-g\sin\epsilon}=\lambda\cos\left\{\left(\frac{4}{a}\cdot\frac{a\gamma^2-g\sin\epsilon}{1+4\cos^2\epsilon}\right)^{\frac{1}{2}}t+\mu\right\}:$$

but, initially, $\eta=0$, and $\dfrac{d\eta}{dt}=0$: hence $\mu=0$, and

$$\lambda=\frac{g\cos\epsilon}{a\gamma^2-g\sin\epsilon}:$$

whence $$\eta=-\frac{g\cos\epsilon}{a\gamma^2-g\sin\epsilon}\text{ vers}\left\{\left(\frac{4}{a}\,\frac{a\gamma^2-g\sin\epsilon}{1+4\cos^2\epsilon}\right)^{\frac{1}{2}}\right\},$$

and therefore, z being the height of the centre of the disc above the smooth plane,

$$z=a\sin\theta=a(\sin\epsilon+\eta\cos\epsilon)$$

$$=a\sin\epsilon-\frac{ag\cos^2\epsilon}{a\gamma^2-g\sin\epsilon}\text{vers}\left\{\left(\frac{4}{a}\cdot\frac{a\gamma^2-g\sin\epsilon}{1+4\cos^2\epsilon}\right)^{\frac{1}{2}}t\right\},$$

which determines completely the motion of the centre of gravity of the disc, and shews that the period of its oscillations is equal to

$$\frac{\pi\{a(1+4\cos^2\epsilon)\}^{\frac{1}{2}}}{2(a\gamma^2-g\sin\epsilon)^{\frac{1}{2}}}.$$

(6) A rigid body is revolving about a fixed point: the angular velocities ω_1, ω_2, ω_3, of the body about its principal axes at the point are, respectively, $\alpha\sin nt$, $\alpha\cos nt$, β, where α and β are constant quantities: to determine the nature of the motion in space.

If the fixed axes be so chosen that $\phi = 0$ when $t = 0$, the motion in space is determined by the equations

$$\frac{d\theta}{dt} = \alpha, \quad \frac{d\phi}{dt} = \beta, \quad \frac{d\psi}{dt} = 0.$$

Griffin; *Solutions of the Examples on the Motion of a Rigid Body*, p. 34.

(7) A body is moveable about a fixed point: two of the principal moments of inertia at the point are each equal to A, C being the other principal moment: the moments of the couples of the impressed forces about these principal axes are, respectively, $a \sin nt$, $a \cos nt$, 0: supposing the instantaneous axis to have coincided initially with the principal axis to which C belongs, to find the angular velocities of the body about the principal axes at any time.

The angular velocity ω_3 is constant: the angular velocities ω_1 and ω_2 are respectively equal to

$$\frac{a}{A(m-n)} \cdot (\cos nt - \cos mt), \quad \frac{a}{A(m-n)} \cdot (\sin mt - \sin nt),$$

where $$m = \frac{A - C}{A} \omega_3.$$

Griffin: *Ib.* p. 39.

(8) A cube, the centre of gravity of which is fixed, is put in motion about any proposed axis: to determine its subsequent motion.

It will continue to revolve about the initial axis of rotation.

Griffin: *Ib.* p. 40.

(9) A right circular cone, the altitude of which is equal to the diameter of its base, is moveable about its centre of gravity, which is fixed: if the cone be put in motion about an axis inclined at a given angle to its axis of figure, to determine the subsequent motion of the cone.

The cone will revolve permanently about the initial axis of rotation. Griffin: *Ib.* p. 40.

(10) A motion is impressed upon a right circular cone about a given axis through its centre of gravity: to find the position of the invariable plane.

Let the centre of gravity be the origin of co-ordinates, and the axis of the cone be the axis of z: let l, m, n, be the direction-cosines of the initial axis of revolution, and let $\tan\alpha$ represent the ratio of the diameter of the base to the altitude of the cone. The equation to the invariable plane is

$$lx + my + nz \operatorname{vers} \alpha = 0.$$

Griffin: *Ib.* p. 40.

(11) A quiescent circular plate is moveable about its centre of gravity as a fixed point: if a given angular velocity be impressed upon it about a given axis, to find the motion of the plate.

Let ω be the given angular velocity, α the inclination of the initial axis of rotation to the plane of the plate. The normal to the plane of the plate will retain a constant inclination to a normal to the invariable plane, and will make a revolution in space in a time equal to

$$\frac{2\pi}{\omega\,(1 + 3\sin^2\alpha)^{\frac{1}{2}}}.$$

Griffin: *Ib.* p. 41.

(12) A solid of revolution is revolving initially about an instantaneous axis, which passes through its centre of gravity, and is inclined at an angle γ to its axis of figure, which is initially inclined at an angle α to an axis fixed in space: given that $A\cot\alpha = C\cot\gamma$, where C, A, are the moments of inertia about the axis of figure and a perpendicular axis through the centre of gravity, to determine the motion of the axis of figure.

The axis of figure will describe a circular right cone about the fixed axis. Griffin: *Ib.* p. 41.

(13) A solid of revolution, moveable about its centre of gravity, is originally put in motion about an axis the moment

of inertia about which is given: to determine the nature of the subsequent motion.

Let Q be the given moment of inertia, C the moment of inertia about the axis of figure, and A that about an axis, perpendicular to the axis of figure, through the centre of gravity. Then

(1) The instantaneous axis will describe in a space a circular cone, the vertical angle of which is greatest when Q is a harmonic mean between A and C.

(2) The axis of figure will describe in space a circular cone the vertical angle of which is equal to

$$2\tan^{-1}\left\{\frac{A}{C}\left(\frac{Q-C}{A-Q}\right)^{\frac{1}{2}}\right\}.$$

(3) The instantaneous axis will describe a circular cone, relatively to the axis of figure, the vertical angle of which is equal to

$$2\tan^{-1}\left(\frac{Q-C}{A-Q}\right)^{\frac{1}{2}}.$$

Griffin: *Ib.* p. 42.

(14) A right circular cone is moving about its centre of gravity as a fixed point: supposing the initial inclination of the instantaneous axis to the axis of figure to be known, to find the path of the vertex of the cone.

Let α be the initial inclination of the instantaneous axis to the axis of figure, β the semi-angle of the cone, and h its altitude. The vertex of the cone will describe in space a circle the radius of which is equal to

$$\frac{3}{32}h\tan\alpha\,(4+\cot^2\beta).$$

(15) A body is moving about a fixed point, two of the principal moments of inertia at the fixed point being equal: supposing the body to be acted on only by a couple, the moment of which is an explicit function of the time, about the

axis of unequal moment, to find the angular velocities about the axes at any time.

Let a be the moment of inertia about one of the axes, b that about each of the other two: let ω_1, ω_2, ω_3, be the angular velocities at any time t, α_1, α_2, α_3, being the initial values of these velocities: let T denote the moment of the couple. Then, α^2 denoting $\alpha_1^2 + \alpha_2^2$,

$$\omega_1 = \alpha_1 + \frac{1}{a}\int_0^t T dt,$$

$$b \sin^{-1}\frac{\omega_2}{\alpha} = b \sin^{-1}\frac{\alpha_2}{\alpha} + (a-b)\,\alpha_1 t + \frac{1}{a}\int_0^t\int_0^t T dt^2,$$

$$b \sin^{-1}\frac{\omega_3}{\alpha} = b \sin^{-1}\frac{\alpha_3}{\alpha} + (b-a)\,\alpha_1 t - \frac{1}{a}\int_0^t\int_0^t T dt^2.$$

(16) A rigid lamina, in the form of a loop of a lemniscate, not acted on by any force, is started with a given angular velocity about one of the tangent lines at its nodal point, the nodal point being fixed: to find the ratio of its greatest to its least angular velocity.

The required ratio is equal to

$$\left(1 + \frac{4}{3\pi}\right)^{\frac{1}{2}}.$$

Sect. 3. *Several Bodies.*

(1) To a wheel and axle are attached weights P and Q, (fig. 200), which are not in equilibrium: to determine their motion and the tensions of the strings by which the weights are suspended.

Through C, the centre of the wheel and axle, draw the horizontal line ACB meeting the strings in A and B; let $AC = a$, $BC = a'$; m = the mass of P, m' = that of Q, μ = that of the wheel and axle together; k = the radius of gyration of the wheel and axle about their common axis; $AP = x$, $BQ = x'$, T = the tension of AP, T' = the tension of BQ; θ = the angle

through which the wheel and axle have revolved at the end of the time t about their common axis.

Then, for the motion of P, we have

$$m\frac{d^2x}{dt^2} = mg - T \quad\text{.......................(1)};$$

for the motion of Q,

$$m'\frac{d^2x'}{dt^2} = m'g - T' \quad\text{.......................(2)};$$

and, for the rotation of the wheel and axle,

$$\mu k^2\frac{d^2\theta}{dt^2} = Ta - T'a' \quad\text{.....................(3)}.$$

But, from the geometry, it is clear that

$$\frac{dx}{dt} = a\frac{d\theta}{dt}; \quad \frac{dx'}{dt} = -a'\frac{d\theta}{dt}:$$

hence, from (1) and (2),

$$ma\frac{d^2\theta}{dt^2} = mg - T \quad\text{........................(4)},$$

$$-m'a'\frac{d^2\theta}{dt^2} = m'g - T' \quad\text{.......................(5)}.$$

Substituting the values of T and T' from (4) and (5) in the equation (3), we get

$$\mu k^2\frac{d^2\theta}{dt^2} = ma\left(g - a\frac{d^2\theta}{dt^2}\right) - m'a'\left(g + a'\frac{d^2\theta}{dt^2}\right),$$

$$(ma^2 + m'a'^2 + \mu k^2)\frac{d^2\theta}{dt^2} = g\,(ma - m'a') \quad\text{.........(6)};$$

whence θ is immediately obtained in terms of t, the initial values of θ and $\frac{d\theta}{dt}$ being supposed to be known.

From (4) and (6) we have

$$T = mg - \frac{mag\,(ma - m'a')}{ma^2 + m'a'^2 + \mu k^2};$$

and, from (5) and (6),

$$T' = m'g + \frac{m'a'g\,(ma - m'a')}{ma^2 + m'a'^2 + \mu k^2}.$$

(2) Two equal uniform rods AC, BC, (fig. 201), having a compass joint at C, are laid in a straight line upon a horizontal plane. A string CDP, to one end of which is attached a weight P greater than that of either rod, passes over a smooth pin D above the plane, and is fastened at its other end to C, which is vertically beneath the pin: to determine the motion of P.

Let $AC = 2a = BC$, $\angle CAB = \theta$; $R =$ the reaction of the plane at each of the points A and B; $T =$ the tension of the string; $S =$ the mutual action of the two rods at the joint, which will evidently take place in a horizontal line parallel to AB; $m =$ the mass of each of the rods, $\mu =$ the mass of the weight P. Let G be the centre of gravity of the rod AC; draw GH, CE, at right angles to AB; let $EH = x$, $GH = y$, $k =$ the radius of gyration of AC about G.

Then, for the motion of the rod AC, we have, resolving forces horizontally,

$$m\frac{d^2x}{dt^2} = S \text{..........................} (1);$$

resolving vertically, $\frac{1}{2}T$ being the force exerted by the string on each rod,

$$m\frac{d^2y}{dt^2} = R + \tfrac{1}{2}T - mg \text{..................} (2);$$

and, taking moments about G,

$$mk^2\frac{d^2\theta}{dt^2} = Sa\sin\theta + \tfrac{1}{2}Ta\cos\theta - Ra\cos\theta \text{..............} (3).$$

Also, for the motion of P, the increment of DP being double that of GH,

$$2\mu\frac{d^2y}{dt^2} = \mu g - T \text{..........................} (4).$$

Multiplying the equation (2) by $a\cos\theta$, we have

$$ma\cos\theta\left(\frac{d^2y}{dt^2} + g\right) = (R + \tfrac{1}{2}T)\,a\cos\theta,$$

and therefore, adding this equation to the equation (3),

$$mk^2\frac{d^2\theta}{dt^2} + ma\cos\theta\left(\frac{d^2y}{dt^2} + g\right) = Sa\sin\theta + Ta\cos\theta:$$

hence, from (1) and (4),

$$mk^2\frac{d^2\theta}{dt^2}+ma\cos\theta\left(\frac{d^2y}{dt^2}+g\right)=ma\sin\theta\frac{d^2x}{dt^2}+\mu a\cos\theta\left(g-2\frac{d^2y}{dt^2}\right);$$

and therefore, since $x=a\cos\theta$, and $y=a\sin\theta$,

$$mk^2\frac{d^2\theta}{dt^2}+ma^2\left(\cos\theta\frac{d^2\sin\theta}{dt^2}-\sin\theta\frac{d^2\cos\theta}{dt^2}\right)+2\mu a^2\cos\theta\frac{d^2\sin\theta}{dt^2}$$
$$=ag(\mu-m)\cos\theta,$$

$$(ma^2+mk^2)\frac{d^2\theta}{dt^2}+2\mu a^2\cos\theta\frac{d^2\sin\theta}{dt^2}=ag(\mu-m)\cos\theta.$$

Multiplying both sides of this equation by $2\frac{d\theta}{dt}$, and integrating, we have, since $\frac{d\theta}{dt}=0$ when $\theta=0$,

$$(ma^2+mk^2+2\mu a^2\cos^2\theta)\frac{d\theta^2}{dt^2}=2ag(\mu-m)\sin\theta,$$

which determines the angular velocity of the rods for any position.

The value of $\frac{d\theta}{dt}$ and therefore of $\frac{d^2\theta}{dt^2}$ being known in terms of θ, we may readily obtain the values of R, S, and T, from the equations (1), (2), (4), in terms of the same angle.

(3) A tube, moveable in a horizontal plane about a vertical axis, is charged with any number of balls at assigned intervals: supposing a given angular velocity to be communicated to the tube, it is required to determine the motion of the tube and of the balls.

Let a, a', a'',... be the initial distances of the balls from the fixed axis, and r, r', r'',... their distances at any time t from the commencement of the motion. Let m, m', m'',... be the masses of the balls, μ of the tube; and let θ be the angle through which the tube has revolved at the end of the time t. Let R, R', R'',... denote the mutual actions and reactions of the balls and the tube. Then, μk^2 denoting the moment of inertia of the

tube about the vertical axis, we have, for the motion of the tube,

$$\mu k^2 \frac{d^2\theta}{dt^2} = Rr + R'r' + R''r'' + \ldots\ldots\ldots\ldots\ldots \quad (1).$$

Also, for the motion of the balls, we have

$$m\,\frac{d^2x}{dt^2} = R\,\frac{y}{r}, \qquad m\,\frac{d^2y}{dt^2} = -R\,\frac{x}{r}\ldots\ldots\ldots\ldots\ldots(2),$$

$$m'\,\frac{d^2x'}{dt^2} = R'\frac{y'}{r'}, \qquad m'\,\frac{d^2y'}{dt^2} = -R'\,\frac{x'}{r'}\ldots\ldots\ldots\ldots\ldots(3),$$

$$m''\frac{d^2x''}{dt^2} = R''\frac{y''}{r''}, \qquad m''\frac{d^2y''}{dt^2} = -R''\,\frac{x''}{r''}\ldots\ldots\ldots\ldots\ldots(4),$$

. . . .

. . . .

where (x, y), (x', y'), (x'', y''),... are the rectangular co-ordinates of the balls at the time t.

Multiplying the former and the latter of the equations (2) by y and x respectively, and subtracting the latter from the former of the resulting equations, we get

$$m\left(y\frac{d^2x}{dt^2} - x\,\frac{d^2y}{dt^2}\right) = Rr\,;$$

and therefore, since $x = r\cos\theta$, $y = r\sin\theta$, the axis of x being supposed to coincide with the initial position of the tube, we may readily obtain, by substitution,

$$m\,\frac{d}{dt}\left(r^2\,\frac{d\theta}{dt}\right) = -Rr.$$

In like manner, from the equations of (3), (4),... we may get

$$m'\,\frac{d}{dt}\left(r'^2\,\frac{d\theta}{dt}\right) = -R'r',$$

$$m''\,\frac{d}{dt}\left(r''^2\,\frac{d\theta}{dt}\right) = -R''r'',$$

. . . . .

. . . . .

Hence from (1) we have

$$\mu k^2 \frac{d^2\theta}{dt^2} = -\frac{d}{dt}\left(mr^2 \frac{d\theta}{dt} + m'r'^2 \frac{d\theta}{dt} + m''r''^2 \frac{d\theta}{dt} + \dots\right):$$

integrating, we get

$$\mu k^2 \frac{d\theta}{dt} = C - (mr^2 + m'r'^2 + m''r''^2 + \dots\dots) \frac{d\theta}{dt}:$$

but, supposing ω to be the initial value of $\frac{d\theta}{dt}$, we have

$$\mu k^2 \omega = C - (ma^2 + m'a'^2 + m''a''^2 + \dots)\, \omega:$$

hence $$\frac{d\theta}{dt} = \frac{\mu k^2 + ma^2 + m'a'^2 + m''a''^2 + \dots}{\mu k^2 + mr^2 + m'r'^2 + m''r''^2 + \dots}\, \omega \dots\dots\dots\dots (5).$$

Again, from the equations (2), we have

$$x \frac{d^2x}{dt^2} + y \frac{d^2y}{dt^2} = 0;$$

and thence, substituting for x and y their values in r and θ,

$$\frac{d^2r}{dt^2} = r \frac{d\theta^2}{dt^2} \dots\dots\dots\dots\dots\dots\dots\dots (6).$$

In the same way, from (3), we may get

$$\frac{d^2r'}{dt^2} = r' \frac{d\theta^2}{dt^2};$$

and therefore, eliminating $\frac{d\theta}{dt}$ between these two equations,

$$r' \frac{d^2r}{dt^2} = r \frac{d^2r'}{dt^2}:$$

integrating and bearing in mind that both $\frac{dr}{dt}$ and $\frac{dr'}{dt}$ are initially equal to zero,

$$r' \frac{dr}{dt} = r \frac{dr'}{dt},$$

and therefore $$\frac{dr}{r} = \frac{dr'}{r'}:$$

integrating again we have, a, a', being the initial values of r, r',

$$\frac{r}{a} = \frac{r'}{a'}, \quad r' = \frac{a'}{a} r.$$

In precisely the same way it may be shewn that

$$r'' = \frac{a'}{a} r, \quad r''' = \frac{a'''}{a} r, \ \ldots\ldots$$

Hence from (5) we have

$$\frac{d\theta}{dt} = \frac{\mu k^2 + ma^2 + m'a'^2 + m''a''^2 + \ldots\ldots\ldots}{\mu k^2 + \left(ma^2 + m'a'^2 + m''a''^2 + \ldots\right)\dfrac{r^2}{a^2}}\,\omega \ldots\ldots\ldots\ldots\ldots (7).$$

From (6) and (7) we obtain

$$\frac{d^2r}{dt^2} = \omega^2 r \left\{\frac{\mu k^2 + ma^2 + m'a'^2 + m''a''^2 + \ldots\ldots\ldots}{\mu k^2 + \left(ma^2 + m'a'^2 + m''a''^2 + \ldots\right)\dfrac{r^2}{a^2}}\right\}^2.$$

Multiplying both sides of this equation by $2\dfrac{dr}{dt}$, integrating, and bearing in mind that $\dfrac{dr}{dt} = 0$ when $r = a$, we shall easily see that

$$\frac{dr^2}{dt^2} = \omega^2 (r^2 - a^2) \frac{\mu k^2 + ma^2 + m'a'^2 + m''a''^2 + \ldots\ldots\ldots}{\mu k^2 + (ma^2 + m'a'^2 + m''a''^2 + \ldots)\dfrac{r^2}{a^2}} \ldots\ldots\ldots\ldots (8).$$

The equations (7) and (8) will give us, for any assigned distance of the ball m from the axis of rotation, the angular velocity of the tube and the velocity of the ball m within it. If between (7) and (8) we eliminate dt, we shall obtain the differential equation in polar co-ordinates to the path of m in the horizontal plane passing through the axis of the tube. Similar results may evidently be obtained for the other balls with which the tube is charged.

COR. If $\mu = 0$, the equations (7) and (8) become

$$\frac{d\theta}{dt} = \frac{a^2\omega}{r^2}, \qquad \frac{dr^2}{dt^2} = (r^2 - a^2)\frac{a^2\omega^2}{r^2},$$

and therefore, eliminating dt,

$$\frac{dr^2}{d\theta^2} = (r^2 - a^2)\frac{r^2}{a^2}, \qquad d\theta = \frac{a\,dr}{r(r^2 - a^2)^{\frac{1}{2}}} = -\frac{d\dfrac{a}{r}}{\left(1 - \dfrac{a^2}{r^2}\right)^{\frac{1}{2}}}:$$

integrating, and remembering that $\theta = 0$ when $r = a$,

$$\theta = \cos^{-1}\frac{a}{r}, \quad \frac{a}{r} = \cos\theta.$$

Again, to determine the relation between r and t, we have

$$dt = \frac{1}{\omega a}\frac{rdr}{(r^2 - a^2)^{\frac{1}{2}}}, \quad t = \frac{1}{\omega a}(r^2 - a^2)^{\frac{1}{2}},$$

and therefore $$\frac{r^2}{a^2} = 1 + \omega^2 t^2.$$

Similar relations holding good for the other balls, we have, for the equations to their paths,

$$\frac{a}{r} = \frac{a'}{r'} = \frac{a''}{r''} = \ldots\ldots = \cos\theta;$$

which shew that they all move in straight lines at right angles to the initial position of the tube; and for their distances from the axis of rotation at any time,

$$\frac{r^2}{a^2} = \frac{r'^2}{a'^2} = \frac{r''^2}{a''^2} = \ldots\ldots = 1 + \omega^2 t^2.$$

Clairaut; *Mém. de l'Acad. des Sciences de Paris*, 1742, p. 48. Daniel Bernoulli; *Mém. de l'Acad. des Sciences de Berlin*, 1745, p. 54. Euler; *Opuscula, de motu corporum tubis mobilibus inclusorum*, p. 71.

(4) A heavy particle P descends down a smooth inclined plane BA, (fig. 202), forming the upper surface of a solid BAC, which is capable of sliding freely along a smooth horizontal plane OAx: to determine the motion of the particle and of the body, both of which are supposed to have initially no motion.

Let PM be at right angles to Ox, and let B be the point in the inclined plane which the particle occupies initially; let A be supposed to coincide with O at the commencement of the motion. Let $OM = x$, $PM = y$, $OA = s$, $AB = a$, $BP = s'$, $\angle BAC = \alpha$; and $R =$ the action and reaction of the plane and

the particle. Then, m denoting the mass of the particle and m' of the body, we shall have

$$m\frac{d^2y}{dt^2} = R\cos\alpha - mg \quad\text{................ (1)},$$

$$m\frac{d^2x}{dt^2} = -R\sin\alpha \quad\text{..................... (2)},$$

$$m'\frac{d^2s}{dt^2} = R\sin\alpha \quad\text{........................(3)}.$$

But $y = (a - s')\sin\alpha$, $x = s + (a - s')\cos\alpha$: hence, from (1),

$$m\sin\alpha\frac{d^2s'}{dt^2} = mg - R\cos\alpha \quad\text{........... ... (4)},$$

and, from (2),

$$m\frac{d^2s}{dt^2} - m\cos\alpha\frac{d^2s'}{dt^2} = -R\sin\alpha \quad\text{.....................(5)}.$$

Adding together (3) and (5), we have

$$(m + m')\frac{d^2s}{dt^2} - m\cos\alpha\frac{d^2s'}{dt^2} = 0 \quad\text{.................. (6)}.$$

Multiplying (3) by $\cos\alpha$, and (4) by $\sin\alpha$, we have, adding together the resulting equations,

$$m'\cos\alpha\frac{d^2s}{dt^2} + m\sin^2\alpha\frac{d^2s'}{dt^2} = mg\sin\alpha \quad\text{............... (7)}.$$

Multiplying (6) by $\sin^2\alpha$, (7) by $\cos\alpha$, and adding together the resulting equations,

$$(m\sin^2\alpha + m')\frac{d^2s}{dt^2} = mg\sin\alpha\cos\alpha:$$

integrating twice with respect to t, and bearing in mind that $s = 0$ and $\frac{ds}{dt} = 0$ when $t = 0$, we obtain

$$s = \tfrac{1}{2}gt^2\frac{m\sin\alpha\cos\alpha}{m\sin^2\alpha + m'} \quad\text{....................(8)}.$$

Again, multiplying (7) by $m + m'$, (6) by $m'\cos\alpha$, and sub-

tracting the latter of the resulting equations from the former, we have

$$m\,(m\sin^2\alpha + m')\,\frac{d^2s'}{dt^2} = mg\sin\alpha\,(m+m'),$$

$$\frac{d^2s'}{dt^2} = \frac{g\sin\alpha\,(m+m')}{m\sin^2\alpha + m'}:$$

integrating twice, and recollecting that $s'=0$ and $\frac{ds'}{dt}=0$ when $t=0$, we get

$$s' = \tfrac{1}{2}gt^2\,\frac{(m+m')\sin\alpha}{m\sin^2\alpha+m'} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (9).$$

The equation (8) gives the position of the moveable inclined plane, and (9) the place of the particle on the plane at any time.

Again, by (3),

$$R = \frac{m'}{\sin\alpha}\,\frac{d^2s}{dt^2} = \frac{mm'g\cos\alpha}{m\sin^2\alpha + m'},$$

which gives the value of the mutual pressure of the particle and the plane; the value of which, therefore, is invariable.

John Bernoulli; *Comment. Acad. Petrop.* 1730, p. 11. *Opera,* Tom. III. p. 365. Euler; *Opuscula, de motu corporum tubis mobilibus inclusorum,* p. 28.

(5) A particle is placed within a thin tube APB (fig. 203), rigidly attached to a plane vertical lamina ABC, the base BC of which is rectilinear and is moveable along a line $OBCE$ on a smooth horizontal plane: supposing the particle and the lamina to be initially at rest, to find their subsequent motions.

Let A be the point of the tube at which the particle is placed, and O the initial position of the point B of the lamina; let $OB=s$; the arc AP of the tube $=s'$; $OM=x$, $PM=y$, where PM is vertical; $\phi=$ the inclination to the horizon of an element of the tube at P; $m=$ the mass of the particle and $m'=$ the mass of the lamina; $R=$ the action and reaction of the

tube and the particle. Then, for the motion of the particle, P being its position at the end of any time t, we have

$$m\frac{d^2x}{dt^2} = -R\sin\phi \quad\text{.....................(1)},$$

$$m\frac{d^2y}{dt^2} = R\cos\phi - mg \quad\text{................(2)};$$

and, for the motion of the lamina,

$$m'\frac{d^2s}{dt^2} = R\sin\phi \quad\text{........................(3)}.$$

Again, from the geometry it is evident that

$$dx = ds - \cos\phi\, ds' \quad\text{........................(4)},$$

and

$$dy = -\sin\phi\, ds' \quad\text{..........................(5)}.$$

From (1) and (3) we have

$$m\frac{d^2x}{dt^2} + m'\frac{d^2s}{dt^2} = 0:$$

integrating, we get

$$m\frac{dx}{dt} + m'\frac{ds}{dt} = C;$$

where C is an arbitrary constant; and therefore, by (4),

$$(m+m')\frac{ds}{dt} - m\cos\phi\frac{ds'}{dt} = C:$$

but, by the conditions of the problem, $\frac{ds}{dt} = 0$, and $\frac{ds'}{dt} = 0$, simultaneously, and therefore $C = 0$: hence

$$(m+m')\frac{ds}{dt} - m\cos\phi\frac{ds'}{dt} = 0 \quad\text{....................(6)}.$$

Again, from (2) and (3),

$$m'\cos\phi\frac{d^2s}{dt^2} - m\sin\phi\frac{d^2y}{dt^2} = mg\sin\phi,$$

and therefore, by (5),

$$m'\cos\phi\frac{d^2s}{dt^2} + m\sin\phi\frac{d}{dt}\left(\sin\phi\frac{ds'}{dt}\right) = mg\sin\phi:$$

but, from (6), $\frac{d^2s}{dt^2} = \frac{m}{m+m'}\frac{d}{dt}\left(\cos\phi\frac{ds'}{dt}\right)$:

hence we have

$$\frac{m'}{m+m'}\cos\phi\,\frac{d}{dt}\left(\cos\phi\,\frac{ds'}{dt}\right)+\sin\phi\,\frac{d}{dt}\left(\sin\phi\,\frac{ds'}{dt}\right)=g\sin\phi.$$

Multiply both sides of this equation by $2\,\frac{ds'}{dt}$ and integrate; then

$$\frac{m'}{m+m'}\left(\cos\phi\,\frac{ds'}{dt}\right)^2+\left(\sin\phi\,\frac{ds'}{dt}\right)^2=2g\int\sin\phi\,ds',$$

$$(m\sin^2\phi+m')\,\frac{ds'^2}{dt^2}=2g\,(m+m')\int\sin\phi\,ds',$$

$$\frac{ds'}{dt}=\{2g\,(m+m')\}^{\frac{1}{2}}\left\{\frac{\int\sin\phi\,ds'}{m\sin^2\phi+m'}\right\}^{\frac{1}{2}}\;\ldots\ldots\ldots\ldots(7):$$

from this equation, when the form of the tube and therefore the relation between ϕ and s' is known, we may determine the relation between s' and t.

Again, for the determination of the relation between s and t, we have, from (6) and (7),

$$\frac{ds}{dt}=\left(\frac{2g}{m+m'}\right)^{\frac{1}{2}}m\cos\phi\left\{\frac{\int\sin\phi\,ds'}{m\sin^2\phi+m'}\right\}^{\frac{1}{2}}:$$

the integral $\int\sin\phi ds'$, which enters into these formulæ, must evidently be so taken as to vanish when $s'=0$.

Clairaut; *Mémoires de l'Académie des Sciences de Paris*, 1742, p. 41. Euler; *Opuscula, de motu corporum tubis mobilibus inclusorum*, p. 48.

(6) A smooth groove ALA' (fig. 204) is carved on the upper edge of a vertical lamina $ACBA'$, which is placed upon a smooth horizontal plane, along which it is able to slide freely: to find the form of the groove in order that a particle may oscillate in it tautochronously, the time of an oscillation being given.

Let L be the lowest point of the groove, LM a vertical line through L, meeting a horizontal line PM through the place P of the particle at any time t; let $PM=x'$, $ML=y'$, arc $LP=s'$. Then by the equation (7) of the preceding problem, putting

$-ds'$ in place of ds', and retaining in other respects the same notation, we have

$$\frac{ds'}{dt} = -\{2g(m+m')\}^{\frac{1}{2}}\left\{\frac{-\int \sin\phi\, ds'}{m\sin^2\phi + m'}\right\}^{\frac{1}{2}}:$$

but $$-\int \sin\phi\, ds' = k - y',$$

if k be the initial value of y': hence

$$\frac{ds'}{dt} = -\{2g(m+m')\}^{\frac{1}{2}}\left\{\frac{k-y'}{m\frac{dy'^2}{ds'^2} + m'}\right\}^{\frac{1}{2}},$$

and therefore, if τ be the time of half an oscillation,

$$\tau = -\frac{1}{\{2g(m+m')\}^{\frac{1}{2}}}\int_k^0 \frac{\frac{ds'}{dy'}}{(k-y')^{\frac{1}{2}}}\left(m\frac{dy'^2}{ds'^2} + m'\right)^{\frac{1}{2}} dy' \dots\dots\dots (1),$$

a quantity which, by the nature of the problem, must be independent of k: hence

$$\int \frac{\frac{ds'}{dy'}}{(k-y')^{\frac{1}{2}}}\left(m\frac{dy'^2}{ds'^2} + m'\right)^{\frac{1}{2}} dy'$$

must be of zero dimensions in y' and k; and therefore

$$\frac{\frac{ds'}{dy'}}{(k-y')^{\frac{1}{2}}}\left(m\frac{dy'^2}{ds'^2} + m'\right)^{\frac{1}{2}}$$

must evidently be of -1 dimensions in k and y': but it is clear that

$$\frac{ds'}{dy'}\left(m\frac{dy'^2}{ds'^2} + m'\right)^{\frac{1}{2}}$$

does not involve k; hence this expression must be of $-\frac{1}{2}$ dimensions in y', and therefore, a being some constant quantity,

$$\frac{ds'}{dy'}\left(m\frac{dy'^2}{ds'^2} + m'\right)^{\frac{1}{2}} = \frac{a}{y'^{\frac{1}{2}}} \dots\dots\dots\dots\dots\dots (2),$$

$$m + m'\frac{ds'^2}{dy'^2} = \frac{a^2}{y'},$$

$$m'\frac{dx'^2}{dy'^2} = \frac{a^2 - (m+m')y'}{y'},$$

$$\frac{dx'}{dy'} = \left(\frac{m+m'}{m'}\right)^{\frac{1}{2}} \left(\frac{2a-y'}{y'}\right)^{\frac{1}{2}},$$

where $2a = \dfrac{a^2}{m+m'}$, : integrating, we get for the equation to the curve,

$$x' = \left(\frac{m+m'}{m'}\right)^{\frac{1}{2}} \left\{(2ay' - y'^2)^{\frac{1}{2}} + a \operatorname{vers}^{-1}\frac{y'}{a}\right\} \ldots\ldots\ldots (3),$$

which is an elongated cycloid, which may be constructed from the ordinary cycloid by increasing the distance of each point of the curve from the axis in the ratio of $(m+m')^{\frac{1}{2}}$ to $m'^{\frac{1}{2}}$.

Again, from (1) and (2), we have

$$\tau = -\frac{a}{\{2g\,(m+m')\}^{\frac{1}{2}}}\int_k^0 \frac{dy'}{(ky'-y'^2)^{\frac{1}{2}}}$$

$$= \frac{\pi a}{\{2g\,(m+m')\}^{\frac{1}{2}}};$$

whence $$a^2 = \frac{2g\tau^2}{\pi^2}(m+m'),$$

and therefore $$2a = \frac{a^2}{m+m'} = \frac{2g\tau^2}{\pi^2}, \quad a = \frac{g\tau^2}{\pi^2}.$$

Hence (3) may be written

$$x' = \left(\frac{m+m}{m'}\right)^{\frac{1}{2}} \left\{\left(\frac{2g\tau^2}{\pi^2}y' - y'^2\right)^{\frac{1}{2}} + \frac{g\tau^2}{\pi^2}\operatorname{vers}^{-1}\frac{\pi^2 y'}{g\tau^2}\right\}.$$

Euler; *Opuscula, de motu corporum tubis mobilibus inclusorum*, p. 51.

(7) A particle P descends down a smooth inclined plane CA (fig. 205), forming the upper edge of a thin vertical lamina CAE which is capable of sliding freely down a smooth inclined plane $OAEB$, with which its whole lower edge is in contact: to determine the motion of the particle and of the lamina, both of which are supposed to be initially at rest, and the pressure of the particle on the lamina.

Let C be the initial position of the particle on the lamina, and O the initial position of the point A of the lamina; let $\angle BAC = \alpha$; $\angle OBF = \beta$, BF being horizontal; $OA = s$, $CP = s'$, $R =$ the required pressure; $m =$ the mass of the particle, $m' =$ that of the lamina; $t =$ the time from the commencement of the motion. Then

$$s = \tfrac{1}{2} gt^2 \frac{(m + m') \sin\beta + m \cos\alpha \sin(\alpha - \beta)}{m \sin^2\alpha + m'},$$

$$s' = \tfrac{1}{2} gt^2 \frac{(m + m') \sin\alpha \cos\beta}{m \sin^2\alpha + m'},$$

$$R = \frac{mm'g \cos\alpha \cos\beta}{m \sin^2\alpha + m'}.$$

Euler; *Ibid.* p. 35.

(8) Any number of particles, P, P', P'',... (fig. 206), are descending down a smooth inclined plane BA, forming the upper edge of a thin vertical lamina BAC capable of sliding freely along a smooth horizontal plane OE, with which its whole lower edge is in contact: to determine the motion of the particles and of the lamina, and the pressures which the particles exert upon the lamina.

Let R, R', R'', ... be the pressures, and m, m', m'', ... the masses of the particles P, P', P'', ...; let $OA = S$, $BP = s$, $BP' = s'$, $BP'' = s''$,..., O being a fixed point in the line OE, and B on the inclined plane BA; $\angle BAC = \alpha$; $M =$ the mass of the lamina. Then

$$S = \tfrac{1}{2} gt^2 \frac{(m + m' + m'' + \ldots) \sin\alpha \cos\alpha}{M + (m + m' + m'' + \ldots\ldots) \sin^2\alpha} + At + B,$$

$$s = \tfrac{1}{2} gt^2 \frac{(M + m + m' + m'' + \ldots\ldots) \sin\alpha}{M + (m + m' + m'' + \ldots\ldots) \sin^2\alpha} + at + b,$$

$$s' = \tfrac{1}{2} gt^2 \frac{(M + m + m' + m'' + \ldots\ldots) \sin\alpha}{M + (m + m' + m'' + \ldots\ldots) \sin^2\alpha} + a't + b',$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{R}{m} = \frac{R'}{m'} = \frac{R''}{m''} = \ldots\ldots = \frac{Mg \cos\alpha}{M + (m + m' + m'' + \ldots) \sin^2\alpha}.$$

The quantities A, B, a, b, a', b', ... are arbitrary constants, to be determined from the initial circumstances of the motion.

Euler; *Ibid.* p. 40.

(9) If a chain of considerable length be suspended from the top of a tower, its lower end touching the earth, and then let fall, to find the velocity at any time.

Let x = the length and v = the velocity of the portion of the chain which is at any time in motion, a = the initial value of x, and r = the radius of the earth: then

$$v^2 = 2gr \log\left(\frac{a+r}{x+r}\right).$$

(10) A thin hollow ring, the plane of which is vertical, and which contains a bead, is placed upon a smooth horizontal plane: to find the period of the bead's oscillation, supposing it to have been placed initially near the lowest point of the ring.

If a be the radius of the ring, μ its mass, and m the mass of the bead, it will oscillate isochronously with a perfect pendulum the length of which is equal to

$$\frac{\mu a}{m+\mu}.$$

Mackenzie and Walton; *Solutions of the Cambridge Problems for* 1854.

(11) The higher ends, A, B, of two equal uniform rods AC, BC, are moveable by little rings along a fixed horizontal rod: the lower ends of the rods are connected at C by a little hinge: supposing C to be placed initially in contact with the fixed horizontal rod, to find the pressures on the horizontal rod and the pressure at the hinge for any subsequent position of the descending rods.

Let θ be the inclination of either rod to the vertical at any time, and W be the weight of each: then the pressure of each ring on the horizontal rod is equal to

$$\tfrac{1}{8}W(11 + 9\cos 2\theta),$$

and that at the hinge to $\tfrac{3}{8}W\sin 2\theta$.

(12) Two solid cylinders descend from rest directly down the two faces of two smooth inclined planes, over the common summit of which passes a thin inelastic string which goes under and round the central transverse sections of the cylinders, to which the ends of the string are fastened: to find the tension of the string and to ascertain how much it will have slid along the planes at the end of any time.

If W, W', be the weights of the cylinders, α, α', the inclinations of the respective planes, the required tension is equal to

$$\tfrac{1}{3}\, WW' . \frac{\sin \alpha + \sin \alpha'}{W + W'};$$

and the space, through which the string has slid at the end of a time t, to

$$\tfrac{1}{3}\, gt^2 . \frac{W \sin \alpha - W' \sin \alpha'}{W + W'}.$$

(13) A circular tube lies on a smooth table: an included globe, the diameter of which is equal to that of a transverse section of the tube, is projected along the tube with a given velocity: to determine the motion of the tube and globe, and to ascertain the horizontal pressure exerted by the globe on the tube.

Let a be the radius of the axis of the tube, m and m' the respective masses of the globe and tube, and v the velocity of projection of the globe: then the centre of gravity of the globe and tube will move, in a direction parallel to the projection of the globe, with a velocity equal to

$$\frac{mv}{m + m'},$$

the globe's centre revolving round the said centre of gravity with an angular velocity $\dfrac{v}{a}$; and the required horizontal pressure will be equal to

$$\frac{v^2 mm'}{a\,(m + m')}.$$

John Bernoulli in a letter to Euler, on the subject of the motion of the globe and tube, says: "Quod attinet ad problema

hoc, miror te tam magnifice de eo sentire, ut illud vocare audeas *argumentum prorsus novum et adhuc intactum*, cum tamen nihil aliud sit quam casus particularis theorematis tritissimi de corpore vel systemate corporum plurium gyrando progrediente super plano horizontali, ubi id semper obtinetur ut commune centrum gravitatis totius systematis progrediatur in linea recta et quidem velocitate uniformi, dum interim reliqua puncta systematis describunt singula aliquam ex cycloidibus sive ordinariam sive protractam seu contractam. Hæc te monere volui, Vir Celeb., ne te præcipites protrudendo in publicum magna pompa rem quandam leviculam, quæ ansam daret inimicis tuis (nam et tu tales habes, præsertim inter scurras Anglicanos qui omnes extraneos odio prosequuntur) carpendi indiscriminatim omnia tua elegantissima inventa, atque hac occasione imprimis in te torquendi Ciceronis proverbium *Laureolam in mustaceo quærere*[1]."

[1] *Correspondance Mathématique et Physique de quelques célèbres Géomètres du XVIIIème Siècle par* P. H. Fuss, Tome II. p. 84.

CHAPTER IX.

MOTION OF RIGID BODIES. ROUGH SURFACES.

SECT. 1. *Single Body.*

(1) A CYLINDER descends down a perfectly rough inclined plane by the action of gravity, its axis being horizontal: to determine the motion of the cylinder and the friction at any time of its descent.

Let G (fig. 207) be the centre of gravity of the cylinder at any instant of its descent; OA the course of the point of contact H of the circular section of the cylinder through G down the inclined plane; let $OH = x$, $\alpha =$ the angle of inclination of OA to the horizon, $\theta =$ the whole angle through which the cylinder has revolved about its centre of gravity in moving from O to H; $a =$ the radius of the cylinder, $k =$ its radius of gyration about its axis. Let F denote the friction of the plane on the cylinder, which, from the signification of perfect roughness, is supposed to be sufficient to prevent sliding, and m the mass of the cylinder.

Then for the motion of the cylinder we have, resolving forces parallel to OA,

$$m\frac{d^2x}{dt^2} = mg \sin \alpha - F \quad\ldots\ldots\ldots\ldots\ldots\ldots(1);$$

and, taking moments about G,

$$mk^2\frac{d^2\theta}{dt^2} = Fa \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

But, since F is sufficiently great to secure perfect rolling, it is evident that $x = a\theta$; and therefore, by (2),

$$mk^2\frac{d^2x}{dt^2} = Fa^2;$$

hence from (1) we get

$$ma^2 \frac{d^2x}{dt^2} = ma^2 g \sin \alpha - mk^2 \frac{d^2x}{dt^2},$$

$$(a^2 + k^2) \frac{d^2x}{dt^2} = a^2 g \sin \alpha,$$

or, since $k^2 = \frac{1}{2}a^2$, $\qquad 3 \frac{d^2x}{dt^2} = 2g \sin \alpha;$

and therefore, integrating twice, and supposing $\frac{dx}{dt} = 0$ when H is at O, we have

$$x = \tfrac{1}{3} g t^2 \sin \alpha,$$

and therefore $\qquad \theta = \frac{gt^2 \sin \alpha}{3a}$ ....(3):

hence we have also, from (2) and (3),

$$F = \frac{mk^2}{a} \frac{d^2\theta}{dt^2} = \frac{2mgk^2 \sin \alpha}{3a^2} = \tfrac{1}{3} mg \sin \alpha.$$

(2) A globe descends from instantaneous rest down the surface of a perfectly rough hemispherical bowl, the centre of the globe always remaining in the same vertical plane: to determine the velocity of the globe at any position of its descent.

Let ABA' (fig. 208) be the vertical section of the bowl made by the plane in which the centre C of the globe is always situated, O being the centre of the bowl and OA a horizontal radius. Let M be the point at which the globe touches the bowl at any time of its motion, B being the initial position of M. Draw the radii OB, OCM; and let $C'C$ be the circular arc described by the centre of gravity of the globe. Let $\angle AOM = \theta$, $\angle AOB = \alpha$, $C'C = s$, $a =$ the radius of the globe, $r =$ the radius of the bowl, $\phi =$ the angle which the globe has described about its centre of gravity in the motion from B to M, $m =$ the mass of the globe; $F =$ the friction of the bowl upon the globe at the point M, which is supposed to be sufficiently great to prevent all sliding.

Then for the motion of the centre of gravity of the globe, which will not be affected by our supposing all the impressed forces to be applied at C in their proper directions,

$$m\frac{d^2s}{dt^2} = -F + mg\cos\theta \dots\dots\dots\dots\dots\dots (1);$$

and, for the motion of the globe about its centre of gravity,

$$mk^2\frac{d^2\phi}{dt^2} = Fa \dots\dots\dots\dots\dots\dots\dots\dots (2).$$

From the points B and M, draw two indefinite straight lines BkB' and MkT, tangents to the section of the hemispherical bowl; along BB' measure a length Bm equal to the circular arc BM; then, if we were to conceive the globe to roll from B along the length Bm, and then Bm to be applied along BM so as to coincide with it, mB' being, as soon as m coincides with M, a tangent both to the circle AMA' and to the globe; it is evident that the globe would have revolved about its centre through the same angle as by its actual motion of rolling down the arc BM. Now by rolling along Bm it would have revolved about its centre through an angle $\frac{Bm}{a} = \frac{BM}{a} = \frac{r}{a}(\theta - \alpha)$; and, by the transference of m to M, it would have revolved through an angle equal to $\angle B'kM = \angle BOM = \theta - \alpha$, in an opposite direction. Hence we see that the whole actual angle through which the globe revolves about its centre in its actual motion from B to M, is equal to $\frac{r-a}{a}(\theta - \alpha) = \phi$.

Hence, putting for ϕ its value in (2), we have

$$\frac{mk^2}{a}(r-a)\frac{d^2\theta}{dt^2} = Fa \dots\dots\dots\dots\dots\dots\dots (3).$$

Again, it is clear from the geometry that $s = (r-a)(\theta - \alpha)$, and therefore, from (1),

$$m(r-a)\frac{d^2\theta}{dt^2} = -F + mg\cos\theta \dots\dots\dots\dots (4).$$

Eliminating F between (3) and (4), we obtain

$$m(r-a)\frac{d^2\theta}{dt^2} + \frac{mk^2}{a^2}(r-a)\frac{d^2\theta}{dt^2} = mg\cos\theta,$$

$$\left(1+\frac{k^2}{a^2}\right)(r-a)\frac{d^2\theta}{dt^2}=g\cos\theta,$$

or, since k^2 is equal to $\frac{2}{5}a^2$,

$$\frac{d^2\theta}{dt^2}=\frac{5g\cos\theta}{7(r-a)} \quad\quad (5):$$

multiplying by $2\dfrac{d\theta}{dt}$, and integrating,

$$\frac{d\theta^2}{dt^2}=C+\frac{10g\sin\theta}{7(r-a)}:$$

but, when $\theta=\alpha$, $\dfrac{d\theta}{dt}$ is equal to zero: hence

$$0=C+\frac{10g\sin\alpha}{7(r-a)},$$

and therefore
$$\frac{d\theta^2}{dt^2}=\frac{10g(\sin\theta-\sin\alpha)}{7(r-a)};$$

whence
$$\frac{ds^2}{dt^2}=\frac{10g}{7}(r-a)(\sin\theta-\sin\alpha);$$

and therefore also, if $s'=$ the arc BM,

$$\frac{ds'^2}{dt^2}=r^2\frac{d\theta^2}{dt^2}=\frac{10r^2g(\sin\theta-\sin\alpha)}{7(r-a)}.$$

For the magnitude of the friction at any time, we have, from (3),

$$F=\frac{mk^2}{a^2}(r-a)\frac{d^2\theta}{dt^2}$$

$$=\frac{5mgk^2\cos\theta}{7a^2}, \text{ by the equation (5).}$$

(3) A heterogeneous sphere rolls along a perfectly rough horizontal plane, its rotatory motion taking place always about an instantaneous axis normal to the vertical plane which passes through its geometrical centre and its centre of gravity: to determine its angular velocity for any position in its path.

Let C (fig. 209) be the geometrical centre and G the centre of gravity of the sphere at any time; S the point of contact of the vertical section of the sphere containing C and G with the horizontal plane; OSE the rectilinear locus of the points of

contact; CGA a radius of the sphere; GM, CS, perpendiculars upon the plane.

Let F denote the friction of the plane at any time upon the sphere, estimated in the direction EO, and R the vertical reaction of the plane; let m = the mass of the sphere; k = the radius of gyration about an axis through G at right angles to the vertical section containing C and G; $OM = x$, $GM = y$, $CS = CA = a$, $\angle AGM = \angle ACS = \phi$, $CG = c$.

Then for the motion of the sphere we have, resolving forces parallel to OE,

$$m\frac{d^2x}{dt^2} = -F \qquad\qquad (1);$$

resolving forces vertically,

$$m\frac{d^2y}{dt^2} = R - mg \qquad\qquad (2);$$

and, taking moments about the centre of gravity,

$$mk^2\frac{d^2\phi}{dt^2} = Fy - Rc\sin\phi \qquad\qquad (3).$$

Now, since the friction is supposed to be sufficiently rough to prevent all sliding, we have from the geometry,

$$x + c\sin\phi = b + a\phi,$$

b being the value of x when $\phi = 0$; and therefore

$$\frac{dx}{dt} = a\frac{d\phi}{dt} - c\cos\phi\frac{d\phi}{dt},$$

$$\frac{d^2x}{dt^2} = (a - c\cos\phi)\frac{d^2\phi}{dt^2} + c\sin\phi\frac{d\phi^2}{dt^2}.$$

Again, from the geometry,

$$y = a - c\cos\phi,$$

and therefore

$$\frac{dy}{dt} = c\sin\phi\frac{d\phi}{dt}, \qquad \frac{d^2y}{dt^2} = c\sin\phi\frac{d^2\phi}{dt^2} + c\cos\phi\frac{d\phi^2}{dt^2}.$$

Hence, from (1),

$$F = -m\left\{(a - c\cos\phi)\frac{d^2\phi}{dt^2} + c\sin\phi\frac{d\phi^2}{dt^2}\right\},$$

and, from (2),

$$R = m\left(\frac{d^2y}{dt^2} + g\right) = m\left(c\sin\phi\frac{d^2\phi}{dt^2} + c\cos\phi\frac{d\phi^2}{dt^2} + g\right).$$

Substituting these expressions for R and F in (3), we get

$$k^2\frac{d^2\phi}{dt^2} = -\left\{(a - c\cos\phi)\frac{d^2\phi}{dt^2} + c\sin\phi\frac{d\phi^2}{dt^2}\right\}(a - c\cos\phi)$$
$$-c\sin\phi\left(c\sin\phi\frac{d^2\phi}{dt^2} + c\cos\phi\frac{d\phi^2}{dt^2} + g\right),$$

and, by simplification,

$$(a^2 + k^2 + c^2 - 2ac\cos\phi)\frac{d^2\phi}{dt^2} + ac\sin\phi\frac{d\phi^2}{dt^2} = -cg\sin\phi;$$

multiplying by $2\frac{d\phi}{dt}$, and integrating,

$$(a^2 + k^2 + c^2 - 2ac\cos\phi)\frac{d\phi^2}{dt^2} = C + 2cg\cos\phi.$$

Let $t = 0$ when $\phi = 0$, and let ω be the initial angular velocity about G; then

$$(a^2 + k^2 + c^2 - 2ac)\,\omega^2 = C + 2cg,$$

and therefore

$$(a^2 + c^2 + k^2 - 2ac\cos\phi)\frac{d\phi^2}{dt^2} = \{(a-c)^2 + k^2\}\,\omega^2 - 2cg\,(1 - \cos\phi),$$

which gives the angular velocity about G at any time in terms of the whole angle described.

Euler; *Nova Acta Acad. Petrop.* 1783; p. 119.

(4) A pendulum of any figure is firmly attached to a solid circular cylinder as an axis; this axis is placed horizontally within a hollow circular horizontal cylinder of larger diameter, and of which the surface is perfectly rough; in the hollow cylinder there is a slit, through which the pendulum hangs: supposing the initial position of the pendulum to be very nearly a position of equilibrium, to find the length of an isochronous simple pendulum.

Let g (fig. 210) denote the position of the centre of gravity of the pendulum and its axis, regarded as one mass, at any instant of the motion; let O, c, denote the centres of the circular

sections of the hollow and the solid cylinders made by a vertical plane through g; produce OA and cg to meet at f: let gp meet at right angles the vertical line OAp, which cuts at A the circular section MAN of the hollow cylinder; join Oc and produce the line to a, the point of contact of the sections of the solid and the hollow cylinders; if e be the point of contact of the section of the solid cylinder in its lowest position with that of the hollow one, the arc Aa will be equal to the arc ea. Let $Op = x$, $gp = y$, $ce = b = ca$, $AO = a = Oa$, $\phi = \angle cfO$, $\angle AOa = \theta$, $cg = c$, $\angle ecf = \beta$; $m =$ the mass of the pendulum and its axis together, $k =$ the radius of gyration about g of the pendulum and axis regarded as one mass, $R =$ the pressure of the hollow upon the solid cylinder, $F =$ the friction of the hollow cylinder upon the solid one in the direction of a tangent to the arc aN.

Then for the motion of the system we have, resolving forces vertically,

$$m\frac{d^2x}{dt^2} = mg - R\cos\theta - F\sin\theta;$$

resolving forces horizontally,

$$m\frac{d^2y}{dt^2} = -R\sin\theta + F\cos\theta,$$

and, taking moments about g,

$$mk^2\frac{d^2\phi}{dt^2} = -Rc\sin(\theta+\phi) + F\{c\cos(\theta+\phi) - b\}.$$

Now, θ and ϕ being by the hypothesis small angles, we may neglect their second and higher powers in the equations, and we get

$$m\frac{d^2x}{dt^2} = mg - R - F\theta,$$

$$m\frac{d^2y}{dt^2} = -R\theta + F,$$

$$mk^2\frac{d^2\phi}{dt^2} = -Rc(\theta+\phi) + F(c-b).$$

Eliminating R and F between these three equations, we shall finally obtain, as far as the first order of small quantities,

$$k^2 \frac{d^2\phi}{dt^2} + cg(\theta + \phi) = \left(\frac{d^2y}{dt^2} + g\theta\right)(c - b) \text{ .... (1).}$$

But, from the geometry, it is clear that

$$y = (a - b)\sin\theta - c\sin\phi = (a - b)\theta - c\phi \qquad \text{nearly,}$$

and therefore, putting for brevity $a - b = e$,

$$\frac{d^2y}{dt^2} = e\frac{d^2\theta}{dt^2} - c\frac{d^2\phi}{dt^2}:$$

hence the equation (1) becomes

$$(k^2 + c^2 - cb)\frac{d^2\phi}{dt^2} - e(c - b)\frac{d^2\theta}{dt^2} + cg\phi + bg\theta = 0 \text{ ...... (2).}$$

Now, since there is no sliding, we may shew, by precisely the same method as in the case of problem (2), that

$$\phi + \beta = \frac{a - b}{b}\theta = \frac{e}{b}\theta,$$

$\phi + \beta$ being evidently the whole angle described by the solid cylinder about its axis in rolling from a to A. Hence, from (2), we have

$$\{k^2 + (c - b)^2\}\frac{d^2\phi}{dt^2} + \frac{g(b^2 + ce)}{e}\phi + \frac{b^2\beta g}{e} = 0:$$

let
$$\frac{g(b^2 + ce)}{e}\phi + \frac{b^2\beta g}{e} = \frac{g(b^2 + ce)}{e}\psi:$$

then $\frac{d^2\phi}{dt^2} = \frac{d^2\psi}{dt^2}$, and the equation becomes

$$\{k^2 + (c - b)^2\}\frac{d^2\psi}{dt^2} + \frac{g(b^2 + ce)}{e}\psi = 0,$$

or
$$\frac{e\{k^2 + (c - b)^2\}}{b^2 + ce}\frac{d^2\psi}{dt^2} + g\psi = 0.$$

Hence, if l denote the length of a perfect pendulum isochronous with the period of ϕ, and therefore of ψ, we shall have

$$l = \frac{e\{k^2 + (c - b)^2\}}{b^2 + ce} = \frac{(a - b)\{k^2 + (c - b)^2\}}{b^2 + c(a - b)}.$$

Euler; *Acta Acad. Petrop.* 1780; P. 2; p: 164.

(5) At the extremities A and B of a uniform beam AB (fig. 211), are two small rings, capable of sliding along the horizontal and vertical rods Ox, Oy; the friction between the ends of the beam and the rods is equal to the normal pressure on each: to determine the motion of the beam.

Let G be the centre of gravity of the beam; draw GH at right angles to Ox; let $OH = x$, $GH = y$, $AG = a = BG$, $\angle BAO = \theta$; let R, S, be the normal reactions of the rods Ox, Oy, and therefore R, S, the frictions along xO, Oy; let $m =$ the mass of the beam, $k =$ the radius of gyration about G.

Then, for the motion of the beam,

$$m\frac{d^2x}{dt^2} = S - R \quad \text{.................................} \quad (1),$$

$$m\frac{d^2y}{dt^2} = S + R - mg \quad \text{.............................} \quad (2),$$

$$mk^2\frac{d^2\theta}{dt^2} = (S - R)\,a\cos\theta + (S + R)\,a\sin\theta \quad \text{.........} \quad (3).$$

Substituting the values of $S - R$ and $S + R$ from (1) and (2) in the equation (3), we get

$$k^2\frac{d^2\theta}{dt^2} = a\cos\theta\frac{d^2x}{dt^2} + \left(g + \frac{d^2y}{dt^2}\right)a\sin\theta \quad \text{............} \quad (4).$$

But $$x = a\cos\theta, \quad \frac{dx}{dt} = -a\sin\theta\frac{d\theta}{dt},$$

$$\frac{d^2x}{dt^2} = -a\cos\theta\frac{d\theta^2}{dt^2} - a\sin\theta\frac{d^2\theta}{dt^2};$$

and $$y = a\sin\theta, \quad \frac{dy}{dt} = a\cos\theta\frac{d\theta}{dt},$$

$$\frac{d^2y}{dt^2} = -a\sin\theta\frac{d\theta^2}{dt^2} + a\cos\theta\frac{d^2\theta}{dt^2}.$$

Substituting these values of $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, in the equation (4), we have

$$k^2\frac{d^2\theta}{dt^2} + a^2\frac{d\theta^2}{dt^2} = ag\sin\theta;$$

or, changing the independent variable from t to θ,

$$-k^2\frac{\dfrac{d^2t}{d\theta^2}}{\dfrac{dt^3}{d\theta^3}}+\frac{a^2}{\dfrac{dt^2}{d\theta^2}}=ag\sin\theta:$$

multiplying both sides of the equation by $2\epsilon^{\frac{2a^2}{k^2}\theta}$, we obtain

$$\frac{d}{d\theta}\left\{\frac{k^2\epsilon^{\frac{2a^2}{k^2}\theta}}{\dfrac{dt^2}{d\theta^2}}\right\}=2ag\epsilon^{\frac{2a^2}{k^2}\theta}\sin\theta;$$

integrating,

$$\frac{k^2\epsilon^{\frac{2a^2}{k^2}\theta}}{\dfrac{dt^2}{d\theta^2}}=C+2ag\int\epsilon^{\frac{2a^2}{k^2}\theta}\sin\theta\,d\theta \quad\text{.............. (5).}$$

Now, integrating by parts, we shall get

$$\int\epsilon^{\frac{2a^2}{k^2}\theta}\sin\theta\,d\theta=\frac{k^4}{k^4+4a^4}\left\{\frac{2a^2}{k^2}\sin\theta-\cos\theta\right\}\epsilon^{\frac{2a^2}{k^2}\theta}:$$

hence, from (5),

$$k^2\epsilon^{\frac{2a^2}{k^2}\theta}\frac{d\theta^2}{dt^2}=\frac{2agk^4}{k^4+4a^4}\left\{C'+\left(\frac{2a^2}{k^2}\sin\theta-\cos\theta\right)\epsilon^{\frac{2a^2}{k^2}\theta}\right\},$$

$$\epsilon^{\frac{2a^2}{k^2}\theta}\frac{d\theta^2}{dt^2}=\frac{2agk^2}{k^4+4a^4}\left\{C'+\left(\frac{2a^2}{k^2}\sin\theta-\cos\theta\right)\epsilon^{\frac{2a^2}{k^2}\theta}\right\}.$$

Since $a^2=3k^2$, we have

$$\epsilon^{6\theta}\frac{d\theta^2}{dt^2}=\frac{6g}{37a}\{C'+(6\sin\theta-\cos\theta)\,\epsilon^{6\theta}\}.$$

Suppose that α, ω, are simultaneous values of θ, $\dfrac{d\theta}{dt}$; then

$$\epsilon^{6\alpha}\omega^2=\frac{6g}{37a}\{C'+(6\sin\alpha-\cos\alpha)\,\epsilon^{6\alpha}\};$$

hence

$$\epsilon^{6\theta}\frac{d\theta^2}{dt^2}-\epsilon^{6\alpha}\omega^2=\frac{6g}{37a}\{(6\sin\theta-\cos\theta)\,\epsilon^{6\theta}-(6\sin\alpha-\cos\alpha)\,\epsilon^{6\alpha}\},$$

which determines the angular velocity of the beam at every position in its descent.

(6) To determine the motion of a cylinder, from a position of instantaneous rest, upon a perfectly rough plane of indefinite extent, which, having been initially horizontal, revolves with a constant angular velocity about a fixed horizontal axis within itself, to which the line of contact of the cylinder and plane is parallel, and with which it initially coincides.

Let a vertical plane at right angles to the axis of revolution, and passing through the centre of gravity C of the cylinder, cut the revolving plane at any instant of the motion in the line OA, (fig. 212); let Ox be the initial position of OA; draw CM, CN, at right angles to Ox, OA; let $OM = x$, $CM = y$, $ON = r$, $CN = a$, $\omega =$ the angular velocity of OA about O, $\angle AOx = \omega t$, $k =$ the radius of gyration of the cylinder about its axis, $R =$ the normal reaction of the plane in the direction NC, $T =$ the tangential reaction along NO, $\theta =$ the angle through which the cylinder has revolved about its centre of gravity at the end of the time t, $m =$ the mass of the cylinder.

Then, for the motion of the cylinder, we have

$$m\frac{d^2x}{dt^2} = R\sin\omega t - T\cos\omega t \text{......................(1)},$$

$$m\frac{d^2y}{dt^2} = mg - R\cos\omega t - T\sin\omega t \text{..............(2)},$$

$$mk^2\frac{d^2\theta}{dt^2} = Ta \text{......................................(3)}.$$

Again, the angle through which the cylinder would revolve about C, by rolling along Ox through a space r, would be $\dfrac{r}{a}$, and that due to the motion of Ox into the position OA would be ωt; hence $\dfrac{r}{a} + \omega t$ is the angle through which it has actually revolved at the end of the time t; or

$$\theta = \frac{r}{a} + \omega t \text{.............................(4)}.$$

From the equations (1) and (2), we have

$$\sin\omega t\frac{d^2x}{dt^2} - \cos\omega t\frac{d^2y}{dt^2} = \frac{R}{m} - g\cos\omega t \text{..........(5)};$$

and, from (1), (2), (3),

$$\cos\omega t\frac{d^2x}{dt^2}+\sin\omega t\frac{d^2y}{dt^2}=g\sin\omega t-\frac{k^2}{a}\frac{d^2\theta}{dt^2};$$

but, from (4), we get $\qquad a\dfrac{d^2\theta}{dt^2}=\dfrac{d^2r}{dt^2}$;

hence $$\cos\omega t\frac{d^2x}{dt^2}+\sin\omega t\frac{d^2y}{dt^2}=g\sin\omega t-\frac{k^2}{a^2}\frac{d^2r}{dt^2}\text{.........(6)}.$$

From the geometry it is clear that

$$x=r\cos\omega t+a\sin\omega t,\quad y=r\sin\omega t-a\cos\omega t:$$

differentiating these expressions twice with respect to t, we shall get

$$\frac{d^2x}{dt^2}=\cos\omega t\frac{d^2r}{dt^2}-2\omega\sin\omega t\frac{dr}{dt}-\omega^2r\cos\omega t-a\omega^2\sin\omega t,$$

$$\frac{d^2y}{dt^2}=\sin\omega t\frac{d^2r}{dt^2}+2\omega\cos\omega t\frac{dr}{dt}-\omega^2r\sin\omega t+a\omega^2\cos\omega t:$$

substituting these expressions for $\dfrac{d^2x}{dt^2}$, $\dfrac{d^2y}{dt^2}$, in the equations (5) and (6), we obtain

$$2\omega\frac{dr}{dt}+a\omega^2=g\cos\omega t-\frac{R}{m}\text{....................(7)},$$

and $$\frac{a^2+k^2}{a^2}\frac{d^2r}{dt^2}-\omega^2r=g\sin\omega t\text{...............(8)}.$$

Since $a^2=2k^2$, the equation (8) becomes

$$\frac{d^2r}{dt^2}-\tfrac{2}{3}\omega^2r=\tfrac{2}{3}g\sin\omega t:$$

the integral of this equation is

$$r=-\frac{2g}{5\omega^2}\sin\omega t+C\epsilon^{\omega' t}+C'\epsilon^{-\omega' t},$$

where $\omega'=(\tfrac{2}{3})^{\frac{1}{2}}\omega$, and C, C', are arbitrary constants. If we determine C and C' from the conditions that $r=0$, $\dfrac{dr}{dt}=0$, initially, we shall have

$$r=-\frac{2g}{5\omega^2}\sin\omega t+\frac{3^{\frac{1}{2}}g}{5\omega^2 2^{\frac{1}{2}}}(\epsilon^{\omega' t}-\epsilon^{-\omega' t}),$$

which determines the position of the cylinder at any time before it detaches itself from the revolving plane. Differentiating with respect to t,

$$\frac{dr}{dt} = -\frac{2g}{5\omega}\cos\omega t + \frac{g}{5\omega}\left(\epsilon^{\omega' t} + \epsilon^{-\omega' t}\right):$$

hence, from (7), we obtain

$$\frac{R}{m} = g\cos\omega t - a\omega^2 + \tfrac{4}{5}\,g\cos\omega t - \frac{2g}{5}\left(\epsilon^{\omega' t} + \epsilon^{-\omega' t}\right)$$

$$= \tfrac{9}{5}\,g\cos\omega t - a\omega^2 - \frac{2g}{5}\left(\epsilon^{\omega' t} + \epsilon^{-\omega' t}\right),$$

which gives the value of R at any time during the motion of the cylinder upon the plane. When $R = 0$, or when the cylinder leaves the plane,

$$9g\cos\omega t = 5a\omega^2 + 2g\left(\epsilon^{\omega' t} + \epsilon^{-\omega' t}\right),$$

an equation which fixes the epoch of the separation.

(7) A sphere is projected directly down an inclined plane with a motion both of translation and of rotation: the motion of rotation is the same in point of direction as that which would correspond to perfect downward rolling, but greater in magnitude: to determine the motion of the sphere, having given the coefficients both of statical and dynamical friction between the sphere and the inclined plane.

Let OA (fig. 213) be the inclined plane, C the position of the sphere's centre, and M its point of contact with OA at the end of a time t from the beginning of the motion. Let $\mu =$ the coefficient of dynamical friction between the sphere and the plane, $a =$ the radius of the sphere, $OM = s$, $\phi =$ the angle through which the sphere has revolved at the end of the time t about its centre of gravity, $R =$ the normal reaction of the plane, $m =$ the mass of the sphere, $\alpha =$ the inclination of the plane to the horizon.

Then, for the motion of the sphere, we have

$$m\frac{d^2s}{dt^2} = mg\sin\alpha + \mu R \;\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$mk^2\frac{d^2\phi}{dt^2} = -\mu a R \;\ldots\ldots\ldots\ldots\ldots\ldots(2):$$

but, since C has no motion at right angles to the plane, we have also $R = mg\cos\alpha$: hence, from (1),

$$\frac{d^2s}{dt^2} = g(\sin\alpha + \mu\cos\alpha) \quad\ldots\ldots\ldots\ldots\ldots (3),$$

and, from (2),

$$k^2\frac{d^2\phi}{dt^2} = -\mu ag\cos\alpha \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Integrating the equation (3) with respect to t, and denoting the initial value of $\frac{ds}{dt}$ by c, we get

$$\frac{ds}{dt} = gt(\sin\alpha + \mu\cos\alpha) + c \quad\ldots\ldots\ldots\ldots\ldots (5):$$

similarly from (4), ω denoting the initial value of $\frac{d\phi}{dt}$,

$$\frac{d\phi}{dt} = \omega - \frac{\mu ag}{k^2}t\cos\alpha \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

As soon as, by the increase of t, $\frac{ds}{dt}$ becomes equal to $a\frac{d\phi}{dt}$, the motion will change its character, and our present equations will cease to be applicable. This event will take place when

$$gt(\sin\alpha + \mu\cos\alpha) + c = a\omega - \frac{\mu a^2 gt}{k^2}\cos\alpha,$$

or
$$t = \frac{k^2(a\omega - c)}{\mu g(a^2 + k^2)\cos\alpha + k^2 g\sin\alpha}.$$

Let t' denote this particular value of t. For all values of t not greater than t', we have from (5) and (6), s and ϕ being considered to be initially zero,

$$s = \tfrac{1}{2}gt^2(\sin\alpha + \mu\cos\alpha) + ct,$$

$$\phi = \omega t - \frac{\mu agt^2}{2k^2}\cos\alpha;$$

the values of s and ϕ at the end of the first period of the motion will be obtained from these expressions by substituting t' in place of t.

When $\frac{ds}{dt}$ becomes equal to $a\frac{d\phi}{dt}$, there evidently exists at that instant no sliding between the plane and the sphere; and therefore, before dynamical friction can again come into play, the statical friction between the sphere and the plane must be overcome. First, let us suppose that the statical friction is sufficiently great to secure perfect rolling; and let F denote the tangential reaction of the plane against the descent of the sphere. The equations of motion will be,

$$m\frac{d^2s}{dt^2} = mg\sin\alpha - F \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$mk^2\frac{d^2\phi}{dt^2} = aF \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Also, there being no sliding, it is clear that

$$a\frac{d\phi}{dt} = \frac{ds}{dt}, \qquad a\frac{d^2\phi}{dt^2} = \frac{d^2s}{dt^2}:$$

hence from (2) there is

$$mk^2\frac{d^2s}{dt^2} = a^2F,$$

and therefore, from (1),

$$(a^2 + k^2)\frac{d^2s}{dt^2} = a^2g\sin\alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3);$$

and therefore, also,

$$(a^2 + k^2)\frac{d^2\phi}{dt^2} = ag\sin\alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Integrating, we get from (3) and (4),

$$\frac{ds}{dt} = \frac{a^2g\sin\alpha}{a^2+k^2}t + c',$$

$$\frac{d\phi}{dt} = \frac{ag\sin\alpha}{a^2+k^2}t + \omega';$$

where c', ω', are the values of $\frac{ds}{dt}$, $\frac{d\phi}{dt}$, at the end of the first stage of the motion, the time being now reckoned from the commencement of the second period: c' is clearly equal to $a\omega'$.

Integrating again, we get

$$s = \frac{\frac{1}{2}a^2gt^2 \sin\alpha}{a^2+k^2} + c't + s',$$

$$\phi = \frac{\frac{1}{2}agt^2 \sin\alpha}{a^2+k^2} + \omega' t + \phi',$$

s', ϕ', being the values of ϕ, s, at the end of the first period.

Also
$$F = \frac{mk^2}{a}\frac{d^2\phi}{dt^2} = \frac{mk^2 g \sin\alpha}{a^2+k^2},$$

which is the value of the statical friction necessary to secure perfect rolling in the second stage of the motion.

If the statical friction be less than this, dynamical friction will arise, and will evidently exert itself *up* the plane. Hence, for the motion,

$$\frac{d^2s}{dt^2} = g(\sin\alpha - \mu\cos\alpha) \;\ldots\ldots\ldots\ldots\ldots\ldots\; \text{(A)},$$

$$k^2\frac{d^2\phi}{dt^2} = \mu a g \cos\alpha \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; \text{(B)}.$$

It may be easily ascertained that the coefficient of g in the expression for $\frac{d^2s}{dt^2}$ is positive: for the coefficient of friction necessary for perfect rolling

$$= \frac{F}{R} = \frac{F}{mg\cos\alpha} = \frac{k^2\tan\alpha}{a^2+k^2};$$

and, since μ is less than this by hypothesis, we have

$$\mu \angle \frac{k^2}{a^2+k^2}\tan\alpha, \text{ and therefore } \mu \angle \tan\alpha,\ \mu\cos\alpha \angle \sin\alpha.$$

From (A) and (B) we have

$$\frac{ds}{dt} = gt(\sin\alpha - \mu\cos\alpha) + c',$$

$$\frac{d\phi}{dt} = \frac{\mu a g t}{k^2}\cos\alpha + \omega'.$$

It may be readily seen that $\frac{ds}{dt} - a\frac{d\phi}{dt}$ never becomes zero in

the second stage of the motion, but is always positive; for, bearing in mind that $c' = a\omega'$,

$$\frac{ds}{dt} - a\frac{d\phi}{dt} = gt\left(\sin\alpha - \mu\cos\alpha - \mu\frac{a^2}{k^2}\cos\alpha\right)$$

$$= gt\cos\alpha\,\frac{a^2+k^2}{k^2}\left(\frac{k^2}{a^2+k^2}\tan\alpha - \mu\right):$$

hence the sphere will always rotate too slowly, in comparison with the velocity of translation, to correspond to perfect rolling.

If s' denote the space through which the sphere descends along the plane in the second stage of the motion, in consequence of sliding,

$$\frac{ds'}{dt} = \frac{ds}{dt} - a\frac{d\phi}{dt} = gt\left(\sin\alpha - \mu\cos\alpha\,\frac{a^2+k^2}{k^2}\right),$$

$$s' = \tfrac{1}{2}gt^2\left(\sin\alpha - \mu\cos\alpha\,\frac{a^2+k^2}{k^2}\right).$$

Euler; *Acta Acad. Petrop.* P. II. p. 131; 1781.

(8) A sphere, revolving about a horizontal diameter, is placed upon an imperfectly rough plane, parallel to the diameter and inclined to the horizon at an angle $\tan^{-1}\mu$, where μ is the coefficient of dynamical friction, the direction of the sphere's rotation being opposite to that which would correspond to perfect downward rolling: to determine the motion of the sphere.

The notation remaining the same as in the preceding problem, we shall have, for the motion of the sphere,

$$m\frac{d^2s}{dt^2} = mg\sin\alpha - \mu R,$$

$$mk^2\frac{d^2\phi}{dt^2} = -\mu a R:$$

but $R = mg\cos\alpha$: hence we have

$$\frac{d^2s}{dt^2} = g(\sin\alpha - \mu\cos\alpha) = 0, \text{ by hypothesis;}$$

and $$\frac{d^2\phi}{dt^2} = -\frac{\mu a g\cos\alpha}{k^2}.$$

Integrating with respect to t, we get

$$\frac{ds}{dt}=C, \quad \frac{d\phi}{dt}=C'-\frac{\mu agt}{k^2}\cos\alpha,$$

where C, C', are arbitrary constants: but, initially, $\frac{ds}{dt}=0$, $\frac{d\phi}{dt}=\omega$; and therefore $C=0$, $C'=\omega$: hence

$$\frac{ds}{dt}=0, \quad \frac{d\phi}{dt}=\omega-\frac{\mu agt}{k^2}\cos\alpha.$$

In the preceding investigation we have supposed the friction of the plane upon the sphere to act upwards: this will cease to be the case when $\frac{ds}{dt}+a\frac{d\phi}{dt}$, or, since $\frac{ds}{dt}=0$, when $\frac{d\phi}{dt}$ becomes zero. Hence we see that, for a time equal to

$$\frac{k^2\omega}{\mu ag\cos\alpha},$$

the centre of the sphere will remain stationary, and that at the end of this time the angular velocity of the sphere will become zero.

Before proceeding to investigate the nature of the motion after the end of the stationary period of the centre of the sphere, it will be necessary to determine the amount of upward friction requisite to cause the sphere subsequently to assume a motion of perfect rolling. The coefficient of friction requisite for this purpose, (see the preceding problem,) is equal to

$$\frac{k^2\tan\alpha}{a^2+k^2}.$$

But μ is equal to $\tan\alpha$, and therefore exceeds the requisite magnitude. The sphere will therefore proceed subsequently to roll down the plane without sliding; and the space described by its centre, at the end of a time t from the termination of its stationary interval, will be equal to

$$\frac{\frac{1}{2}a^2gt^2\sin\alpha}{a^2+k^2},$$

which, since $k^2 = \frac{2}{5}a^2$, is equal to $\frac{5}{14}gt^2 \sin\alpha$. Also, putting for k^2 its value, the stationary interval will have for its value $\dfrac{2a\omega}{5g\sin\alpha}$.

Euler; *Acta Acad. Petrop.* P. II. p. 131; 1781.

(9) A homogeneous sphere, attracted towards a given centre of force varying directly as the distance, is projected with a given velocity along a plane passing through that centre, friction being such as to prevent all sliding: to determine the path described by the sphere.

Let O, (fig. 214), the centre of force, be taken as the origin of co-ordinates, and let Ox, Oy, which are at right angles to each other and in the plane along which the sphere rolls, be taken as the co-ordinate axes. Let C be the centre of the sphere at any time of its motion, P its point of contact with the plane xOy; join CO, CP, and draw PM parallel to yO; draw also Oz at right angles to the plane xOy. Let $OC = r$, $CP = c$, $OM = x$, $PM = y$, $m =$ the mass of the sphere, $\mu =$ the attraction of the central force upon a unit of mass collected at a point at a unit of distance; let X, Y, denote the friction of the plane on the sphere, estimated parallel to Ox, Oy; let ω_x, ω_y, be the angular velocities of the sphere at any time about diameters parallel to Ox, Oy, estimated in the directions indicated by the arrows in the planes yOz, zOx; let $k =$ the radius of gyration of the sphere about a diameter.

It may be readily ascertained that the attraction on the whole sphere will be equal to a force μmr in the direction CO, and the resolved parts of this force parallel to Ox, Oy are evidently $-\mu mx$, $-\mu my$. Hence, for the motion of the sphere, we have

$$m\frac{d^2x}{dt^2} = X - \mu mx \quad\text{.............................}(1),$$

$$m\frac{d^2y}{dt^2} = Y - \mu my \quad\text{.............................}(2),$$

$$mk^2\frac{d\omega_x}{dt} = cY \quad\text{.............................}(3),$$

$$mk^2\frac{d\omega_y}{dt} = -cX \quad\text{.............................}(4).$$

But, since the friction is sufficiently great to prevent all sliding, it is clear that

$$\frac{dx}{dt} = c\omega_y, \quad \frac{dy}{dt} = -c\omega_x:$$

hence, from (3) and (4), we get

$$Y = -\frac{mk^2}{c^2}\frac{d^2y}{dt^2}, \quad X = -\frac{mk^2}{c^2}\frac{d^2x}{dt^2},$$

and therefore, from (1) and (2),

$$\frac{c^2+k^2}{c^2}\frac{d^2x}{dt^2} = -\mu x,$$

$$\frac{c^2+k^2}{c^2}\frac{d^2y}{dt^2} = -\mu y;$$

or, since $k^2 = \frac{2}{5}c^2$,

$$\frac{d^2x}{dt^2} = -\frac{5\mu}{7}x, \quad \frac{d^2y}{dt^2} = -\frac{5\mu}{7}y:$$

the integrals of these equations are

$$x = A\sin(\lambda t + \epsilon),$$

$$y = A'\sin(\lambda t + \epsilon'),$$

A, A', ϵ, ϵ', being arbitrary constants, and λ denoting $\left(\frac{5\mu}{7}\right)^{\frac{1}{2}}$.

Let a, b, be the initial values of x, y, and α, β, of $\frac{dx}{dt}$, $\frac{dy}{dt}$; then

$$x = a\cos(\lambda t) + \frac{\alpha}{\lambda}\sin(\lambda t),$$

$$y = b\cos(\lambda t) + \frac{\beta}{\lambda}\sin(\lambda t).$$

From the last two equations we have,

$$\beta x - \alpha y = (a\beta - b\alpha)\cos(\lambda t),$$

and $$\lambda(bx - ay) = -(a\beta - b\alpha)\sin(\lambda t).$$

Squaring the last two equations and adding, we obtain, for the equation to the path of the sphere,

$$(\beta x - \alpha y)^2 + \lambda^2(bx - ay)^2 = (a\beta - b\alpha)^2.$$

Thus we see that the sphere will describe an ellipse the centre of which coincides with the origin of co-ordinates.

(10) A sphere, the centre of which is moving with a given velocity, and which is spinning with a given angular velocity round a horizontal diameter inclined at a given angle to the direction of motion of the centre, is placed on an imperfectly rough horizontal table: to determine the subsequent motion of the centre.

Let two rectangular axes Ox, Oy, be taken in the plane of the table. Let x, y, be at any time the co-ordinates of the projection of the centre of the sphere on the table, and let ω_x, ω_y, be the angular velocities of the sphere about axes through its centre parallel to Ox, Oy. Let μ be the coefficient of friction between the sphere and the table, and θ the inclination of the friction at any time to the axis of x. Let a be the radius of the sphere.

Then for the motion of the sphere we have

$$\frac{2}{5}a\frac{d\omega_x}{dt} = -\mu g \sin\theta \qquad (1),$$

$$\frac{2}{5}a\frac{d\omega_y}{dt} = \mu g \cos\theta \qquad (2),$$

$$\frac{d^2x}{dt^2} = -\mu g \cos\theta \qquad (3),$$

$$\frac{d^2y}{dt^2} = -\mu g \sin\theta. \qquad (4).$$

The velocity of the point of contact of the sphere, resolved parallel to Ox, is equal to $\frac{dx}{dt} - a\omega_y$, and, resolved parallel to Oy, to $\frac{dy}{dt} + a\omega_x$. Hence

$$\tan\theta = \frac{\frac{dy}{dt} + a\omega_x}{\frac{dx}{dt} - a\omega_y} \qquad (5).$$

Let the co-ordinate axes be so chosen that the axis of x is parallel to the axis of the sphere's initial rotation: let ω be the initial angular velocity of the sphere; $V\cos\alpha$, $V\sin\alpha$, the

components of the initial velocity of its centre. Also let the origin of co-ordinates coincide with the initial point of contact of the sphere with the table.

From (1) and (4), we have

$$\frac{2}{5}a\frac{d\omega_x}{dt}=\frac{d^2y}{dt^2},$$

and therefore

$$\frac{2}{5}a\omega_x=\tfrac{2}{5}a\omega-V\sin\alpha+\frac{dy}{dt}\;\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

From (2) and (3),

$$\frac{2}{5}a\frac{d\omega_y}{dt}=-\frac{d^2x}{dt^2},$$

and therefore

$$\frac{2}{5}a\omega_y=V\cos\alpha-\frac{dx}{dt}\;\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

From (1), (2), (5), (6), (7), we see that

$$\frac{d\omega_x}{\frac{7}{5}a\omega_x+V\sin\alpha-\frac{2}{5}a\omega}=\frac{d\omega_y}{\frac{7}{5}a\omega_y-V\cos\alpha},$$

and therefore, λ being a constant,

$$\frac{7}{5}a\omega_x+V\sin\alpha-\frac{2}{5}a\omega=\lambda\left(\frac{7}{5}a\omega_y-V\cos\alpha\right).$$

Since, initially, $\omega_x=\omega$, and $\omega_y=0$, we have

$$V\cos\alpha\left(\frac{7}{5}a\omega_x+V\sin\alpha-\frac{2}{5}a\omega\right)$$

$$=\left(V\cos\alpha-\frac{7}{5}a\omega_y\right)(a\omega+V\sin\alpha),$$

and therefore

$$V\cos\alpha\,d\omega_x+(V\sin\alpha+a\omega)\,d\omega_y=0,$$

and thence, by (1) and (2),

$$\tan\theta=\frac{V\sin\alpha+a\omega}{V\cos\alpha},$$

which shews that θ is constant: hence, by (1) and (2),

$$x = Vt\cos\alpha - \frac{1}{2}\mu g t^2 \cos\theta,$$

$$y = Vt\sin\alpha - \frac{1}{2}\mu g t^2 \sin\theta;$$

whence

$$x\sin\theta - y\cos\theta = Vt\sin(\theta - \alpha)$$

and $$x\sin\alpha - y\cos\alpha = \frac{1}{2}\mu g t^2 \sin(\theta - \alpha),$$

and therefore the equation to the projection on the table of the path of the centre of the sphere is

$$(x\sin\theta - y\cos\theta)^2 = \frac{2V^2}{\mu g}\sin(\theta - \alpha)\cdot(x\sin\alpha - y\cos\alpha),$$

which is therefore a parabola.

If $\omega = 0$, then $\theta = \alpha$, and the equation becomes $x\sin\alpha = y\cos\alpha$: if $V = 0$, then $\theta = \frac{\pi}{2}$, and the equation becomes $x = 0$.

(11) A sphere is laid upon a rough inclined plane: to determine whether it will slide or not.

It will slide if the coefficient of friction be less than $\frac{2}{7}\tan\alpha$, where α is the inclination of the plane; and not otherwise.

(12) A rough uniform rod is placed on a table at right angles to its edge, beyond which more than half the rod projects: to ascertain whether, during the ensuing motion, the rod will slide at the edge.

If $2a$ be the length of the rod, b the initial distance of its centre of gravity from the edge, and μ the coefficient of friction, it will begin to slide when it has turned through an angle the tangent of which is equal to

$$\frac{\mu a^2}{a^2 + 9b^2}.$$

(13) A wheel, the centre of gravity of which is at a distance c from its centre, and of which the radius is a, rolls on

a perfectly rough horizontal plane: to find the velocity of the centre, when the centre of gravity is vertically below it, in order that, when it is vertically above the centre, the normal pressure on the plane may be zero: to determine also the friction in these two positions of the centre of gravity, and the normal pressure when the centre of gravity is vertically below the centre.

Let k denote the radius of gyration about the centre of gravity, v the required velocity of the centre; F the friction, when the centre of gravity is lowest, and F', when it is highest; R the normal pressure when the centre of gravity is lowest: then

$$v^2 = \frac{a^2 g}{c} \cdot \frac{4c^2 + (a+c)^2 + k^2}{(a-c)^2 + k^2}, \quad F = 0, \quad F' = 0,$$

$$R = 2mg \cdot \frac{a^2 + 3c^2 + k^2}{(a-c)^2 + k^2}.$$

For a complete discussion of the Rolling motion of a cylinder on a rough plane and other interesting problems, the student is referred to a memoir by the Rev. Henry Moseley, in the *Philosophical Transactions of London;* Part 2, for 1851.

(14) A sphere is projected obliquely up a perfectly rough inclined plane: to find the equation to the path of the point of contact between the sphere and plane.

Let α = the inclination of the plane to the horizon, V = the velocity of projection, β = the inclination of the direction of projection to the horizontal line Ox in the inclined plane through the point O of projection; let Oy be drawn up the inclined plane at right angles to Ox.

Then the equation to the path of the point of contact will be

$$y = x \tan \beta - \frac{5}{14} \cdot \frac{gx^2 \sin \alpha}{V^2 \cos^2 \beta}.$$

(15) A sphere is held in contact with the inner surface of a perfectly rough vertical cylinder: it is then projected horizontally in a direction parallel to the tangent plane at the point of contact: to determine the motion of the sphere.

Let c be the distance between the centre of the sphere and the axis of the cylinder, and v the initial velocity of the centre of the sphere: then, at the end of any time t, the plane through the axis of the cylinder and the centre of the sphere will have revolved through an angle equal to $\frac{vt}{c}$, and the centre of the sphere will have descended through a vertical space equal to

$$\frac{5gc^2}{v^2}\sin^2\left(\frac{vt}{c\sqrt{14}}\right).$$

(16) A perfectly rough heavy sphere is placed in contact with a perfectly rough vertical plane, which is made to revolve with a uniform angular velocity about a vertical axis in its own plane: to determine the motion of the sphere.

Let a be the radius of the sphere; c the initial distance and r the distance, after any time t, of its point of contact with the plane from the axis of revolution; ω the angular velocity of the plane; z the descent of the centre of the sphere during the time t: then

$$2r=\left(c+a\sqrt{\frac{7}{5}}\right)e^{\omega t\sqrt{\frac{2}{7}}}+\left(c-a\sqrt{\frac{7}{5}}\right)e^{-\omega t\sqrt{\frac{2}{7}}},$$

$$z=\frac{5g}{\omega^2}\sin^2\frac{\omega t}{\sqrt{14}}.$$

Routh; *Rigid Dynamics*, p. 414.

SECT. 2. *Several Bodies.*

(1) A cylinder rolls directly down a perfectly rough inclined plane, while a string coils round it which unwinds from an equal parallel cylinder revolving about its axis, which is fixed, the position of the latter cylinder being such that the string is parallel to the plane: to find the acceleration of descent, the tension of the string, and the friction of the inclined plane.

Let O (fig. 215) be the centre of gravity of the descending cylinder at any time of its motion down the plane BA, M being its point of contact with the plane; let C be the centre of

gravity of the other cylinder; join CO. Let $CO=x$ at any time t; $a=$ the radius of each of the cylinders, $\alpha=$ the inclination of the plane BA to the horizon, $T=$ the tension of the uncoiled string, $F=$ the friction of the inclined plane exerted upon the cylinder O at M in the direction MB; $m=$ the mass of each of the cylinders, and $k=$ the radius of gyration of each about its axis; let θ, θ', denote the angles through which the cylinders O, C, have revolved about their axes at the end of the time t.

Then, for the motion of the cylinder O, we have

$$m\frac{d^2x}{dt^2}=mg\sin\alpha-F-T\quad\text{..................(1)},$$

$$mk^2\frac{d^2\theta}{dt^2}=(F-T)\,a\quad\text{........................(2)};$$

and, for the motion of the cylinder C,

$$mk^2\frac{d^2\theta'}{dt^2}=Ta\quad\text{..............................(3)}.$$

Multiplying the equations (1) and (3) by a and 2 respectively, and adding the resulting equations to the equation (2), we get

$$k^2\frac{d^2\theta}{dt^2}+2k^2\frac{d^2\theta'}{dt^2}+a\frac{d^2x}{dt^2}=ag\sin\alpha:$$

but, from the geometry, it is evident that

$$a\frac{d^2\theta}{dt^2}=\frac{d^2x}{dt^2},\qquad a\frac{d^2\theta'}{dt^2}=2\frac{d^2x}{dt^2}:$$

hence we have $\qquad (a^2+5k^2)\dfrac{d^2x}{dt^2}=a^2g\sin\alpha,$

or, since $2k^2=a^2$, $\qquad \dfrac{d^2x}{dt^2}=\frac{2}{7}g\sin\alpha.$

Again, from (3),

$$T=\frac{mk^2}{a}\frac{d^2\theta'}{dt^2}=\frac{2mk^2}{a^2}\frac{d^2x}{dt^2}=m\frac{d^2x}{dt^2}=\tfrac{2}{7}mg\sin\alpha.$$

Lastly, from (2), $\quad F=T+\dfrac{mk^2}{a}\dfrac{d^2\theta}{dt^2}=T+\frac{1}{2}ma\dfrac{d^2\theta}{dt^2}$

$$=T+\tfrac{1}{2}m\frac{d^2x}{dt^2}=\tfrac{2}{7}mg\sin\alpha+\tfrac{1}{7}mg\sin\alpha$$

$$=\tfrac{3}{7}mg\sin\alpha.$$

(2) To one end of a string, which passes through a small fixed ring, is attached a weight: the other end of the string is fixed to a point of a cylinder about a circular section of which, through its centre, the string is wound and to the axis of which the whole string is perpendicular: the cylinder is supported upon a rough horizontal plane, its diameter being equal to the altitude of the ring above the plane: to determine the motion of the weight and of the cylinder, which are supposed to be initially in a state of instantaneous rest.

Let R (fig. 216) be the position of the ring; NRP the free portion of the string meeting at the point O the locus OMA of the point M at which the circular section of the cylinder touches the plane at any time; C the position at any time of the centre of gravity of the cylinder. Let a = the radius of the cylinder, $OP=y$, $OM=x$, ϕ = the angle through which the cylinder has revolved about its axis at the end of the time t, F = the action of the plane on the cylinder estimated in the direction AO, m = the mass of the cylinder, k = its radius of gyration about its axis, m' = the mass of P, T = the tension of the string.

Then, for the motion of the weight, we have

$$m'\frac{d^2y}{dt^2}=m'g-T \qquad (1);$$

and for the motion of the cylinder, both in respect to translation and to rotation,

$$m\frac{d^2x}{dt^2}=-T-F \qquad (2),$$

$$mk^2\frac{d^2\phi}{dt^2}=Ta-Fa \qquad (3).$$

Now, since the horizontal plane along which the cylinder moves may be either perfectly or imperfectly rough, we shall have to consider two cases of the motion. We will first suppose the plane to be perfectly rough, that is, to be sufficiently rough to prevent all sliding.

Since the cylinder rolls without sliding, it is evident that $dx = -ad\phi$; and therefore, by (3),

$$-mk^2\frac{d^2x}{dt^2} = Ta^2 - Fa^2 \text{ .... } (4):$$

also, from (2),

$$-ma^2\frac{d^2x}{dt^2} = Ta^2 + Fa^2 \text{ .... } (5),$$

and therefore, adding together the last two equations,

$$-m(a^2+k^2)\frac{d^2x}{dt^2} = 2Ta^2 \text{ .... } (6);$$

and therefore, by (1),

$$2m'a^2\frac{d^2y}{dt^2} - m(a^2+k^2)\frac{d^2x}{dt^2} = 2m'a^2g:$$

but, if l denote the original length of the free string, it is clear that

$$x + y = l + a\phi, \quad \frac{d^2x}{dt^2} + \frac{d^2y}{dt^2} = a\frac{d^2\phi}{dt^2} = -\frac{d^2x}{dt^2}, \quad \frac{d^2y}{dt^2} = -2\frac{d^2x}{dt^2}:$$

hence we have

$$\{4m'a^2 + m(a^2+k^2)\}\frac{d^2x}{dt^2} = -2m'a^2g \text{ .... } (7):$$

integrating, and bearing in mind that $\frac{dx}{dt} = 0$ when $t = 0$,

$$\{4m'a^2 + m(a^2+k^2)\}\frac{dx}{dt} = -2m'a^2gt:$$

integrating again, and taking c for the initial value of x,

$$x = c - \frac{m'a^2gt^2}{4m'a^2 + m(a^2+k^2)}.$$

We may easily shew also that

$$y = l - c + \frac{2m'a^2gt^2}{4m'a^2 + m(a^2+k^2)};$$

and therefore, since $a\phi = x + y - l$, that

$$\phi = \frac{m'agt^2}{4m'a^2 + m(a^2+k^2)}.$$

Also, from (4) and (5),

$$2Fa^2 = -m(a^2 - k^2)\frac{d^2x}{dt^2},$$

and consequently, from (7),

$$F = \frac{mm'(a^2 - k^2)g}{4m'a^2 + m(a^2 + k^2)}.$$

Also, from (6), we have

$$T = \frac{m(a^2 + k^2)}{2a^2}\frac{d^2x}{dt^2}$$

$$= \frac{mm'g(a^2 + k^2)}{4m'a^2 + m(a^2 + k^2)}.$$

We will now proceed to the consideration of the case when the friction of the plane is not sufficient to prevent sliding: and since the value which we obtained for F, the friction necessary to prevent sliding, is positive, therefore this force would act during the whole motion in the direction AO, which shews that, if the action of the plane on the cylinder be not sufficient to secure perfect rolling, the dynamical friction will be exerted in the direction AO. Let μ denote the coefficient of dynamical friction; then, putting μmg instead of F in the equations (2) and (3), we have

$$m\frac{d^2x}{dt^2} = -T - \mu mg,$$

$$mk^2\frac{d^2\phi}{dt^2} = Ta - \mu mag:$$

from these two equations, together with the equation (1), and the appropriate geometrical relations, we may easily shew that

$$\phi = \tfrac{1}{2}\frac{(1 - 2\mu)m' - \mu m}{mk^2 + m'(a^2 + k^2)}agt^2,$$

$$x = c - \tfrac{1}{2}gt^2\frac{m'(2\mu a^2 + k^2) + m\mu k^2}{mk^2 + m'(a^2 + k^2)},$$

$$y = l - c + \tfrac{1}{2}gt^2\frac{m'(a^2 + k^2) - \mu m(a^2 - k^2)}{mk^2 + m'(a^2 + k^2)},$$

$$T = \frac{\mu a^2 + (1 - \mu)k^2}{mk^2 + m'(a^2 + k^2)}mm'g.$$

In a memoir by Fuss, from which this problem has been extracted, is discussed the more general case when the cylinder descends down an inclined plane, and the ring is replaced by a pully of considerable inertia.

Fuss; *Nova Acta Acad. Petrop.* 1787, p. 176.

(3) A cylinder rolls, without sliding, down a moveable inclined plane, which rests on a perfectly smooth horizontal surface: to determine the motion of the plane and of the cylinder.

The axis of the cylinder is supposed to be horizontal, and a vertical plane, at right angles to the axis, to contain the centre of gravity C (fig. 217) of the cylinder and the centre of gravity of the inclined plane: let Ox be the intersection of this vertical plane with the smooth surface which supports the inclined plane AB. Draw CM at right angles to Ox. Let R be the mutual normal action and reaction of the cylinder and inclined plane, F the action of the plane on the cylinder along AB; let $m =$ the mass of the cylinder, $mk^2 =$ its moment of inertia about its axis; $m' =$ the mass of the inclined plane; $OM = x$, $CM = y$, $OA = x'$; $\theta =$ the angle through which the cylinder has revolved about its axis at the end of the time t; $\alpha =$ the inclination of AB to Ox, $a =$ the radius of the cylinder.

Then, for the motion of the cylinder, we have

$$m\frac{d^2x}{dt^2} = -R\sin\alpha + F\cos\alpha \qquad\text{(1)},$$

$$m\frac{d^2y}{dt^2} = R\cos\alpha + F\sin\alpha - mg \qquad\text{(2)},$$

$$mk^2\frac{d^2\theta}{dt^2} = Fa \qquad\text{(3)};$$

and, for the motion of the inclined plane,

$$m'\frac{d^2x'}{dt^2} = R\sin\alpha - F\cos\alpha \qquad\text{(4)}.$$

From the equations (1) and (4), we get

$$m\frac{d^2x}{dt^2} + m'\frac{d^2x'}{dt^2} = 0 \qquad\text{(5)}.$$

Again, from the geometry, we see that

$$y \cos \alpha = a + (x - x') \sin \alpha:$$

hence
$$\cos \alpha \frac{d^2y}{dt^2} = \sin \alpha \left(\frac{d^2x}{dt^2} - \frac{d^2x'}{dt^2}\right),$$

and therefore, by (5),

$$m' \cos \alpha \frac{d^2y}{dt^2} = (m + m') \sin \alpha \frac{d^2x}{dt^2} \quad \text{................(6).}$$

Also, since no sliding takes place between the cylinder and the plane, it is clear that

$$\frac{dx}{dt} = \frac{dx'}{dt} - a \cos \alpha \frac{d\theta}{dt},$$

and therefore
$$a \cos \alpha \frac{d^2\theta}{dt^2} = \frac{d^2x'}{dt^2} - \frac{d^2x}{dt^2}:$$

hence, by the aid of (5), we have

$$m'a \cos \alpha \frac{d^2\theta}{dt^2} = -(m + m') \frac{d^2x}{dt^2} \quad \text{................(7).}$$

Again, from (1) and (2),

$$m \cos \alpha \frac{d^2x}{dt^2} + m \sin \alpha \frac{d^2y}{dt^2} = F - mg \sin \alpha,$$

and therefore, by (3),

$$a \cos \alpha \frac{d^2x}{dt^2} + a \sin \alpha \frac{d^2y}{dt^2} = k^2 \frac{d^2\theta}{dt^2} - ag \sin \alpha:$$

substituting in this equation the expressions for $\frac{d^2y}{dt^2}$ and $\frac{d^2\theta}{dt^2}$ given in (6) and (7), we obtain

$$\{m'a^2 \cos^2 \alpha + (m + m')(a^2 \sin^2 \alpha + k^2)\} \frac{d^2x}{dt^2} = -m'a^2g \sin \alpha \cos \alpha:$$

the value of $\frac{d^2x}{dt^2}$ is therefore constant during the whole motion; the values of $\frac{d^2x'}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2\theta}{dt^2}$, may now be readily obtained by the aid of the equations (5), (6), (7), and will be constant during the whole motion. Knowing the values of $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}$,

$\frac{d^2\theta}{dt^2}$, $\frac{d^2x'}{dt^2}$, we may immediately obtain the values of x, y, θ, x', in terms of t, if the initial values of x, x', $\frac{dx}{dt}$, $\frac{dx'}{dt}$, be given. The values of R and F may also be readily obtained from the equations (1), (2), (3), (4).

(4) A bullet is fired with a given velocity into a body in a direction passing through the centre of gravity of the body; the body is initially at rest and is capable of free motion, not being under the action of any forces: to determine the velocities of the bullet and of the body when the bullet has traversed any space within the body, the resistance of the body to the motion of the bullet being supposed to be a constant force.

Let k denote the constant retarding force, m the mass of the bullet, μ of the body, β the initial velocity of the bullet; then, if u and v denote the velocities of the bullet and of the body when the bullet has traversed a space x within the body,

$$u = \frac{m\beta}{m+\mu} + \frac{\mu}{m+\mu}\left\{\beta^2 - \frac{2k}{m\mu}(m+\mu)\,x\right\}^{\frac{1}{2}},$$

$$v = \frac{m\beta}{m+\mu} - \frac{m}{m+\mu}\left\{\beta^2 - \frac{2k}{m\mu}(m+\mu)\,x\right\}^{\frac{1}{2}}.$$

Camus; *Mém. de l'Acad. des Sciences de Paris*, 1738, p. 147.

(5) A rough cylinder, the centre of gravity of which is not in its axis, is placed, in a position nearly coinciding with one of stable equilibrium, on a board resting on a smooth horizontal plane: to find the length of the simple pendulum which vibrates isochronously with the oscillations of the system.

If c be the shortest distance of the centre of gravity of the board from the surface of the cylinder, k the radius of gyration of the cylinder about a line, parallel to its axis, through its centre of gravity, and m, m', the masses of the cylinder and board, then the length l of the pendulum is given by the equation

$$lh = k^2 + \frac{m'c^2}{m+m'}.$$

CHAPTER X.

DYNAMICAL PRINCIPLES.

SECT. 1. *Vis Viva.*

THE term Vis Viva was first introduced into the language of Mechanics by Leibnitz, in a memoir published in the *Acta Eruditorum* for the year 1695, entitled *Specimen dynamicum pro admirandis naturæ legibus circa corporum vires et mutuas actiones detegendis et ad suas causas revocandis:* it was intended by its author to signify the force of a body in actual motion, called otherwise its Vis Motrix or Moving Force, as distinguished from the statical pressure of a body, which has merely a tendency to motion, against a fixed obstacle; the statical force of a body he designated by the appellation of Vis Mortua. Leibnitz contended, in opposition to the received doctrine of the Cartesians, that the proper measure of the Vis Viva or Moving Force of a body, is the product of its mass into the square of its velocity, the measure adopted by the disciples of Descartes having been the same as that of the Quantity of Motion, namely, the product of the mass and the first power of the velocity. This contrariety of opinion in respect to the estimation of Moving Force, gave rise to one of the most memorable controversies in the annals of philosophy; almost all the mathematicians of Europe ultimately arranging themselves as partizans, either of the Cartesian or of the Leibnitzian doctrine. Among the adherents of Leibnitz may be mentioned John and Daniel Bernoulli, Poleni, Wolff, 'sGravesande, Camus, Muschenbroek, Papin, Hermann, Bulfinger, Kœnig, and eventually Madame du Châtelet; while in the opposite ranks may be named Maclaurin, Clarke, Stirling, Desaguliers, Catalan, Robins, Mairan, and Voltaire. The Vis Motrix, or, as Leibnitz

expressed it, the Vis Viva of a moving body was regarded as a power inherent in the body, by which it is able to encounter a certain amount of resistance before losing the whole of its velocity: the question reduced itself, therefore, to the determination of an appropriate measure of this amount of resistance, to which the Moving Force was supposed to be proportional. Leibnitz regarded the product of the mass of the body and the space through which it must move, under the action of a given retarding force, to lose the whole of its velocity, as the correct measure of the whole resistance expended in the destruction of its motion, and therefore as a proper representative of the Vis Motrix or Vis Viva of the body. Now, by the theory of uniform acceleration, $mv^2 = 2mfs$, m being the mass of the body, and s the space which it must describe, under the action of a constant retarding force f, to lose the whole of its velocity v: hence it is evident that, according to the doctrine of Leibnitz, mv^2 will represent the body's Vis Viva. On the other hand, the Cartesians estimated the whole resistance necessary for the destruction of the body's velocity by the product of the mass of the body and the whole time of the action of the given retarding force; and therefore, by the formula $mv = mft$, it would follow that mv is the proper measure of the Vis Motrix, or, in the language of Leibnitz, of the Vis Viva of the body. The memorable controversy of the Vis Viva, after raging for the space of about thirty years, was finally set to rest by the luminous observations of D'Alembert in the preface to his *Dynamique*, who declared the whole dispute to be a mere question of terms, and as having no possible connection with the fundamental principles of Mechanics. Since the publication of D'Alembert's work, the term Vis Viva has been used to signify merely the algebraical product of the mass of a moving body and the square of its velocity, while the words Moving Force have been universally employed, agreeably to the definition given by Newton in the *Principia*, in the signification of the product of the mass of a body and the accelerating force to which it is conceived to be subject, no physical theory whatever in regard to the absolute nature of force being supposed to be involved in these definitions. For

additional information respecting the controversy of the Vis Viva, the reader is referred to Montucla's *Histoire des Mathématiques*, Tom. III.; Hutton's *Mathematical Dictionary* under the word 'Force'; and Whewell's *History of the Inductive Sciences*.

The Principle of the Conservation of Vis Viva is comprehended in the following proposition: *If a system of particles, any number of which are rigidly connected together, move from one position to another, either with or without constraint, under the action of finite accelerating forces, external or internal; the change of the vis viva of the whole system will be independent of the actions of the particles arising from their mutual connections, and will be equal to the sum of the changes which would be experienced by the vires vivæ of the particles, were each to move unconnectedly from its original to its new position through a thin smooth fixed tube, under the action of the very accelerating forces to which it is subject in the actual state of the motion.* This Principle immediately furnishes us with a first integral of the differential equations of motion, which is frequently of great use; especially if the co-ordinates of the position of the moving system involve only one independent variable, as in the problem of the Centre of Oscillation, when the Principle is sufficient for the complete determination of the motion.

The Principle employed by Huyghens[1] as the basis of his investigations on the problem of the Centre of Oscillation, constitutes under an indirect form a particular instance of the Principle of the Conservation of Vis Viva. John Bernoulli[2], however, was the first who enunciated the theory of the Conservation of Vis Viva, a name which he gave to the Principle, as a general law of nature, from which he deduced that of

[1] Si pendulum e pluribus ponderibus compositum, atque e quiete dimissum, partem quamcunque oscillationis integræ confecerit, atque inde porro intelligantur pondera ejus singula, relicto communi vinculo, celeritates acquisitas sursum convertere, ac quousque possunt ascendere; hoc facto, centrum gravitatis ex omnibus compositæ, ad eandem altitudinem reversum erit, quam ante inceptam oscillationem obtinebat. *Horolog. Oscillator.* p. 126.

[2] *Opera*, passim.

Huyghens as a particular case. Daniel Bernoulli[1] afterwards extended the application of the Principle to the motion of bodies subject to mutual attraction, or solicited towards fixed centres by forces varying as any functions of the distances. A demonstration of the Principle in particular cases was first given by D'Alembert[2] by the aid of his general Principle of Dynamics, the same method of proof being, it was evident, of general application.

If m be the mass and v the velocity of a particle of a material system, $\Sigma\,(mv^2)$, called by Leibnitz the Vis Viva of the system, was termed its Energy by Young[3]. Rankine[4] designated the expression $\frac{1}{2}\Sigma\,(mv^2)$ by the term Actual Energy, in order to distinguish it from Potential Energy, a term invented by him to denote the amount of work which the mutual forces of the system perform during its passage from any initial configuration to the configuration at any subsequent instant. What Rankine has termed Actual Energy has been called by Thomson[5] Dynamical Energy, and by Thomson[6] and Tait[6] Kinetic Energy. What Rankine has called Potential Energy had been named the Sum of the Tensions by Helmholtz[7] and Statical Energy by Thomson. The terms Kinetic and Potential Energy are now usually adopted. On the subject of Energetics the student is referred also to Tyndall's *Heat considered as a Mode of Motion;* Balfour Stewart's *Elementary Treatise on Heat;* Maxwell's *Theory of Heat;* and Tait's *Thermodynamics.*

(1) A uniform rod AB (fig. 218) moves in a vertical plane, within a hemisphere: to determine its angular velocity in any of its positions, its initial position being one of instantaneous rest.

[1] *Mémoires de l'Académie des Sciences de Berlin,* 1748.
[2] *Traité de Dynamique, Seconde Partie,* chap. IV. p. 252.
[3] *Lectures on Natural Philosophy,* 1807; Vol. I. p. 78; Vol. II. p. 52.
[4] *Edinburgh New Philosophical Journal,* 1855, Vol. II. p. 120.
[5] *Edinburgh New Philosophical Journal,* 1855, Vol. I. p. 90.
[6] *Treatise on Natural Philosophy,* Vol. I. p. 163.
[7] Berlin, 1847. Translated in Taylor's *Scientific Memoirs,* Feb. 1853.

Let O be the centre of the sphere; G the middle point of AB, which will be its centre of gravity; GH a perpendicular from G upon the horizontal radius through O, which is in the plane of the rod's motion; let $OG = c$, $AG = a = BG$, $k =$ the radius of gyration about G; $OH = x$, $GH = y$, and $\theta =$ the angle of inclination of AB to the horizon at any time t. Then, by the Principle of the Conservation of Vis Viva, m being the mass of the rod,

$$m\left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + k^2\frac{d\theta^2}{dt^2}\right) = C + 2mgy:$$

let h be the initial value of y; then, since $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{d\theta}{dt}$, are initially zero, we have $\quad 0 = C + 2mgh$:

hence
$$\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + k^2\frac{d\theta^2}{dt^2} = 2g(y - h).$$

But from the geometry it is plain that

$$x = c\sin\theta, \qquad y = c\cos\theta,$$

whence
$$\frac{dx}{dt} = c\cos\theta\frac{d\theta}{dt}, \qquad \frac{dy}{dt} = -c\sin\theta\frac{d\theta}{dt}:$$

we have, therefore,

$$(c^2 + k^2)\frac{d\theta^2}{dt^2} = 2cg(\cos\theta - \cos\alpha),$$

α being the initial value of θ: hence, putting for k^2 its value $\frac{1}{3}a^2$, we have, for the angular velocity of the rod in any of its positions,

$$\frac{d\theta^2}{dt^2} = \frac{6cg}{3c^2 + a^2}(\cos\theta - \cos\alpha).$$

(2) A rod PQ (fig. 219) is kept in a vertical position by means of two small rings A and A'; its lower end P is supported on an inclined plane BC, which is at liberty to move freely on a horizontal plane: to determine the motion of the rod and the plane.

Produce QP to meet the horizontal plane at the point O; let $OP = y$, $OB = x$, at any time of the motion; $h =$ the initial

value of y, $a=$ the inclination of the inclined plane to the vertical, $m=$ the mass of the rod, $m'=$ the mass of the inclined plane.

Then, by the Principle of the Conservation of Vis Viva,

$$m'\frac{dx^2}{dt^2}+m\frac{dy^2}{dt^2}=C-2mgy:$$

but, supposing the rod and the plane to be initially in a state of instantaneous rest,

$$0=C-2mgh:$$

hence $$m'\frac{dx^2}{dt^2}+m\frac{dy^2}{dt^2}=2mg(h-y):$$

but, from the geometry,

$$x=y\tan a,\quad \frac{dx}{dt}=\tan a\frac{dy}{dt}:$$

hence we have

$$(m'\tan^2 a+m)\frac{dy^2}{dt^2}=2mg(h-y),$$

$$(m'\tan^2 a+m)^{\frac{1}{2}}\frac{dy}{(h-y)^{\frac{1}{2}}}=-(2mg)^{\frac{1}{2}}dt,$$

the negative sign being taken, because y decreases as t increases: therefore, by integration,

$$2(m'\tan^2 a+m)^{\frac{1}{2}}(h-y)^{\frac{1}{2}}=C+(2mg)^{\frac{1}{2}}t:$$

but $y=h$ when $t=0$; and therefore $C=0$: hence

$$2(m'\tan^2 a+m)(h-y)=mgt^2,$$

and therefore, at any instant of the motion,

$$y=h-\frac{\frac{1}{2}mgt^2}{m+m'\tan^2 a},$$

and

$$x=h\tan a-\frac{\frac{1}{2}mgt^2\tan a}{m+m'\tan^2 a}.$$

(3) AB (fig. 220) is a uniform beam, capable of moving freely about a hinge A; the extremity B rests upon an inclined plane CE, which forms the upper surface of a body ECD; the

body rests with a flat base upon a smooth horizontal plane passing through A, the vertical plane which contains AB being supposed to cut the plane surface of the body CED at right angles, and to pass through its centre of gravity: to determine the motion of the beam and the body.

Let G be the centre of gravity of AB; draw GH at right angles to the straight line ACD; let m, m', denote the masses of the beam and of the body; let $AH = x$, $GH = y$, $\angle BAC = \theta$, $\angle ECD = \alpha$, $AC = x'$, $k =$ the radius of gyration of AB about G. Then, by the Principle of the Conservation of Vis Viva,

$$m\left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right) + mk^2\frac{d\theta^2}{dt^2} + m'\frac{dx'^2}{dt^2} = C - 2mgy:$$

but from the geometry we see that

$$x = a\cos\theta, \quad y = a\sin\theta, \quad x' = \frac{2a}{\sin\alpha}\sin(\alpha - \theta)\ldots\ldots(1):$$

hence we have

$$m(a^2 + k^2)\frac{d\theta^2}{dt^2} + \frac{4a^2}{\sin^2\alpha}m'\cos^2(\alpha - \theta)\frac{d\theta^2}{dt^2} = C - 2mga\sin\theta,$$

$$\{m(a^2 + k^2)\sin^2\alpha + 4m'a^2\cos^2(\alpha - \theta)\}\frac{d\theta^2}{dt^2} = \sin^2\alpha\,(C - 2mga\sin\theta);$$

let β be the value of θ when $\frac{d\theta}{dt} = 0$; then

$$0 = \sin^2\alpha\,(C - 2mga\sin\beta),$$

and therefore we get

$$\{m(a^2 + k^2)\sin^2\alpha + 4m'a^2\cos^2(\alpha - \theta)\}\frac{d\theta^2}{dt^2} = 2mag\sin^2\alpha\,(\sin\beta - \sin\theta),$$

which gives the value of $\frac{d\theta}{dt}$ for any assigned value of θ; whence, by the aid of the equations (1), we may obtain the values of $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dx'}{dt}$, for any position of the beam.

(4) A uniform lever ACB, (fig. 221), of which the arms AC and BC are at right angles to each other, is in equilibrium when AC is inclined at a given angle to the horizon: if AC be raised to a horizontal position, C being fixed, to find the angle through which it will fall.

Let $CA = 2a$, $CB = 2a'$, $m =$ the mass of AC, $m' =$ the mass of BC; let θ, θ', be the inclinations of CA, CB, to the horizon, at any time of the motion.

Then the vis viva of the lever will be equal to

$$2mag \sin \theta + 2m'a'g \sin \theta' + C:$$

but, when $\theta = 0$ and therefore $\theta' = \frac{1}{2}\pi$, the vis viva is equal to zero; hence

$$0 = 2m'a'g + C:$$

hence the vis viva for any position of the lever is equal to

$$2mag \sin \theta + 2m'a'g \sin \theta' - 2m'a'g.$$

Now, when the value of θ is a maximum, the vis viva will again become zero; hence, for the required value of θ,

$$ma \sin \theta + m'a' \sin \theta' = m'a' \dots\dots\dots\dots\dots\dots (1).$$

Let β, β', be the values of θ, θ', for the equilibrium of the lever; then

$$ma \cos \beta = m'a' \cos \beta':$$

hence from (1) there is

$$\cos \beta' \sin \theta + \cos \beta \sin \theta' = \cos \beta,$$

or, since $\beta' = \frac{1}{2}\pi - \beta$, $\theta' = \frac{1}{2}\pi - \theta$,

$$\sin \beta \sin \theta + \cos \beta \cos \theta = \cos \beta, \quad \cos (\theta - \beta) = \cos \beta;$$

and therefore $\theta = 2\beta$, the angle through which CA falls.

(5) To determine the motion of a pendulum, the axis of which is a cylinder resting upon two perfectly rough planes which coincide with the same horizontal plane, the cylindrical axis being thus capable of rolling along the planes.

Let C (fig. 222) be the centre of a circular section of the cylindrical axis made by a plane containing the centre of gravity of the pendulum; C may be regarded as the centre of gravity of the axis. Let G be the centre of gravity of the pendulum and cylinder together, and mk^2 their moment of inertia about a horizontal line through G parallel to the axis, m denoting the sum of their masses. Let GH be drawn at right angles to the horizontal plane along which the axis rolls; let O be the point of contact of the section C of the axis with this plane at any time of the motion, A being the position of O corresponding to the

equilibrium of the system. Let $CO = c$, $CG = a$, $\angle GCK = \phi$, CK being the vertical line, $AH = x$, $GH = y$.

Then the vis viva of the system at the time t due to the motion of G will be $m\left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right)$, and the vis viva due to rotation about G will be $mk^2\frac{d\phi^2}{dt^2}$: hence the whole vis viva of the system will be equal to

$$m\left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + k^2\frac{d\phi^2}{dt^2}\right):$$

also the sum of the products of the mass of each molecule of the system into the vertical space through which it has descended, will be equal to my together with some constant quantity depending upon the initial circumstances of the system. Hence, by the Principle of Conservation of Vis Viva,

$$m\left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + k^2\frac{d\phi^2}{dt^2}\right) = C + 2gmy:$$

but from the geometry it is evident that

$$x = a\sin\phi - c\phi, \quad y = a\cos\phi - c,$$

and therefore

$$\frac{dx}{dt} = (a\cos\phi - c)\frac{d\phi}{dt}, \quad \frac{dy}{dt} = -a\sin\phi\frac{d\phi}{dt}:$$

hence we have

$$(a^2 + c^2 + k^2 - 2ac\cos\phi)\frac{d\phi^2}{dt^2} = C' + 2g(a\cos\phi - c):$$

let α be the maximum value of ϕ; then

$$0 = C' + 2g(a\cos\alpha - c),$$

and therefore

$$(a^2 + c^2 + k^2 - 2ac\cos\phi)\frac{d\phi^2}{dt^2} = 2ga(\cos\phi - \cos\alpha),$$

which gives the angular velocity of the pendulum for every position which it assumes during its motion.

For the period of a semi-oscillation we have

$$T = \frac{1}{(2ag)^{\frac{1}{2}}}\int_0^\alpha \frac{(a^2 + c^2 + k^2 - 2ac\cos\phi)^{\frac{1}{2}}}{(\cos\phi - \cos\alpha)^{\frac{1}{2}}}d\phi \ldots\ldots\ldots\ldots(1).$$

This integration cannot be effected except, which we will suppose to be the case, the amplitude of the pendulum's oscillation is very small.

Assume then $\quad \sin\frac{\phi}{2} = s, \quad \sin\frac{\alpha}{2} = b;$

whence
$$\cos\phi = 1 - 2\sin^2\frac{\phi}{2} = 1 - 2s^2,$$
$$\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2} = 1 - 2b^2,$$

and
$$d\phi = \frac{4sds}{\sin\phi} = \frac{4sds}{(1-\cos^2\phi)^{\frac{1}{2}}} = \frac{2ds}{(1-s^2)^{\frac{1}{2}}}.$$

Hence, from (1),
$$T = \frac{1}{(2ag)^{\frac{1}{2}}}\int_0^b \frac{(a^2+c^2+k^2-2ac+4acs^2)^{\frac{1}{2}}}{(2b^2-2s^2)^{\frac{1}{2}}} \cdot \frac{2ds}{(1-s^2)^{\frac{1}{2}}},$$
and therefore, putting $(a-c)^2 + k^2 = h^2$,
$$T = \frac{1}{(ag)^{\frac{1}{2}}}\int_0^b \frac{(h^2+4acs^2)^{\frac{1}{2}}\,ds}{(1-s^2)^{\frac{1}{2}}(b^2-s^2)^{\frac{1}{2}}};$$
but, s being a small quantity, we have, neglecting small quantities of orders higher than the second,
$$\frac{1}{(1-s^2)^{\frac{1}{2}}} = (1-s^2)^{-\frac{1}{2}} = 1 + \tfrac{1}{2}s^2,$$
$$(h^2+4acs^2)^{\frac{1}{2}} = h\left(1 + \frac{2ac}{h^2}s^2\right),$$
and therefore
$$\frac{(h^2+4acs^2)^{\frac{1}{2}}}{(1-s^2)^{\frac{1}{2}}} = h\left(1 + \frac{4ac+h^2}{2h^2}s^2\right) \text{ nearly:}$$
hence
$$T = \frac{h}{(ag)^{\frac{1}{2}}}\int_0^b ds\left\{\frac{1}{(b^2-s^2)^{\frac{1}{2}}} + \frac{4ac+h^2}{2h^2}\,\frac{s^2}{(b^2-s^2)^{\frac{1}{2}}}\right\}$$
$$= \frac{h}{(ag)^{\frac{1}{2}}}\left\{\frac{\pi}{2} + \frac{4ac+h^2}{2h^2}\,\frac{\pi b^2}{4}\right\}$$
$$= \frac{\pi h}{2(ag)^{\frac{1}{2}}} + \frac{\pi b^2(4ac+h^2)}{8h(ag)^{\frac{1}{2}}};$$

and therefore the period of a complete oscillation is equal to

$$\frac{\pi h}{(ag)^{\frac{1}{2}}} + \frac{\pi b^2 (4ac + h^2)}{4h (ag)^{\frac{3}{2}}}.$$

Euler has discussed this problem, starting with the general equations of motion, and investigated the pressure on the plane at any time, as well as the horizontal action of the plane upon the cylinder which shall be sufficient to prevent sliding.

Euler; *Nova Acta Acad. Petrop.* 1788; p. 145.

(6) Two equal particles are attached to the extremities A, A', (fig. 223), of a straight lever, without weight, of which C is the fulcrum, and of which the arms CA, CA', are equal to each other: at a fixed point O, vertically above C, there is a centre of force varying inversely as the cube of the distance: CO is equal to either arm of the lever: supposing the lever to be placed in a given position, to determine after what time it will become vertical.

Let $CO = a$; $OA = r$ and $OA' = r'$, at any time of the motion; $\angle ACO = \theta$; m = the mass of each of the particles, μ = the attraction of the central force upon a unit of matter collected at a point at a unit of distance; α = the initial value of θ.

Then the vis viva of the two particles together will be, at any time t, $2ma^2 \frac{d\theta^2}{dt^2}$: hence

$$2ma^2 \frac{d\theta^2}{dt^2} = 2\int\left(-\frac{\mu m}{r^3}\,dr - \frac{\mu m}{r'^3}\,dr'\right) + C,$$

and therefore

$$2a^2 \frac{d\theta^2}{dt^2} = \frac{\mu}{r^2} + \frac{\mu}{r'^2} + C'.$$

Now from the geometry we have

$$r^2 = 2a^2 (1 - \cos\theta), \quad r'^2 = 2a^2 (1 + \cos\theta):$$

hence
$$2a^2 \frac{d\theta^2}{dt^2} = \frac{\mu}{2a^2}\left(\frac{1}{1-\cos\theta} + \frac{1}{1+\cos\theta}\right) + C'$$

$$= \frac{\mu}{a^2}\frac{1}{\sin^2\theta} + C':$$

but when $\theta = \alpha$, $\frac{d\theta}{dt} = 0$: hence

$$0 = \frac{\mu}{a^2}\frac{1}{\sin^2\alpha} + C',$$

and therefore

$$2a^2\frac{d\theta^2}{dt^2}=\frac{\mu}{a^2\sin^2\alpha}\,\frac{\sin^2\alpha-\sin^2\theta}{\sin^2\theta}:$$

hence, $\frac{d\theta}{dt}$ being negative because θ decreases with the increase of t,

$$\left(\frac{2}{\mu}\right)^{\frac{1}{2}}a^2\frac{\sin\theta\,d\theta}{(\sin^2\alpha-\sin^2\theta)^{\frac{1}{2}}}=-\frac{dt}{\sin\alpha},$$

$$\left(\frac{2}{\mu}\right)^{\frac{1}{2}}a^2\frac{d\cos\theta}{(\cos^2\theta-\cos^2\alpha)^{\frac{1}{2}}}=\frac{dt}{\sin\alpha}:$$

integrating we get

$$\left(\frac{2}{\mu}\right)^{\frac{1}{2}}a^2\log\{\cos\theta+(\cos^2\theta-\cos^2\alpha)^{\frac{1}{2}}\}=\frac{t}{\sin\alpha}+C:$$

but $\theta=\alpha$ when $t=0$: hence $C=\left(\frac{2}{\mu}\right)^{\frac{1}{2}}a^2\log\cos\alpha$; and therefore

$$\left(\frac{2}{\mu}\right)^{\frac{1}{2}}a^2\log\frac{\cos\theta+(\cos^2\theta-\cos^2\alpha)^{\frac{1}{2}}}{\cos\alpha}=\frac{t}{\sin\alpha}.$$

When ACA' becomes vertical, $\theta=0$, and therefore the required value of t is equal to

$$\left(\frac{2}{\mu}\right)^{\frac{1}{2}}a^2\sin\alpha\log\frac{1+\sin\alpha}{\cos\alpha}=\left(\frac{2}{\mu}\right)^{\frac{1}{2}}a^2\sin\alpha\log\tan\frac{\pi+2\alpha}{4}.$$

(7) *BFG* (fig. 224) is a body of any form, of which C is the centre of gravity; an inextensible string, attached to a fixed point E, is wound about the circumference of a circle ALH, having C for its centre, and representing an axle: EA is vertical: to determine the velocity of C when the body has descended from rest through a given altitude, under the action of gravity, by the uncoiling of the string.

Let a be the radius of the axle, k the radius of gyration of the body about C; v the velocity acquired by C, after descending through a space x. Then

$$v^2=\frac{2ga^2x}{a^2+k^2}.$$

This problem is one of the 'Theoremata Selecta,' given by John Bernoulli, '*pro conservatione virium vivarum demonstranda et experimentis confirmanda.*'

Comment. Acad. Petrop. 1727, p. 200. *Opera,* Tom. III. p. 127.

(8) A particle A (fig. 225) descends down the curve CKA, drawing a particle B up the curve CLB by means of a string passing over the point C: to determine the velocities of the particles after moving from rest through any corresponding spaces.

Let m, m', be the masses of A, B, respectively; v, v', their velocities after moving through vertical spaces equal to y, y'; then, ds, ds', denoting elements of the two curves,

$$v^2 = 2g(my - m'y')\frac{ds^2}{mds^2 + m'ds'^2}, \quad v'^2 = 2g(my - m'y')\frac{ds'^2}{mds^2 + m'ds'^2}.$$

John Bernoulli; *Act. Erudit. Lips.* 1735. Mai. p. 210; *Opera,* Tom. III. p. 257. Hermann; *Mémoires de St. Pétersbourg,* Tom. II. D'Alembert; *Traité de Dynamique,* p. 123; Seconde Edition.

(9) Two equal spheres, starting simultaneously from rest, descend down two equally inclined planes, the one plane quite smooth, the other perfectly rough: to find the ratio of the vis viva of the former to that of the latter sphere at the end of any time.

The required ratio is $\frac{7}{5}$.

(10) A uniform straight plank rests with its middle point upon a rough horizontal cylinder, their directions being perpendicular to each other: supposing the plank to be slightly displaced, so as to remain always in contact with the cylinder without sliding, to determine the period of an oscillation.

If $2a$ = the length of the plank, and r = the radius of the circle, the time of an oscillation is equal to

$$\frac{\pi a}{(3gr)^{\frac{1}{2}}}.$$

(11) Two equal weights P, P, are tied to the ends of a fine string which passes over two pullies without mass in a horizontal line: supposing a weight W, less than $2P$, to be fixed to the middle point of the horizontal portion of the string, to determine how far it will descend.

If a = the distance between the two pullies, W will fall through a space equal to

$$\frac{2PWa}{4P^2 - W^2}.$$

(12) A solid cylinder is freely moveable about its axis, which is fixed horizontally, and weights W, W', are hung at the ends of a string wound round it and attached to it at some point so as to prevent slipping: after W' has descended from rest for t seconds, it is suddenly cut off, and the system comes to rest in t seconds more: to find the weight of the cylinder.

The weight of the cylinder is equal to

$$\frac{4W^2}{W' - 2W}.$$

(13) A thin uniform smooth tube is balancing horizontally about its middle point, which is fixed: a uniform rod, such as just to fit the bore of the tube, is placed end to end in a line with the tube, and then shot into it with such a horizontal velocity that its middle point shall only just reach that of the tube: supposing the velocity of projection to be known, to find the angular velocity of the tube and rod at the moment of the coincidence of their middle points.

If v be the velocity of the rod's projection, m the mass of the rod, m' that of the tube, $2a$, $2a'$, their respective lengths, and ω the required angular velocity; then

$$\omega^2 = \frac{3mv^2}{ma^2 + m'a'^2}.$$

(14) A circular wire ring, carrying a small bead, lies on a smooth horizontal table: one end of an elastic thread, the natural length of which is less than the diameter of the ring, is attached to the bead and the other to a point in the wire: the

bead is placed initially so that the thread coincides very nearly with a diameter of the ring: to find the vis viva of the system when the string has contracted to its natural length.

If c be the diameter of the ring, a the natural length of the thread, and μ the modulus of elasticity, the required vis viva is equal to

$$\frac{\mu}{a}\cdot(c-a)^2.$$

SECT. 2. *Vis Viva and the Conservation of the Motion of the Centre of Gravity.*

The Principle of the Conservation of the Motion of the Centre of Gravity, under its most general form, asserts that, *the motion of the centre of gravity of a free system of bodies, disposed relatively to each other in any conceivable manner, is the same as if the bodies were all united at their centre of gravity, and each of them were animated by the same accelerating forces as in its actual state.* The discovery of the Principle is due to Newton[1], by whom it received a demonstration in the particular case where the system is subject to no external force, when the centre of gravity will either remain at rest or move in a straight line with a uniform velocity. D'Alembert[2] afterwards extended the Principle to the case where each body is supposed to be solicited by a constant accelerating force, acting in parallel lines, or directed towards a fixed point and varying as the distance. Finally, Lagrange[3] expressed the Principle under its most general form for every law of force to which the bodies can be subject.

(1) A smooth groove KAL (fig. 226) is carved in a vertical plane in the body $KBCL$, which is placed upon a smooth horizontal plane, along which it is able to slide freely: to find the form of the groove in order that a particle, placed within it, may oscillate in it tautochronously, the time of an oscillation being given.

[1] *Principia; Axiomata sive Leges Motus*, Cor. 4.
[2] *Traité de Dynamique, Seconde Partie*, Chap. II.
[3] *Mécanique Analytique*, Tom. I. p. 257, &c.

Let P be the place of the particle in the groove at any time; draw PN vertically to meet the horizontal plane at N, which will lie in the line OE formed by the intersection of a vertical plane through the groove with the horizontal plane. Let A be the lowest point of the groove; draw AM horizontally, AA' vertically. Let O be a fixed point in OE; let $OA'=x'$, $ON=x_1$, $PN=y_1$, $AM=x$, $PM=y$; let k_1, k, be the initial values of y_1, y; let m = the mass of the particle, m' = the mass of the body.

Then, by the Principle of the Conservation of the Motion of the Centre of Gravity, since no forces act upon the particle and body parallel to OE,

$$m'\frac{dx'}{dt}+m\frac{dx_1}{dt}=0 \text{..................(1)}.$$

Also, by the Principle of the Conservation of Vis Viva,

$$m'\frac{dx'^2}{dt^2}+m\left(\frac{dx_1^2}{dt^2}+\frac{dy_1^2}{dt^2}\right)=2mg\,(k_1-y_1) \text{.........(2)}.$$

But, from the geometry, it is evident that

$$\frac{dx_1}{dt}=\frac{dx'}{dt}+\frac{dx}{dt} \text{........................(3)},$$

and $$k_1-y_1=k-y,\qquad \frac{dy_1}{dt}=\frac{dy}{dt} \text{..................(4)}.$$

From (1) and (3) we have

$$\frac{dx_1}{dt}=\frac{m'}{m+m'}\frac{dx}{dt},\qquad \frac{dx'}{dt}=-\frac{m}{m+m'}\frac{dx}{dt} \text{.........(5)}.$$

Hence, from (2), (4), (5), we see that

$$\frac{m'}{m+m'}\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}=2g\,(k-y)\,;$$

and therefore, if τ denote the time of a semi-oscillation,

$$\tau=-\frac{1}{(2g)^{\frac{1}{2}}}\int_k^0\frac{\left(\frac{m'}{m+m'}\frac{dx^2}{dy^2}+1\right)^{\frac{1}{2}}}{(k-y)^{\frac{1}{2}}}dy \text{............(6)}.$$

This value of τ must be independent of k in order that the particle may oscillate tautochronously, and therefore we must

have, it being necessary that the coefficient of dy be of -1 dimensions in y and k,

$$\left(\frac{m'}{m+m'}\frac{dx^2}{dy^2}+1\right)^{\frac{1}{2}}=\frac{a^{\frac{1}{2}}}{y^{\frac{1}{2}}}\text{......................}(7),$$

where a is a constant quantity: hence

$$\frac{dx}{dy}=\left(\frac{m+m'}{m'}\right)^{\frac{1}{2}}\left(\frac{a-y}{y}\right)^{\frac{1}{2}},$$

and therefore, by integration,

$$x=\left(\frac{m+m'}{m'}\right)^{\frac{1}{2}}\left\{(ay-y^2)^{\frac{1}{2}}+\tfrac{1}{2}a\operatorname{vers}^{-1}\frac{2y}{a}\right\}\text{..........}(8).$$

But, from (6) and (7),

$$\tau=-\frac{1}{(2g)^{\frac{1}{2}}}\int_k^0\frac{a^{\frac{1}{2}}dy}{(ky-y^2)^{\frac{1}{2}}}=\frac{\pi a^{\frac{1}{2}}}{(2g)^{\frac{1}{2}}},\quad a=\frac{2g\tau^2}{\pi^2},$$

and therefore from (8) we get, for the equation to the groove,

$$x=\left(\frac{m+m'}{m'}\right)^{\frac{1}{2}}\left\{\left(\frac{2g\tau^2}{\pi^2}y-y^2\right)^{\frac{1}{2}}+\frac{g\tau^2}{\pi^2}\operatorname{vers}^{-1}\frac{\pi^2 y}{g\tau^2}\right\}.$$

Clairaut; *Mémoires de l'Académie des Sciences de Paris*, 1742, p. 41. Euler; *Opuscula, de motu corporum tubis mobilibus inclusorum*, p. 48.

(2) In a smooth circular tube are placed two equal particles, which are connected together by an elastic string, the natural length of which is two-thirds of the length of the circumference: the string is stretched until the particles are in contact and the tube is placed upon a smooth horizontal table and left to itself: to determine the ratio of the kinetic energy of the two particles to the work done in stretching the string, when the string resumes its natural length.

Let m be the mass of either particle, m' that of the tube. Let Ox be the line of motion of the centre of the tube, O being a fixed point: let C be the position of the centre of the tube at the end of any time t, P being the corresponding position of either particle. Let $OC=x$, $\angle PCx=\theta$, $CP=a$. Then, by

the Principle of the Conservation of the Motion of the Centre of Gravity,

$$2m\frac{d}{dt}(x + a\cos\theta) + m'\frac{dx}{dt} = 0,$$

whence

$$\frac{\dfrac{dx}{dt}}{\dfrac{d\theta}{dt}} = \frac{2ma\sin\theta}{2m+m'} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Again, by the Principle of the Conservation of Vis Viva,

$$2m\left\{\frac{d}{dt}(x + a\cos\theta)\right\}^2 + 2m\left\{\frac{d}{dt}(a\sin\theta)\right\}^2 + m'\frac{dx^2}{dt^2} = 4a\int_\theta^\pi T d\theta.$$

The kinetic energy of the two particles, that is, half their vis viva, is equal to

$$m\left\{\frac{d}{dt}(x + a\cos\theta)\right\}^2 + m\left(\frac{d}{dt}a\sin\theta\right)^2.$$

Hence the ratio of the kinetic energy of the two particles to the work done in stretching the string, viz. $2a\int_\theta^\pi T d\theta$, is equal to

$$\frac{2m\left\{\dfrac{d}{dt}(x + a\cos\theta)\right\}^2 + 2m\left(\dfrac{d}{dt}a\sin\theta\right)^2}{2m\left\{\dfrac{d}{dt}(x + a\cos\theta)\right\}^2 + 2m\left(\dfrac{d}{dt}a\sin\theta\right)^2 + m'\dfrac{dx^2}{dt^2}},$$

which, by (1), is equal to

$$\frac{2m\,.\,(am'\sin\theta)^2 + 2m\,.\,\{a\cos\theta\,(2m+m')\}^2}{2m\,(am'\sin\theta)^2 + 2m\,.\,\{a\cos\theta\,(2m+m')\} + m'\,.\,(2ma\sin\theta)^2},$$

and therefore, since $\theta = \dfrac{2\pi}{3}$, is equal to

$$\frac{2\,(m^2 + mm' + m'^2)}{(2m+m')\,(2m'+m)}.$$

(3) A rigid quiescent wire, in the form of a semicircle, is suspended from its ends by little rings, moveable along a hori-

zontal rod: if a bead, moveable along the wire, be placed at one of its higher ends, to find the velocity of the bead, relatively to the wire, at any point of its descent.

If m, μ, be the respective masses of the wire and bead, a the radius of the circle, θ the inclination of the bead's distance from the centre of the circle to the horizon, and v the required velocity,

$$v^2 = \frac{2ga\,(m+\mu)\sin\theta}{m+\mu\cos^2\theta}.$$

(4) To the highest extremity of a uniform rod is attached a ring, moveable along a smooth fixed horizontal rod: to the former rod, resting initially in a vertical position, an angular velocity is communicated in the vertical plane containing the fixed rod: to find its angular velocity in any subsequent position.

Let $2a$ be the length of the moveable rod, ω its initial angular velocity, ω' its angular velocity when inclined to the vertical at any angle θ: then

$$\omega'^2 = \frac{a\omega^2 - 12g\sin^2\frac{\phi}{2}}{a\,(1+3\sin^2\phi)}.$$

(5) A thin spherical shell, the radius of which is a, rests upon a smooth horizontal plane: a particle, of the same mass as the shell, is placed at the lowest point of its internal surface, which is smooth: to determine to what height the particle will ascend, supposing the shell to be projected with a horizontal velocity $2\,(ga)^{\frac{1}{2}}$.

The particle will ascend just as high as the centre of the shell, and then descend.

(6) A narrow tube, in the form of a common helix, is wound round an upright cylinder, initially at rest, which is pierced by two smooth fixed rods, parallel to each other and horizontal: supposing a molecule to be placed within the tube, at a point of which the distance from the axis of the cylinder is parallel to the rods, prove that, m, m', being the masses of the molecule and cylinder, the velocities which the cylinder has acquired, at

the successive arrivals of the molecule at points most distant from the plane in which the axis of the cylinder moves, will have their greatest values when the inclination of the helix to the horizon is equal to

$$\tan^{-1}\left(\frac{m'}{m+m'}\right)^{\frac{1}{2}}.$$

SECT. 3. *Vis Viva and the Conservation of Areas.*

The Principle of the Conservation of Areas asserts, that *if a system of particles be subject only to mutual actions, the sum of the products of the mass of every particle into the projection (on any proposed plane) of the area described by its radius vector round any assigned point, is proportional to the time.* The same principle holds good also if the system be subject to external forces, provided that they be such that the algebraical sum of their moments about a line through the assigned point at right angles to the proposed plane be zero. This principle, which is in fact a generalization of Newton's theorem respecting the areas described by a single body about a centre of force, was discovered, about the same time, by Euler[1], Daniel Bernoulli[2], and D'Arcy[3]; the enunciation of the Principle given by Euler and Bernoulli being expressed under a form somewhat different from that given by D'Arcy, under which it is now generally expressed. The discovery of the Principle was suggested to these three mathematicians by the consideration of the problem of the motion of several bodies within a tube of given form, moving about a fixed point.

(1) P, Π, (fig. 227), are two material particles attached to an inflexible straight line $PO\Pi$, moveable in a horizontal plane about a fixed point O; the particle Π is fixed to the inflexible line, while the particle P is capable of sliding along it: to determine the path described by P, corresponding to any initial velocities of the particles.

[1] *Opuscula, de motu corporum tubis mobilibus inclusorum*, p. 48, 1746.

[2] *Mémoires de l'Académie des Sciences de Berlin*, 1745, p. 54.

[3] *Mémoires de l'Académie des Sciences de Paris*, 1747, p. 348.

Let OE be an immoveable straight line passing through O; let $PO=r$, $\Pi O=a$, $m=$ the mass of P, $\mu=$ the mass of Π, $\angle POE=\theta$. Then, by the Principle of the Conservation of Areas, since the only force to which the moving system is subject is the reaction of the fixed point O, we have

$$(mr^2+\mu a^2)\frac{d\theta}{dt}=C \quad\text{.....................}\quad (1),$$

where C is some constant quantity.

Again, by the Principle of the Conservation of Vis Viva,

$$m\left(\frac{dr^2}{dt^2}+r^2\frac{d\theta^2}{dt^2}\right)+\mu a^2\frac{d\theta^2}{dt^2}=C',$$

$$m\frac{dr^2}{dt^2}+(mr^2+\mu a^2)\frac{d\theta^2}{dt^2}=C' \quad\text{..................}\quad (2),$$

C' being a constant quantity.

Eliminating dt between (1) and (2), we obtain

$$m\frac{dr^2}{d\theta^2}+mr^2+\mu a^2=\frac{C'}{C^2}(mr^2+\mu a^2)^2,$$

$$m\frac{dr^2}{d\theta^2}=\left\{\frac{C'}{C^2}(mr^2+\mu a^2)-1\right\}(mr^2+\mu a^2),$$

which is the differential equation to P's path.

In order to determine C and C', suppose that b, ω, u, are the initial values of r, $\frac{d\theta}{dt}$, $\frac{dr}{dt}$, respectively. Then, from (1),

$$(mb^2+\mu a^2)\,\omega=C,$$

which determines C; and, from (2),

$$mu^2+(mb^2+\mu a^2)\,\omega^2=C',$$

which determines C'.

Clairaut; *Mém. de l'Acad. des Sciences de Paris*, 1742, p. 22. D'Arcy: *Mém. de l'Acad. des Sciences de Paris*, 1747, p. 351. D'Alembert; *Traité de Dynamique*, p. 104, seconde edit.

(2) A straight rod PQ, (fig. 228), subject to the condition of always passing through a small fixed ring at O, is in motion on a horizontal plane: to determine the path of its centre of gravity G.

At any time t of the motion, let $OG = r$, $\angle GOE = \theta$, OE being a fixed line in the plane. Let m be the mass of an element of the rod at any distance ρ from O, and let μ be the mass of the whole rod.

Then, by the Principle of the Conservation of Areas, the only force which acts on the rod being the reaction of the ring,

$$C = \Sigma\left(m\rho^2 \frac{d\theta}{dt}\right) = \Sigma\,(m\rho^2)\frac{d\theta}{dt} = \mu\,(r^2 + k^2)\frac{d\theta}{dt} \quad\ldots\ldots\ldots\ldots(1),$$

k being the radius of gyration of the rod about its centre of gravity, and C a constant quantity.

Again, by the Principle of the Conservation of Vis Viva, the ring being considered perfectly smooth,

$$C' = \mu\left(\frac{dr^2}{dt^2} + r^2\frac{d\theta^2}{dt^2}\right) + \mu k^2 \frac{d\theta^2}{dt^2}$$

$$= \mu\frac{dr^2}{dt^2} + \mu\,(r^2 + k^2)\frac{d\theta^2}{dt^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

C' being a constant quantity.

Eliminating dt between (1) and (2), we have

$$\frac{dr^2}{d\theta^2} + r^2 + k^2 = \frac{\mu C'}{C^2}(r^2 + k^2)^2,$$

$$\frac{dr^2}{d\theta^2} = \left\{\frac{\mu C'}{C^2}(r^2 + k^2) - 1\right\}(r^2 + k^2) \quad\ldots\ldots\ldots\ldots(3).$$

Suppose that a, u, ω, are the initial values of r, $\frac{dr}{dt}$, $\frac{d\theta}{dt}$, respectively; then, by (1) and (2),

$$C = \mu\,(a^2 + k^2)\,\omega, \qquad C' = \mu u^2 + \mu\,(a^2 + k^2)\,\omega^2:$$

hence the equation (3) becomes

$$\frac{dr^2}{d\theta^2}=\left\{\frac{u^2+(a^2+k^2)\,\omega^2}{(a^2+k^2)^2\,\omega^2}(r^2+k^2)-1\right\}(r^2+k^2)$$

$$=\frac{r^2+k^2}{\omega^2\,(a^2+k^2)^2}\{(r^2+k^2)\,u^2+\omega^2\,(a^2+k^2)\,(r^2-a^2)\},$$

which is the differential equation to the path of G.

Clairaut; *Mémoires de l'Acad. des Sciences de Paris*, 1742, p. 38—41.

(3) Two equal particles P, P, (fig. 229), are attached to the extremities of a rod PP, the middle point O of which is fixed: to determine the motion of the particles corresponding to any initial circumstances, supposing the mass of the rod to be so small that it may be neglected.

Through the point O draw a straight line AOB; with O as a centre and radius equal to OP, describe the two indefinite circular arcs APk, Al, the latter of which is supposed to lie within an assigned plane. Let $OP=a$, $\angle AOP=\phi$, $\angle kAl=\theta$; $m=$ the mass of each of the particles. Then, t denoting the corresponding time, we shall have, by the Principle of the Conservation of Vis Viva, whether the particles be subject to the action of gravity or not,

$$2ma^2\left(\frac{d\phi^2}{dt^2}+\sin^2\phi\,\frac{d\theta^2}{dt^2}\right)=C,$$

or

$$\frac{d\phi^2}{dt^2}+\sin^2\phi\,\frac{d\theta^2}{dt^2}=c\,\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

C, c, being constant quantities.

Again, by the Principle of the Conservation of Areas, we have

$$2ma^2\sin^2\phi\,\frac{d\theta}{dt}=C_1,$$

or

$$\sin^2\phi\,\frac{d\theta}{dt}=c_1\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

C_1, c_1, being constants.

Eliminating $\dfrac{d\theta}{dt}$ between the equations (1) and (2) we get

$$\frac{d\phi^2}{dt^2} + \frac{c_1^2}{\sin^2\phi} = c,$$

and therefore $\qquad \sin\phi \, d\phi = (c \sin^2\phi - c_1^2)^{\frac{1}{2}} \, dt,$

$$(c - c_1^2 - c\cos^2\phi)^{\frac{1}{2}} \, dt = - d\cos\phi :$$

integrating, and adding an arbitrary constant c_2,

$$t + c_2 = \frac{1}{c^{\frac{1}{2}}} \cos^{-1} \frac{c^{\frac{1}{2}} \cos\phi}{(c - c_1^2)^{\frac{1}{2}}} \text{......................(3)},$$

$$\cos\phi = \frac{(c - c_1^2)^{\frac{1}{2}}}{c^{\frac{1}{2}}} \cos\{c \;\; (t + c_2)\} \text{............(4)}.$$

Suppose that when $t = 0$, $\phi = \beta$, $\dfrac{d\theta}{dt} = \omega$, $\dfrac{d\phi}{dt} = \omega'$; then

$$c = \omega'^2 + \omega^2 \sin^2\beta, \text{ from (1)};$$

$$c_1 = \omega \sin^2\beta, \qquad \text{from (2)};$$

and therefore, from (3),

$$c_2 = \frac{1}{(\omega'^2 + \omega^2 \sin^2\beta)^{\frac{1}{2}}} \cos^{-1} \frac{(\omega'^2 + \omega^2\sin^2\beta)^{\frac{1}{2}} \cos\beta}{(\omega'^2 + \omega^2 \sin^2\beta \cos^2\beta)^{\frac{1}{2}}}.$$

The value of $\cos\phi$ being given by (4), we may then, by the aid of (2), get an expression for θ in terms of t.

(4) A spherical shell, the interior radius of which is the n^{th} of the exterior, is filled with fluid the density of which is the same as that of the shell: to compare the space, through which it would roll from rest in a given time down a perfectly rough inclined plane, with that which would be described by a solid sphere of the same size and weight rolling down the same plane.

Let m, m', denote the masses of the shell and fluid respectively; a, a', the exterior and interior radii; k, k', the radii of gyration of the shell and fluid about a diameter of the sphere; α the inclination of the plane to the horizon; θ, θ', the angles through which the shell and fluid have revolved about their common centre of gravity since the beginning of the motion; x the space described by the centre of the sphere.

Then, by the Principle of the Conservation of Vis Viva,

$$m\left(\frac{dx^2}{dt^2}+k^2\frac{d\theta^2}{dt^2}\right)+m'\left(\frac{dx^2}{dt^2}+k'^2\frac{d\theta'^2}{dt^2}\right)=C+2(m+m')\,gx\sin\alpha:$$

but, since the resultant of all the forces which act on the fluid-sphere passes through its centre of gravity, we have, by the Principle of the Conservation of Areas, the fluid having no initial motion,

$$m'k'^2\frac{d\theta'}{dt}=0:$$

hence, from these two equations,

$$m\left(\frac{dx^2}{dt^2}+k^2\frac{d\theta^2}{dt^2}\right)+m'\frac{dx^2}{dt^2}=C+2(m+m')\,gx\sin\alpha:$$

but, since the shell rolls, we have $x=a\theta$: hence, putting $m+m'=\mu$,

$$(\mu a^2+mk^2)\frac{dx^2}{dt^2}=Ca^2+2\mu a^2gx\sin\alpha:$$

differentiating with respect to t, and dividing by $2\dfrac{dx}{dt}$,

$$(\mu a^2+mk^2)\frac{d^2x}{dt^2}=\mu a^2g\sin\alpha\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Now, $\quad mk^2=\frac{2}{5}\mu a^2-\frac{2}{5}m'a'^2=\frac{2}{5}(\mu a^2-m'a'^2):$

but $m'=\dfrac{\mu}{n^3}$ and $a'=\dfrac{a}{n}$; hence

$$mk^2=\tfrac{2}{5}\mu a^2\frac{n^5-1}{n^5}.$$

Substituting this value of mk^2 in (1), we obtain

$$\left(1+\tfrac{2}{5}\frac{n^5-1}{n^5}\right)\frac{d^2x}{dt^2}=g\sin\alpha,$$

$$(7n^5-2)\frac{d^2x}{dt^2}=5n^5g\sin\alpha:$$

integrating, and bearing in mind that $x=0$, $\dfrac{dx}{dt}=0$, when $t=0$, we get

$$(7n^5-2)\,x=\tfrac{5}{2}n^5gt^2\sin\alpha.$$

In the case of the solid sphere, we shall obtain in a similar manner,

$$7x' = \tfrac{5}{2} g t^2 \sin \alpha,$$

x' denoting the space through which it has rolled along the plane at the end of the time t.

Hence finally we get $\qquad \dfrac{x}{x'} = \dfrac{7n^5}{7n^5 - 2}.$

If $n = 2$, we have $\qquad \dfrac{x}{x'} = \dfrac{112}{111}.$

Lady's and Gentleman's Diary, 1842, p. 51.

(5) The particle m (fig. 230) is connected with the particles m' and m'' by means of two fine inextensible strings mOm', mOm'', passing through a small smooth ring at O; m, m', m'', all lie on a smooth horizontal plane passing through O: to determine the tensions of the two strings and the motions of the particles, supposing the particles to have received any initial impulses such as, at least at the commencement of the motion, to keep the strings at full stretch.

Draw through O a straight line OA in the plane of the motions; let $Om = r$, $Om' = r'$, $Om'' = r''$, $\angle mOA = \theta$, $\angle m'OA = \theta'$, $\angle m''OA = \theta''$, at any time t; $T' =$ the tension of the string mOm', and $T'' =$ that of the string mOm''.

Then, since the only forces which act upon the particles pass through O, we have, by a formula in the theory of Central Forces,

$$\left.\begin{aligned} \frac{d^2r'}{dt^2} &= r'\frac{d\theta'^2}{dt^2} - \frac{T'}{m'}, \\ \frac{d^2r''}{dt^2} &= r''\frac{d\theta''^2}{dt^2} - \frac{T''}{m''}, \\ \frac{d^2r}{dt^2} &= r\frac{d\theta^2}{dt^2} - \frac{T' + T''}{m}. \end{aligned}\right\} \text{..................(1).}$$

But, the strings being supposed to be kept at full stretch, we have

$$r + r' = c', \quad r + r'' = c'', \text{..................(2)};$$

where c', c'', denote the lengths of the strings mOm', mOm''; and therefore

$$\frac{d^2r}{dt^2}+\frac{d^2r'}{dt^2}=0, \quad \frac{d^2r}{dt^2}+\frac{d^2r''}{dt^2}=0:$$

hence, by the first and third of the formulæ (1),

$$\frac{T'}{m'}+\frac{T'+T''}{m}=r\frac{d\theta^2}{dt^2}+r'\frac{d\theta'^2}{dt^2}\ldots\ldots\ldots\ldots\ldots(3);$$

and, by the second and third,

$$\frac{T''}{m''}+\frac{T'+T''}{m}=r\frac{d\theta^2}{dt^2}+r''\frac{d\theta''^2}{dt^2}\ldots\ldots\ldots\ldots\ldots(4).$$

Again, since the only forces which act upon the three particles pass through the point O, we have, by the Principle of the Conservation of Areas,

$$r^2\frac{d\theta}{dt}=e, \quad r'^2\frac{d\theta'}{dt}=e', \quad r''^2\frac{d\theta''}{dt}=e'', \ldots\ldots\ldots\ldots\ldots(5);$$

where e, e', e'', are invariable quantities: hence, from (3) and (4), we have

$$\frac{T'}{m'}+\frac{T'+T''}{m}=\frac{e^2}{r^3}+\frac{e'^2}{r'^3},$$

$$\frac{T''}{m''}+\frac{T'+T''}{m}=\frac{e^2}{r^3}+\frac{e''^2}{r''^3}:$$

from these two equations we may readily ascertain that

$$\left.\begin{aligned}(m+m'+m'')\frac{T'}{m'}&=\frac{me^2}{r^3}+\frac{(m+m'')\,e'^2}{r'^3}-\frac{m''e''^2}{r''^3},\\(m+m'+m'')\frac{T''}{m''}&=\frac{me^2}{r^3}+\frac{(m+m')\,e''^2}{r''^3}-\frac{m'e'^2}{r'^3},\\(m+m'+m'')\frac{T'+T''}{m}&=\frac{(m'+m'')\,e^2}{r^3}+\frac{m'e'^2}{r'^3}+\frac{m''e''^2}{r''^3},\end{aligned}\right\}\ldots(6),$$

which give the values of the tensions of the two strings mOm', mOm'', and of the double string Om. It is important to observe that these values for the tensions hold good only so long as both the strings are at full stretch; if either of the strings become slack at any epoch of the motion, these formulæ will no longer

apply; this will be evident when it is considered that in obtaining them we made use of the equations (2) which are grounded on the supposition that the strings are at full stretch. The formulæ themselves will indicate the epoch at which their inapplicability may commence by giving a zero value for either T' or T''.

Again, by the Principle of Vis Viva,

$$m\left(r^2\frac{d\theta^2}{dt^2}+\frac{dr^2}{dt^2}\right)+m'\left(r'^2\frac{d\theta'^2}{dt^2}+\frac{dr'^2}{dt^2}\right)+m''\left(r''^2\frac{d\theta''^2}{dt^2}+\frac{dr''^2}{dt^2}\right)=C,$$

where C is some constant quantity: hence, observing that, by the equations (2), $\frac{dr'}{dt}=-\frac{dr}{dt}=\frac{dr''}{dt}$, we get

$$mr^2\frac{d\theta^2}{dt^2}+m'r'^2\frac{d\theta'^2}{dt^2}+m''r''^2\frac{d\theta''^2}{dt^2}+(m+m'+m'')\frac{dr^2}{dt^2}=C,$$

and therefore, by the equations (5),

$$\frac{me^2}{r^2}+\frac{m'e'^2}{r'^2}+\frac{m''e''^2}{r''^2}+(m+m'+m'')\frac{dr^2}{d\theta^2}\frac{e^2}{r^4}=C\text{ ....(7)},$$

and thence, by the equations (2), putting $m+m'+m''=\mu$,

$$\frac{me^2}{r^2}+\frac{m'e'^2}{(c'-r)^2}+\frac{m''e''^2}{(c''-r)^2}+\frac{\mu e^2}{r^4}\frac{dr^2}{d\theta^2}=C\text{ ....(8)},$$

which is the differential equation to the path of m. Similarly may be obtained the differential equations to the paths of m' and m''. These equations will evidently cease to define the paths of the particles if at any time either of the strings become slack, or either T' or T'' become zero. If either of the strings become slack at any time, then we shall have to investigate the motions of the two particles whose connecting string is not slack, the particle which belongs to the loose string moving along for a time without constraint. From the equation (7) it is evident that none of the quantities r, r', r'', can ever become zero; or that the particles, so long as the strings are tight, will none of them arrive at the point O.

Suppose that ω, ω', ω'', a, β, are the initial values of $\frac{d\theta}{dt}$, $\frac{d\theta'}{dt}$, $\frac{d\theta''}{dt}$, r, $\frac{dr}{dt}$; then, from the equations (5),

$$e = a^2\omega, \quad e' = (c' - a)^2\omega', \quad e'' = (c'' - a)^2\omega'';$$

which give the values of e, e', e'': and then, from (8),

$$C = ma^2\omega^2 + m'(c' - a)^2\omega'^2 + m''(c'' - a)^2\omega''^2 + \mu\beta^2.$$

If, instead of attaching two particles m', m'', to m, we had attached any number of them, the problem would have been essentially of no greater difficulty.

Riccati; *Comment. Bonon.* Tom. V. P. I. p. 150; anno 1767.

(6) The bob of a pendulum is a hollow sphere, smooth internally, which is filled with a fluid or with a solid sphere, fixed to the bob, of the same density as the fluid: to find the length of the equivalent simple pendulum, (1) when the cavity is filled with the solid, (2) when it is filled with the fluid, the rod and cavity being supposed to be rigid and without weight.

Let mk^2 = the moment of inertia of the solid or fluid sphere about a diameter, a = the distance of the centre of the sphere from the point of suspension, r = the radius of the sphere, θ = the inclination of the rod to the vertical at any time t, and ω = the angular velocity of the sphere about a diameter parallel to the axis of suspension.

Then, by the Principle of the Conservation of Vis Viva, we have

$$ma^2\frac{d\theta^2}{dt^2} + mk^2\omega^2 = 2mga\cos\theta + C.$$

Now, in the case of the solid sphere, $\omega = \frac{d\theta}{dt}$, and therefore

$$(a^2 + k^2)\frac{d\theta^2}{dt^2} = 2ag(\cos\theta - \cos\beta),$$

β being the value of θ when $\frac{d\theta}{dt} = 0$.

In the case of the fluid sphere, by the Principle of the Conservation of Areas, ω = a constant, and therefore

$$a^2 \frac{d\theta^2}{dt^2} = 2ag (\cos\theta - \cos\beta).$$

Hence, in the former case, the length of the equivalent pendulum is equal to

$$\frac{a^2 + k^2}{a} = a + \frac{2r^2}{5a},$$

and, in the latter, to a.

(7) A rod is fixed at one end, about which it can move freely in any direction: when it is inclined to the horizon without motion at a given angle, a given horizontal velocity is communicated to its other end: to determine the velocity and direction of motion of the free end at the moment when the rod becomes horizontal.

Let $2a$ be the length of the rod, α its initial inclination to the horizon, V the initial velocity of the free end. Let u and v be the horizontal and vertical components of the free end at the moment when the rod becomes horizontal.

Since the angular velocities of the particles of the rod, about a vertical through the fixed end, in its initial and final state are respectively

$$\frac{V}{2a\cos\alpha}, \quad \frac{u}{2a},$$

and since the initial and final distances of any particle from the vertical are as $\cos\alpha$ to 1, we have, by the Principle of the Conservation of Areas,

$$\frac{V}{2a\cos\alpha} \cdot \cos^2\alpha = \frac{u}{2a},$$

whence

$$u = V\cos\alpha \text{..........................(1)}.$$

Again, the initial vis viva, m being the mass of the rod, is equal to

$$\int_0^{2a} \frac{mdr}{2a} \cdot (r\cos\alpha)^2 \cdot \left(\frac{V}{2a\cos\alpha}\right)^2 = \frac{1}{3}\, mV^2$$

Similarly the vis viva, in the final state of the rod, which is due to the horizontal motion of the end of the rod, is equal to $\frac{1}{3}mu^2$.

The vis viva, in the final state of the rod, which is due to the vertical motion of the end of the rod, is equal to

$$\int_0^{2a} \frac{mdr}{2a} \cdot \left(\frac{rv}{2a}\right)^2 = \frac{1}{3}mv^2.$$

Hence, in the final state of the rod, the whole vis viva is equal to

$$\frac{1}{3}m(u^2 + v^2).$$

But, by the Principle of Vis Viva, the vis viva generated during the motion is equal to $2mga \sin \alpha$: hence

$$\frac{1}{3}m(u^2 + v^2) = \frac{1}{3}mV^2 + 2mga \sin \alpha,$$

and therefore, by (1),

$$v^2 = V^2 \sin^2 \alpha + 6ga \sin \alpha \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The equations (1) and (2) determine the velocity and direction of the motion of the free end of the rod when horizontal.

(8) A uniform rod is moving in a horizontal plane, its ends being attached by little rings to a fixed smooth circular wire the diameter of which is three times the length of the rod: the rod lengthens till it is half as long again as at first: to compare the work done by the internal forces of the rod with the work done in starting the rod.

Let m be the mass of the rod, $2a$ its original length; let ω, ω_1, be its initial and final angular velocities; p, p_1, its initial and final distances from the centre of the circle; k, k_1, its initial and final radii of gyration about its centre of gravity.

The initial vis viva of the rod is equal to

$$mp^2\omega^2 + mk^2\omega^2 = m\omega^2\left(8a^2 + \frac{1}{3}a^2\right) = \frac{25}{3}m\omega^2a^2:$$

the final vis viva of the rod is equal to

$$mp_1^2\omega_1^2 + mk_1^2\omega_1^2 = m\omega_1^2\left(\frac{27}{4}a^2 + \frac{3}{4}a^2\right) = \frac{15}{2}m\omega_1^2a^2.$$

But, by the Principle of the Conservation of Areas, since the external forces acting on the rod pass through the centre of the circle,

$$m(p^2+k^2)\,\omega = m(p_1^2+k_1^2)\,\omega_1,$$

$$\frac{25}{3}\omega = \frac{15}{2}\omega_1, \quad \frac{\omega_1}{\omega} = \frac{10}{9}.$$

Now, the work done in any time being equal to half the vis viva generated in that time, the ratio of the work done by the internal forces of the rod to the work done in starting the rod is equal to

$$\frac{\frac{15}{2}ma^2\omega_1^2 - \frac{25}{3}ma^2\omega^2}{\frac{25}{3}ma^2\omega^2}$$

$$= \frac{9\omega_1^2 - 10\omega^2}{10\omega^2}$$

$$= \frac{9 \times 10^2 - 10 \times 9^2}{10 \times 9^2}$$

$$= \frac{10}{9} - 1 = \frac{1}{9}.$$

(9) A particle is projected horizontally along the internal surface of a fixed hemisphere, the axis of which is vertical and vertex downwards: having given the point of projection, to determine the velocity in order that the particle may ascend exactly to the rim of the hemisphere.

If a = the radius of the sphere, and β = the inclination to the vertical of the particle's initial distance from the sphere's centre, the required velocity is equal to

$$\left(\frac{2ag}{\cos\beta}\right)^{\frac{1}{2}}.$$

(10) A smooth surface is generated by the revolution of the curve $x^2y=c^3$ about the axis of y, which is vertical and below the origin: a particle is projected along the surface with a velocity due to the depth below the horizontal plane through the origin: to determine the course subsequently pursued by the particle.

If h be twice the area conserved round the axis of y, the path of the particle will intersect all the meridians of the surface at an angle ϕ given by the equation

$$\sin^2\phi = \frac{h^2}{2c^3g}.$$

(11) Two particles P, P', (fig. 231), are connected together by a rigid rod without inertia, which passes through a small smooth ring at O; the rod rests upon a horizontal plane: supposing any impulse whatever to have been communicated to the particles, to find the paths which they will describe.

Let OE be a fixed line in the plane of the motion; let $OP=r$, $PP'=l$; let a, a', be the initial values of OP, OP'; m, m', the masses of P, P'; let $\angle POE=\theta$; let ω, β, be the initial values of $\frac{d\theta}{dt}$, $\frac{dr}{dt}$. Then the differential equation to P's path will be

$$\{mr^2+m'(l-r)^2\}\{A[mr^2+m'(l-r)^2]-1\}=(m+m')\frac{dr^2}{d\theta^2},$$

where $$A=\frac{(m+m')\beta^2+(ma^2+m'a'^2)\omega^2}{(ma^2+m'a'^2)^2\omega^2};$$

and similarly for the path of P'.

Clairaut; *Mém. Acad. Paris*, 1742, p. 38. D'Arcy; *Ib.* 1747, p. 352.

(12) A cone is revolving round its axis with a given angular velocity, when the length of the axis begins to be diminished uniformly, and the vertical angle to be increased so that the volume of the cone remains unchanged: to determine the angular velocity of the cone at the end of any time and the number of revolutions it will make before the motion ceases.

Let ω = the initial angular velocity, h = the initial length, and v = the velocity of decrease of the axis of the cone; then the angular velocity, at the end of a time t, will be equal to

$$\omega\left(1-\frac{vt}{h}\right),$$

and the required number of revolutions is equal to

$$\frac{h\omega}{4\pi v}.$$

(13) An indefinitely great number of indefinitely thin cylindrical shells, just fitting one another, are revolving with different angular velocities, but in the same angular direction, about their common axis; the angular velocity of each shell being proportional to a positive power of its radius. If the system of shells be suddenly united into a solid cylinder, to find the angular velocity of the cylinder about its axis.

Let ω be the original angular velocity of the outermost shell, and n the said power: then the required angular velocity is equal to

$$\frac{4\omega}{n+4}.$$

Ferrers and Jackson; *Solutions of the Cambridge Problems*, 1848 to 1851; p. 308.

(14) A series of rough concentric spherical shells fit closely one within another: rotations of given magnitude being impressed upon them about given diameters, no extraneous force acting on the system, to find the ultimate state of the motion.

Let Ox, Oy, Oz, be axes of co-ordinates the directions of which are fixed in space, O being the centre of each shell. Let A be the moment of inertia of a shell about a diameter and ω its initial angular velocity, the direction-cosines of its initial axis of rotation being l, m, n. Then ultimately the whole system will revolve as a solid sphere about an axis the equations to which are

$$\frac{x}{\Sigma(A\omega l)}=\frac{y}{\Sigma(A\omega m)}=\frac{z}{\Sigma(A\omega n)}.$$

Griffin; *Solutions of the Examples on the Motion of a Rigid Body*, p. 70.

(15) The line joining the centres of two equal fixed rings is perpendicular to both their planes: two small rings, the masses of which are m, m', which exert on each other at a distance r a mutual attraction $mm'f(r)$, are placed slightly out of a position of stable equilibrium: to find the time of a small oscillation.

If c be the distance between the centres of the fixed rings, the required time is equal to

$$\frac{\pi c^{\frac{1}{2}}}{\{(m+m')f(c)\}^{\frac{1}{2}}}.$$

(16) A uniform rod can turn freely about one extremity: in its initial position it is horizontal, and is projected horizontally with a given angular velocity: to find the least angle it will make with the vertical during its motion.

Let $2a$ be the length of the rod, and ω the initial angular velocity; then θ, the required inclination, may be found from the equation

$$2a\omega^2 \cos\theta = 3g \sin^2\theta.$$

(17) One extremity of a string is attached to a ring (supposed to have no weight) which slides along a vertical axis, and the other is attached to a particle of equal mass which moves on a horizontal plane: the particle is projected in a direction perpendicular to the plane which passes through the string and axis: to find the position of the string when it has revolved through a horizontal angle of 90°.

The string will be horizontal, whatever be the initial velocity of the particle or position of the ring.

(18) A uniform rod is moving on a horizontal table about one extremity, and driving before it a particle of mass equal to its own, which starts from rest from a point indefinitely near to the fixed extremity: to find the inclination of the rod to the direction of motion of the particle, when the particle has described any proposed distance along the rod.

Let k be the radius of gyration of the rod about its fixed extremity, and r the space described by the particle along the rod at any time. Then the required angle is equal to

$$\tan^{-1} \frac{k}{(r^2 + k^2)^{\frac{1}{2}}}.$$

(19) A screw of Archimedes is capable of turning freely round its axis, which is fixed in a vertical position; a heavy particle is placed at the top of the tube and runs down through it: to determine the whole angular velocity communicated to the screw.

Let $h =$ the height of the screw, $a =$ the radius of the cylinder, $\alpha =$ the angle which an indefinitely small element of the screw makes with the vertical, $\omega =$ the required angular velocity; then, m, m', representing the masses of the screw and particle respectively,

$$\omega^2 = \frac{2m'^2 gh \sin^2 \alpha}{a^2 (m + m')(m + m' \cos^2 \alpha)}.$$

(20) A square formed of four similar uniform rods, jointed freely at their extremities, is laid upon a smooth horizontal table, one of its angular points being fixed: if given angular velocities in the plane of the table be communicated to the two sides terminating at the fixed point, to determine the greatest value of the angle contained between them during the subsequent motion.

If ω, ω', be the given angular velocities and θ the required angle,

$$\cos 2\theta = -\frac{5}{6} \frac{(\omega - \omega')^2}{\omega^2 + \omega'^2}.$$

Frost; *Quarterly Journal of Pure and Applied Mathematics*, Vol. 3, p. 82.

(21) Four equal particles, exercising no attraction on each other, move in an ellipse about a centre of force at the centre: at the commencement of the motion they were situated at the extremities of the major and minor axes: if at any time they

become suddenly connected with each other so as to form a rigid system, to find the angular velocity of the system about the centre of the ellipse.

If μ denote the absolute force, and $2a$, $2b$, the major and minor axes, the system will move about the centre with a constant angular velocity equal to

$$\frac{2ab\mu^{\frac{1}{2}}}{a^2+b^2}.$$

O'Brien and Ellis; *Solutions of the Senate-House Problems for* 1844.

(22) AB, AC, are two equal rods, capable of motion about a fixed point A: BC is a rod the length of which is at first equal to the sum of the lengths of the two former rods, and it joins loosely their extremities, so as to be close to A: in this state an angular velocity is given to the rods about a vertical axis through A: the rod BC then contracts, its centre rising vertically until ABC becomes an equilateral triangle: to find the work done in the contraction of BC.

The required work done is equal to three times the work done in giving the original rotation together with the work which would be done in raising BC and one of the rods AB, AC, to the height of the triangle.

CHAPTER XI.

COEXISTENCE OF SMALL OSCILLATIONS.

CONCEIVE that a particle or a system of particles, subject to certain fixed laws of geometrical connection or constraint, be slightly but generally deranged from a position of stable equilibrium, the invariable elements of the geometry being supposed to be free from particular relations. Then, if in the geometrical equations there be n independent variables, the motion of each member of the system may be represented by the composition of n primary oscillations of different periods, the periods of the n oscillations of any two members of the system being coexistent, while their amplitudes will generally be different. When the periods of the n elementary oscillations are commensurable, the whole system will return to its original state after an interval equal to the least common multiple of these periods; as in the case of vibrating cords and vibrating surfaces. This general property of sympathetic vibrations has been entitled the *Principle of the Coexistence of small Oscillations or Vibrations.*

Should the original derangement of the system from its position of equilibrium, instead of being perfectly general, be effected by peculiar adaptation, we may reduce the n elementary oscillations to any smaller number we may please.

If the fixed geometrical elements of the system be not, as we have supposed, free from particular relations, and if it receive a perfectly general derangement, there will as before arise in the system altogether n classes of oscillations; under these circumstances however a peculiarity occasionally presents itself, viz. that, although as we have supposed the original derangement be quite general, yet into the motion of no single member of the system will all the elementary oscillations enter; this case

will then constitute a failure of the Principle of the Coexistence of small Oscillations.

The Principle of Coexistent Oscillations was first laid down by Daniel Bernoulli, who has written several memoirs on the subject in the St. Petersburgh Transactions. See particularly *Nov. Comment. Petrop.* Vol. XIX. p. 281. The student is referred also to Lagrange, *Mécanique Analytique,* Tom. I. p. 347, and to Poisson, *Traité de Mécanique,* Tom. II. p. 426, where he will find investigations of the Principle based on the first principles of Mechanics.

(1) To determine the nature of the oscillations of a particle within the surface of an ellipsoid, one of the axes of which is vertical, in the neighbourhood of the lower extremity of the vertical axis.

Let $2a$, $2b$, denote the lengths of the two horizontal axes of the ellipsoid, $2c$ representing the length of the vertical one; and let the co-ordinate axes be so chosen that, the origin coinciding with the lower end of the vertical axis of the ellipsoid, the axes of x and y may be parallel to the horizontal axes of the ellipsoid, and the axis of z coincide with the ellipsoid's vertical axis.

Then, by D'Alembert's Principle combined with the Principle of Virtual Velocities, we have for the motion of the particle,

$$\frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \left(\frac{d^2z}{dt^2} + g\right)\delta z = 0 \qquad (1),$$

where x, y, z, denote the co-ordinates of the particle at any time t, and δx, δy, δz, the increments of x, y, z, in passing to any point of the surface infinitesimally distant from the position of the particle.

Again, by the equation to the ellipsoid, we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{(c-z)^2}{c^2} = 1;$$

and therefore, neglecting powers of small quantities beyond the second,

$$c - z = c\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\frac{1}{2}} = c\left(1 - \frac{x^2}{2a^2} - \frac{y^2}{2b^2}\right),$$

$$z = \tfrac{1}{2} c \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right),$$

$$\delta z = c \left(\frac{x\delta x}{a^2} + \frac{y\delta y}{b^2}\right):$$

hence, from (1), neglecting the products and powers, beyond the first, of small quantities in the coefficients of δx, δy, we get

$$\frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \frac{cg}{a^2} x\delta x + \frac{cg}{b^2} y\delta y = 0,$$

and therefore $$\left(\frac{d^2x}{dt^2} + \frac{cg}{a^2} x\right)\delta x + \left(\frac{d^2y}{dt^2} + \frac{cg}{b^2} y\right)\delta y = 0.$$

Equating to zero the coefficients of δx, δy, which are independent of each other, we get

$$\frac{d^2x}{dt^2} + \frac{cg}{a^2} x = 0 \qquad (2),$$

$$\frac{d^2y}{dt^2} + \frac{cg}{b^2} y = 0 \qquad (3).$$

The integral of the equation (2) is

$$x = \beta \sin\left\{\frac{(cg)^{\frac{1}{2}}}{a} t + \epsilon\right\},$$

and that of (3) is

$$y = \gamma \sin\left\{\frac{(cg)^{\frac{1}{2}}}{b} t + \zeta\right\};$$

where β, γ, ϵ, ζ are arbitrary constants, which may be determined from the initial values of x, y, $\frac{dx}{dt}$, $\frac{dy}{dt}$. It may be observed that the oscillation of the particle depends upon two simple oscillations, of which $\frac{\pi a}{(cg)^{\frac{1}{2}}}$, $\frac{\pi b}{(cg)^{\frac{1}{2}}}$, are the periods; the number of independent simple oscillations being the same as the number of independent variables in the geometrical equation to which the position of the particle is subject.

Poisson; *Traité de Mécanique*, Tom. II. p. 439.

(2) A uniform rod AB (fig. 232), which is connected by a string OA with a fixed point O, having been slightly displaced from its position of equilibrium in a vertical plane through O; to investigate the nature of its small oscillations.

Draw vertically the indefinite straight line Ox; take P any point in AB, draw PM at right angles to Ox, and produce BA to meet Ox at C. Let $AB=2a$, $OM=x$, $PM=y$, $AP=s$, $OA=l$, $\angle AOx=\theta$, $\angle BCx=\phi$.

Then for the motion of the rod we have, by D'Alembert's Principle combined with the Principle of Virtual Velocities,

$$\int_0^{2a}\left\{ds\left(\frac{d^2x}{dt^2}-g\right)\delta x\right\}+\int_0^{2a}\left\{ds\,\frac{d^2y}{dt^2}\,\delta y\right\}=0\ldots\ldots\ldots\ldots(1),$$

where dx, dy, denote the small spaces described by the element ds of the rod in the time dt, parallel to the co-ordinate axes; δx, δy, denoting the resolved parts of its virtual velocity.

Now, from the geometry, we have

$$x=l\cos\theta+s\cos\phi,\quad y=l\sin\theta+s\sin\phi;$$

and therefore, our object being to transform the equation (1) into an equation involving θ, ϕ, instead of x, y, and to retain small quantities only as far as the first order in the coefficients of $\delta\theta$, $\delta\phi$, of the new equation, we have approximately

$$x=l\,(1-\tfrac{1}{2}\theta^2)+s\,(1-\tfrac{1}{2}\phi^2),\qquad y=l\theta+s\phi,$$

$$\delta x=-\,l\theta\,\delta\theta-s\phi\,\delta\phi,\qquad \delta y=l\delta\theta+s\delta\phi,$$

$$\frac{d^2x}{dt^2}=0,\qquad \frac{d^2y}{dt^2}=l\,\frac{d^2\theta}{dt^2}+s\,\frac{d^2\phi}{dt^2}:$$

hence, substituting these values of x, y,......in the equation (1), we have

$$\int_0^{2a}\{g\,ds\,(l\theta\,\delta\theta+s\phi\,\delta\phi)\}+\int_0^{2a}\left\{ds\left(l\frac{d^2\theta}{dt^2}+s\,\frac{d^2\phi}{dt^2}\right)(l\delta\theta+s\delta\phi)\right\}=0.$$

Equating to zero the coefficient of $\delta\theta$, we get

$$\left(g\theta+l\frac{d^2\theta}{dt^2}\right)\int_0^{2a}ds+\frac{d^2\phi}{dt^2}\int_0^{2a}s\,ds=0,$$

and therefore $$l\,\frac{d^2\theta}{dt^2}+a\,\frac{d^2\phi}{dt^2}+g\theta=0\ldots\ldots\ldots\ldots\ldots\ldots(2);$$

and, equating to zero the coefficient of $\delta\phi$, we obtain

$$\left(g\phi+l\,\frac{d^2\theta}{dt^2}\right)\int_0^{2a}s\,ds+\frac{d^2\phi}{dt^2}\int_0^{2a}s^2\,ds=0,$$

and therefore $$l\,\frac{d^2\theta}{dt^2}+\tfrac{4}{3}\,a\,\frac{d^2\phi}{dt^2}+g\phi=0\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

In order to integrate the equations (2) and (3), assume

$$\theta = a \sin\left\{\left(\frac{g}{\rho}\right)^{\frac{1}{2}} t + \epsilon\right\}, \quad \phi = \beta \sin\left\{\left(\frac{g}{\rho}\right)^{\frac{1}{2}} t + \epsilon\right\}:$$

substituting these expressions for θ in (2), and dividing by $\sin\left\{\left(\frac{g}{\rho}\right)^{\frac{1}{2}} t + \epsilon\right\}$, we get

$$-\frac{l\alpha}{\rho} - \frac{a\beta}{\rho} + \alpha = 0, \quad \text{or } a\beta = \alpha(\rho - l) \dots\dots\dots (4);$$

and substituting in (3), we get, in the same way,

$$-\frac{l\alpha}{\rho} - \frac{4a\beta}{3\rho} + \beta = 0, \quad \text{or } \beta(3\rho - 4a) = 3l\alpha \dots\dots\dots (5).$$

Eliminating α and β between the two equations (4) and (5), we obtain

$$\frac{3\rho - 4a}{a} = \frac{3l}{\rho - l}, \quad \text{or } 3\rho^2 - (4a + 3l)\rho + al = 0.$$

Let the two values of ρ deducible from this quadratic be denoted by m, m': then the motion of the rod will be completely determined by the equations

$$\theta = \alpha \sin\left\{\left(\frac{g}{m}\right)^{\frac{1}{2}} t + \epsilon\right\} + \alpha' \sin\left\{\left(\frac{g}{m'}\right)^{\frac{1}{2}} t + \epsilon'\right\} \dots\dots\dots\dots (6),$$

$$\phi = \beta \sin\left\{\left(\frac{g}{m}\right)^{\frac{1}{2}} t + \epsilon\right\} + \beta' \sin\left\{\left(\frac{g}{m'}\right)^{\frac{1}{2}} t + \epsilon'\right\} \dots\dots\dots\dots (7).$$

In these two equations there are six arbitrary constants, α, α', $\beta, \beta', \epsilon, \epsilon'$; they are not however all of them independent of each other; in fact, by (4), since α and α' correspond respectively to the values m and m' of the quantity ρ, we have

$$\beta = \frac{\alpha}{a}(m - l), \quad \beta' = \frac{\alpha'}{a}(m' - l):$$

hence, from (7), we see that

$$\phi = \frac{\alpha}{a}(m - l) \sin\left\{\left(\frac{g}{m}\right)^{\frac{1}{2}} t + \epsilon\right\} + \frac{\alpha'}{a}(m' - l) \sin\left\{\left(\frac{g}{m'}\right)^{\frac{1}{2}} t + \epsilon'\right\} \dots (8).$$

The four constants $\alpha, \alpha', \epsilon, \epsilon'$, involved in the two equations (7) and (8), may be determined if the initial circumstances of the rod, or the initial values of $\theta, \frac{d\theta}{dt}, \phi, \frac{d\phi}{dt}$, be given.

If $\alpha' = 0$, $\beta' = 0$, then we have

$$\theta = \alpha \sin \left\{ \left(\frac{g}{m}\right)^{\frac{1}{2}} t + \epsilon \right\}, \qquad \phi = \beta \sin \left\{ \left(\frac{g}{m}\right)^{\frac{1}{2}} t + \epsilon \right\},$$

and the oscillations of θ and ϕ will evidently be regular and isochronous, the time of vibration being equal to $\pi \left(\frac{m}{g}\right)^{\frac{1}{2}}$.

If α', β', be not equal to zero, the oscillations of θ and ϕ will be compounded of two simple and isochronous vibrations.

Suppose that at two different times t', t'', the values of θ and of $\frac{d\theta}{dt}$ are the same. This will manifestly be the case if

$$\left(\frac{g}{m}\right)^{\frac{1}{2}} t'' + \epsilon = \left(\frac{g}{m}\right)^{\frac{1}{2}} t' + \epsilon + 2\lambda\pi,$$

and
$$\left(\frac{g}{m'}\right)^{\frac{1}{2}} t'' + \epsilon' = \left(\frac{g}{m'}\right)^{\frac{1}{2}} t' + \epsilon' + 2\lambda'\pi,$$

λ, λ', being any integers: hence

$$2\lambda\pi \left(\frac{m}{g}\right)^{\frac{1}{2}} = t'' - t' = 2\lambda'\pi \left(\frac{m'}{g}\right)^{\frac{1}{2}},$$

and therefore
$$\lambda m^{\frac{1}{2}} = \lambda' m'^{\frac{1}{2}},$$

or m, m', must be to each other as two square numbers.

It will be observed that, in agreement with the general theory of the Coexistence of small Oscillations, the number of independent oscillations of θ and ϕ is two, which is the same as the number of the independent geometrical variables.

The following is another method of solving this problem.

Let G (fig. 233) be the position of the centre of gravity of the rod at any time t; draw GH at right angles to the vertical line Ox; let m = the mass of the rod, mk^2 = its moment of inertia about G, T = the tension of the string AO, $OH = x$, $GH = y$. Then, the rest of the notation being the same as before, we have, for the motion of the rod,

$$m \frac{d^2x}{dt^2} = mg - T \cos\theta \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$m\frac{d^2y}{dt^2} = -T\sin\theta \qquad\ldots\ldots(2),$$

$$mk^2\frac{d^2\phi}{dt^2} = -aT\sin(\phi-\theta) \qquad\ldots\ldots(3).$$

Eliminating T between (1) and (2), and omitting small quantities of higher orders than the first, we have

$$\frac{d^2y}{dt^2} + g\theta = 0 \qquad\ldots\ldots(4);$$

and, eliminating T between (1) and (3), we get in the same manner

$$k^2\frac{d^2\phi}{dt^2} + ag(\phi-\theta) = 0 \qquad\ldots\ldots(5).$$

But $\qquad y = a\sin\phi + l\sin\theta = a\phi + l\theta$, nearly:
hence (4) becomes

$$l\frac{d^2\theta}{dt^2} + a\frac{d^2\phi}{dt^2} + g\theta = 0;$$

and, putting for k^2 its value $\frac{1}{3}a^2$ in (5), we have

$$\frac{1}{3}a\frac{d^2\phi}{dt^2} + g(\phi-\theta) = 0.$$

The last two equations are equivalent to the equations (2) and (3) in the former investigation.

Daniel Bernoulli; *Novi Comment. Petrop.* 1773, Tom. XVIII. p. 247. Euler; *Ib.* p. 268.

(3) A pendulum of any form is firmly attached to a solid circular cylinder as an axis; this axis is supported in a horizontal position at its two extremities, which rest within two hollow circular cylinders, placed horizontally, of the same dimensions: to investigate the small oscillations of the pendulum corresponding to any initial state of displacement and motion, the surfaces in contact being considered perfectly smooth.

Let G (fig. 234) be the centre of gravity of the pendulum and its axis, regarded as one mass, at any time of the motion; let the plane of the paper represent the vertical plane through G, which cuts the axis of the solid cylinder at right angles at the

point C. Let the circular arc MAN be the common intersection of the two concave cylinders with the plane of the paper, when produced to meet it. From O, the centre of the arc MAN, draw OAx vertically; draw GH at right angles to Ox; produce GC to meet Ox in K; join OC, and produce it to a, which will be the point of contact between MAN and the circular section of the solid cylinder made by the plane of the paper. Let $OH=x$, $GH=y$, $AO=a$, $Ca=b$, $\angle AKC=\phi$, $\angle COx=\theta$, $CG=c$; $m=$ the mass of the pendulum and its axis together; $k=$ their radius of gyration about G; $R=$ the reaction of the hollow cylinders against the axis of the pendulum.

Then, for the motion of the pendulum, we have

$$m\frac{d^2x}{dt^2}=mg-R\cos\theta \text{......................} (1),$$

$$m\frac{d^2y}{dt^2}=-R\sin\theta \text{..........................} (2),$$

$$mk^2\frac{d^2\phi}{dt^2}=-Rc\sin(\phi-\theta) \text{..................} (3).$$

From (1) and (2) we get, as far as small quantities of the first order,

$$\frac{d^2y}{dt^2}+g\theta=0 \text{............................} (4);$$

and, from (1) and (3), to the same degree of approximation,

$$k^2\frac{d^2\phi}{dt^2}+cg(\phi-\theta)=0 \text{..................} (5).$$

Now, from the geometry,

$$\begin{aligned} y &= (a-b)\sin\theta+c\sin\phi \\ &= (a-b)\,\theta+c\phi, \qquad \text{nearly}: \end{aligned}$$

hence from (4) we obtain

$$(a-b)\frac{d^2\theta}{dt^2}+c\frac{d^2\phi}{dt^2}+g\theta=0 \text{..................} (6).$$

Assume $\theta=\alpha\sin\left\{\left(\frac{g}{r}\right)^{\frac{1}{2}}t+\epsilon\right\}$, $\quad\phi=\beta\sin\left\{\left(\frac{g}{r}\right)^{\frac{1}{2}}t+\epsilon\right\}$:

then from (5) we may get

$$\beta\,(cr - k^2) = acr \text{.........................} (7),$$

and, from (6), $\quad c\beta = a\,\{r - (a - b)\}$,

and therefore, eliminating a and β,

$$(cr - k^2)\,(r - a + b) = c^2 r.$$

Let the two roots of this quadratic in r be denoted by m and m'; then, for the general values of θ and ϕ, we have

$$\theta = a \sin\left\{\left(\frac{g}{m}\right)^{\frac{1}{2}} t + \epsilon\right\} + a' \sin\left\{\left(\frac{g}{m'}\right)^{\frac{1}{2}} t + \epsilon'\right\} \text{..........} (8),$$

$$\phi = \beta \sin\left\{\left(\frac{g}{m}\right)^{\frac{1}{2}} t + \epsilon\right\} + \beta' \sin\left\{\left(\frac{g}{m'}\right)^{\frac{1}{2}} t + \epsilon'\right\} \text{..........} (9).$$

From (7) we have, β, β', being the values of β, and a, a', those of a, corresponding to the values m, m', of r,

$$\beta = \frac{acm}{cm - k^2}, \qquad \beta' = \frac{a'cm'}{cm' - k^2}:$$

hence, from (9), we have

$$\phi = \frac{acm}{cm - k^2} \sin\left\{\left(\frac{g}{m}\right)^{\frac{1}{2}} t + \epsilon\right\} + \frac{a'cm'}{cm' - k^2} \sin\left\{\left(\frac{g}{m'}\right)^{\frac{1}{2}} t + \epsilon'\right\} \text{......} (10).$$

In the equations (8) and (10) there are four arbitrary constants, a, a', ϵ, ϵ', which may be determined if the initial values of θ, ϕ, $\dfrac{d\theta}{dt}$, $\dfrac{d\phi}{dt}$, be given.

If $a' = 0$, $\beta' = 0$, we have

$$\theta = a \sin\left\{\left(\frac{g}{m}\right)^{\frac{1}{2}} t + \epsilon\right\}, \qquad \phi = \beta \sin\left\{\left(\frac{g}{m}\right)^{\frac{1}{2}} t + \epsilon\right\};$$

and the oscillations of θ and ϕ will be regular and isochronous, the time of vibration being $\pi \left(\dfrac{m}{g}\right)^{\frac{1}{2}}$.

If a' and β' have finite values, the oscillations of θ and ϕ will be compounded of two simple isochronous oscillations.

Euler; *Acta Acad. Petrop.* 1780, P. II. p. 133.

(4) A string $AEFB$ (fig. 235) is attached to two fixed points A, B, in the same horizontal line. From E, F, points so chosen that AE, EF, FB, are all equal, two masses are suspended by strings EM, FN, of different lengths, the masses being equal: supposing the system to be slightly deranged, in its own plane, from its position of equilibrium, to investigate the nature of its small oscillations.

At any time t let EM, FN, make angles ϕ, ϕ', with the vertical. Let AE, EF, BF, make angles $\alpha+\omega$, ω', $\alpha-\omega''$, with the horizon, the values of these angles being α, 0, α, when the system is in the position of equilibrium. Draw Mm, Nn, horizontally, to meet the vertical line Amn at the points m, n. Let $AE=EF=FB=a$, $EM=k$, $FN=k'$, $Am=x$, $Mm=y$, $An=x'$, $Nn=y'$.

By D'Alembert's Principle and the Principle of Virtual Velocities, we have, for the motion of the system,

$$\left(\frac{d^2x}{dt^2}-g\right)\delta x+\left(\frac{d^2x'}{dt^2}-g\right)\delta x'+\frac{d^2y}{dt^2}\delta y+\frac{d^2y'}{dt^2}\delta y'=0\ldots\ldots(1).$$

Our object is now to express x, y, x', y', in terms of ω, ϕ, ϕ', and to substitute their values in this equation. This computation must be effected as far as small quantities of the second order.

By the geometry it is plain that

$$a\cos(\alpha+\omega)+a\cos\omega'+a\cos(\alpha-\omega'')=2a\cos\alpha+a,$$

and therefore

$$\cos\alpha(1-\tfrac{1}{2}\omega^2)-\sin\alpha.\omega+1-\tfrac{1}{2}\omega'^2+\cos\alpha(1-\tfrac{1}{2}\omega''^2)+\sin\alpha.\omega''=2\cos\alpha+1;$$

whence

$$\cos\alpha.\omega^2+2\sin\alpha.\omega+\omega'^2+\cos\alpha.\omega''^2-2\sin\alpha.\omega''=0\ldots\ldots(2).$$

Again, by the geometry,

$$a\sin(\alpha+\omega)=a\sin\omega'+a\sin(\alpha-\omega''),$$

and therefore

$$\sin\alpha(1-\tfrac{1}{2}\omega^2)+\cos\alpha.\omega=\omega'+\sin\alpha(1-\tfrac{1}{2}\omega''^2)-\cos\alpha.\omega'';$$

whence

$$2\cos\alpha.\omega-\sin\alpha.\omega^2=2\omega'-\sin\alpha.\omega''^2-2\cos\alpha.\omega''\ldots\ldots(3).$$

Now, as far as the first order of small quantities, we have, from (2),

$$2 \sin \alpha \,.\, \omega = 2 \sin \alpha \,.\, \omega'',$$

and therefore $\qquad \omega'' = \omega\,;$

and from (3) we have

$$2 \cos \alpha \,.\, \omega = 2\omega' - 2 \cos \alpha \,.\, \omega'' = 2\omega' - 2 \cos \alpha \,.\, \omega,$$

and therefore $\qquad \omega' = 2 \cos \alpha \,.\, \omega.$

Substituting these values of ω', ω'', in the terms of the second order in (2) and (3), we get

$$(2 \cos \alpha + 4 \cos^2 \alpha)\, \omega^2 + 2 \sin \alpha \,.\, \omega - 2 \sin \alpha \,.\, \omega'' = 0,$$

and $\qquad \cos \alpha \,.\, \omega = \omega' - \cos \alpha \,.\, \omega''$:

from the last two equations we see that

$$(2\cos^2\alpha + 4\cos^2\alpha)\,\omega^2 + 2 \sin\alpha\cos\alpha \,.\, \omega - 2 \sin\alpha \,.\, \omega' + 2 \sin\alpha \cos\alpha \,.\, \omega = 0,$$

and therefore

$$\omega' = 2 \cos \alpha \,.\, \omega + \frac{\cos^2 \alpha + 2 \cos^2 \alpha}{\sin \alpha}\, \omega^2 \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Again, as far as our approximation requires,

$$x = a \sin (\alpha + \omega) + k \cos \phi = a \sin \alpha\,(1 - \tfrac{1}{2}\omega^2) + a \cos \alpha \,.\, \omega + k (1 - \tfrac{1}{2}\phi^2),$$

$$\delta x = - a \sin \alpha\, \omega \delta\omega + a \cos \alpha\, \delta\omega - k\phi\, \delta\phi,$$

$$\frac{d^2x}{dt^2} = a \cos \alpha \frac{d^2\omega}{dt^2}\,;$$

$$y = a \cos (\alpha + \omega) + k \sin \phi = a \cos \alpha\,(1 - \tfrac{1}{2}\omega^2) - a \sin \alpha \,.\, \omega + k\phi,$$

$$\delta y = - a \sin \alpha\, \delta\omega + k\, \delta\phi,$$

$$\frac{d^2y}{dt^2} = - a \sin \alpha \frac{d^2\omega}{dt^2} + k \frac{d^2\phi}{dt^2}\,;$$

$$\begin{aligned} x' &= a \sin (\alpha + \omega) - a \sin \omega' + k' \cos \phi' \\ &= a \sin \alpha\,(1 - \tfrac{1}{2}\omega^2) + a \cos \alpha \,.\, \omega - a\omega' + k'(1 - \tfrac{1}{2}\phi'^2) \\ &= a \sin\alpha - a \cos\alpha \,.\, \omega - a \frac{\sin^2\alpha + 2\cos^2\alpha + 4\cos^2\alpha}{2 \sin \alpha}\, \omega^2 + k'(1 - \tfrac{1}{2}\phi'^2), \end{aligned}$$

by (4);

$$\delta x' = - a \cos \alpha\, \delta\omega - a \frac{\sin^2\alpha + 2 \cos^2\alpha + 4 \cos^2\alpha}{\sin \alpha}\, \omega\, \delta\omega - k'\phi'\delta\phi',$$

$$\frac{d^2x'}{dt^2} = - a \cos \alpha \frac{d^2\omega}{dt^2}\,;$$

$$y' = a\cos(\alpha+\omega) + a\cos\omega' + k'\sin\phi'$$
$$= a\cos\alpha\,(1-\tfrac{1}{2}\omega^2) - a\sin\alpha\,.\,\omega + a\,(1-\tfrac{1}{2}\omega'^2) + k'\phi',$$
$$\delta y' = k'\delta\phi' - a\sin\alpha\,\delta\omega,$$
$$\frac{d^2y'}{dt^2} = k'\frac{d^2\phi'}{dt^2} - a\sin\alpha\frac{d^2\omega}{dt^2}\,.$$

Hence, by the equation (1), there is

$$\left.\begin{array}{l} 2a^2\cos^2\alpha\dfrac{d^2\omega}{dt^2}\delta\omega + kg\phi\delta\phi + ga\dfrac{2+4\cos^2\alpha}{\sin\alpha}\omega\delta\omega \\ \quad + gk'\phi'\delta\phi' + \left(a\sin\alpha\dfrac{d^2\omega}{dt^2} - k\dfrac{d^2\phi}{dt^2}\right)(a\sin\alpha\delta\omega - k\delta\phi) \\ \qquad + \left(k'\dfrac{d^2\phi'}{dt^2} - a\sin\alpha\dfrac{d^2\omega}{dt^2}\right)(k'\delta\phi' - a\sin\alpha\,\delta\omega) \end{array}\right\} = 0.$$

Hence, equating to zero the coefficients of the independent quantities $\delta\phi$, $\delta\phi'$, $\delta\omega$, we get

$$k\frac{d^2\phi}{dt^2} - a\sin\alpha\frac{d^2\omega}{dt^2} + g\phi = 0 \;\ldots\ldots\ldots\ldots (5),$$

$$k'\frac{d^2\phi'}{dt^2} - a\sin\alpha\frac{d^2\omega}{dt^2} + g\phi' = 0 \;\ldots\ldots\ldots\ldots (6),$$

$$2a\frac{d^2\omega}{dt^2} - k\sin\alpha\frac{d^2\phi}{dt^2} - k'\sin\alpha\frac{d^2\phi'}{dt^2} + g\frac{2+4\cos^2\alpha}{\sin\alpha}\omega = 0 \ldots\ldots (7).$$

Eliminating $\dfrac{d^2\phi}{dt^2}$ and $\dfrac{d^2\phi'}{dt^2}$ between (5), (6), (7), we get

$$2a\sin\alpha\cos^2\alpha\frac{d^2\omega}{dt^2} + g\{(2+4\cos^2\alpha)\,\omega + \sin^2\alpha\,.\,\phi + \sin^2\alpha\,.\,\phi'\} = 0 \ldots\ldots (8).$$

Let r denote the length of a pendulum, isochronous with one of the elementary oscillations, and assume accordingly

$$\omega = \Omega\sin\left\{\left(\frac{g}{r}\right)^{\frac{1}{2}}t + \epsilon\right\},$$

$$\phi = F\sin\left\{\left(\frac{g}{r}\right)^{\frac{1}{2}}t + \epsilon\right\},$$

$$\phi' = F'\sin\left\{\left(\frac{g}{r}\right)^{\frac{1}{2}}t + \epsilon\right\}.$$

Then, from (5), (6), (8), we have

$$(k-r)\,F = a\sin\alpha\,.\,\Omega,$$
$$(k'-r)\,F' = a\sin\alpha\,.\,\Omega,$$

$$-2a\sin\alpha\cos^2\alpha\frac{1}{r}\Omega+(2+4\cos^2\alpha)\Omega+\sin^2\alpha.F+\sin^2\alpha.F'=0;$$

and, by eliminating the constants F, F', Ω, we have a cubic equation in r,

$$\frac{2\sin\alpha\cos^2\alpha}{r}+\frac{\sin^2\alpha}{r-k}+\frac{\sin^2\alpha}{r-k'}-\frac{2+4\cos^2\alpha}{a}=0\ldots\ldots\ldots(9).$$

Let l, l', l'', be the three roots of this equation; then, for the complete solution of the problem, we have

$$\omega''=\omega=\Omega\left\{\left(\frac{g}{l}\right)^{\frac{1}{2}}t+\epsilon\right\}+\Omega'\sin\left\{\left(\frac{g}{l'}\right)^{\frac{1}{2}}t+\epsilon'\right\}+\Omega''\sin\left\{\left(\frac{g}{l''}\right)^{\frac{1}{2}}t+\epsilon''\right\},$$

$$\omega'=2\cos\alpha.\omega,$$

$$\phi=F_1\sin\left\{\left(\frac{g}{l}\right)^{\frac{1}{2}}t+\epsilon\right\}+F_2\sin\left\{\left(\frac{g}{l'}\right)^{\frac{1}{2}}t+\epsilon'\right\}+F_3\sin\left\{\left(\frac{g}{l''}\right)^{\frac{1}{2}}t+\epsilon''\right\},$$

$$\phi'=F_1'\sin\left\{\left(\frac{g}{l}\right)^{\frac{1}{2}}t+\epsilon\right\}+F_2'\sin\left\{\left(\frac{g}{l'}\right)^{\frac{1}{2}}t+\epsilon'\right\}+F_3'\sin\left\{\left(\frac{g}{l''}\right)^{\frac{1}{2}}t+\epsilon''\right\}.$$

This problem may be solved also in the following manner, which is Euler's method of considering it.

Let P, Q, be the tensions of the strings EM, FN, and m the mass of each of the bodies.

Then, for the motion of the bodies, we have, approximately,

$$m\frac{d^2x}{dt^2}=mg-P\cos\phi=mg-P\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$m\frac{d^2y}{dt^2}=-P\sin\phi=-mg\phi\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$m\frac{d^2x'}{dt^2}=mg-Q\cos\phi'=mg-Q\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

$$m\frac{d^2y'}{dt^2}=-Q\sin\phi'=-mg\phi'\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4);$$

these four equations being true as far as the first order of small quantities.

Let T denote the tension of the string EF; then, since the three tensions acting upon the point E must be in equilibrium, there is

$$\frac{T}{P}=\frac{\sin\{\phi+\frac{1}{2}\pi+\alpha+\omega\}}{\sin\{\frac{1}{2}\pi-(\alpha+\omega)+\frac{1}{2}\pi-\omega'\}}=\frac{\cos(\alpha+\omega+\phi)}{\sin(\alpha+\omega+\omega')}.$$

Similarly, for the tensions at F, we have

$$\frac{Q}{T}=\frac{\sin(\alpha-\omega''-\omega')}{\cos(\alpha-\omega''-\phi')};$$

hence
$$\frac{Q}{P}=\frac{\sin(\alpha-\omega''-\omega')\cos(\alpha+\omega+\phi)}{\cos(\alpha-\omega''-\phi')\sin(\alpha+\omega+\omega')}.$$

Hence, as far as small quantities of the first order,

$$\begin{aligned}&Q\{\sin\alpha+\cos\alpha\,(\omega+\omega')\}\{\cos\alpha+\sin\alpha\,(\omega''+\phi')\}\\ &=P\{\sin\alpha-\cos\alpha\,(\omega'+\omega'')\}\{\cos\alpha-\sin\alpha\,(\omega+\phi)\},\end{aligned}$$

and therefore

$$\left.\begin{aligned}&Q\{\sin\alpha\cos\alpha+\sin^2\alpha\,(\omega''+\phi')+\cos^2\alpha\,(\omega+\omega')\}\\ &=P\{\sin\alpha\cos\alpha-\sin^2\alpha\,(\omega+\phi)-\cos^2\alpha\,(\omega'+\omega'')\}\end{aligned}\right\}\ldots\ldots(5).$$

Now, by the geometry,

$$\cos(\alpha+\omega)+\cos\omega'+\cos(\alpha-\omega'')=2\cos\alpha+1,$$

and therefore, as far as the first order of small quantities,

$$-\sin\alpha\,.\,\omega+\sin\alpha\,.\,\omega''=0,$$

$$\omega''=\omega\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Also, by the geometry,

$$\sin(\alpha+\omega)=\sin\omega'+\sin(\alpha-\omega''),$$

$$\cos\alpha\,.\,\omega=\omega'-\omega''\cos\alpha=\omega'-\omega\cos\alpha,$$

$$\omega'=2\omega\cos\alpha\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

Hence by (5), (6), (7), we have

$$\left.\begin{aligned}&Q\{\sin\alpha\cos\alpha+\sin^2\alpha\,(\omega+\phi')+\cos^2\alpha\,(1+2\cos\alpha)\,\omega\}\\ &=P\{\sin\alpha\cos\alpha-\sin^2\alpha\,(\omega+\phi)-\cos^2\alpha\,(1+2\cos\alpha)\,\omega\}\end{aligned}\right\}\ldots\ldots(8).$$

Eliminating P and Q between (1), (3), and (8), we have, as far as small quantities of the first order,

$$\left(\frac{d^2x}{dt^2}-\frac{d^2x'}{dt^2}\right)\sin\alpha\cos\alpha=-g\{(2+4\cos^2\alpha)\omega+\sin^2\alpha\,.\,\phi+\sin^2\alpha\,.\,\phi'\}\,..(9).$$

But $\quad x=a\sin(\alpha+\omega)+k\cos\phi=a\cos\alpha\,.\,\omega+\ldots,$

$$y=a\cos(\alpha+\omega)+k\sin\phi=-a\sin\alpha\,.\,\omega+k\phi+\ldots,$$

$$x' = a\sin(\alpha+\omega) - a\sin\omega' + k'\cos\phi' = a\cos\alpha\,.\,\omega - a\omega' + \dots$$
$$= -a\cos\alpha\,.\,\omega + \dots,$$

$$y' = a\cos(\alpha+\omega) + a\cos\omega' + k'\sin\phi' = -a\sin\alpha\,.\,\omega + k'\phi' + \dots;$$

hence, by (9),

$$2a\sin\alpha\cos^2\alpha\,\frac{d^2\omega}{dt^2} + g\,\{(2+4\cos^2\alpha)\,\omega + \phi\sin^2\alpha + \phi'\sin^2\alpha\} = 0:$$

and, by (2) and (4),

$$k\frac{d^2\phi}{dt^2} - a\sin\alpha\,\frac{d^2\omega}{dt^2} + g\phi = 0,$$

$$k'\frac{d^2\phi'}{dt^2} - a\sin\alpha\,\frac{d^2\omega}{dt^2} + g\phi' = 0;$$

which are the same three linear equations as (5), (6), (8), in the former solution.

If k be equal to k', the cubic equation (9) of the former solution will degenerate into a quadratic, and the variations of ω, ϕ, ϕ', will no longer be expressible by the composition of the same elementary vibrations. This will be an instance of the failure of the Principle of the Coexistence of small Oscillations.

Euler; *Act. Acad. Petrop.* 1779, P. II. p. 95.

(5) A hollow circular ring is suspended by a point in its circumference, and a particle is placed inside it: they are both made to oscillate through a small extent from their positions of equilibrium, in the plane of the ring: to determine the number and periods of the coexistent oscillations of the system.

If a denote the radius of the ring, and M, m, the masses of the ring and particle respectively, there will be in the system two coexistent oscillations the periods of which are

$$\pi\left(\frac{2a}{g}\right)^{\frac{1}{2}} \text{ and } \pi\left(\frac{a}{g}\right)^{\frac{1}{2}}\cdot\left(\frac{M}{M+m}\right)^{\frac{1}{2}}.$$

(6) A thin hemispherical bowl rocks slightly on a horizontal plane, sufficiently rough to prevent sliding: a particle, the mass of which is equal to that of the bowl, is fixed to one end of a fine string, the length of which is equal to half that of the radius, the other end of the string being attached to the centre,

fixed in relation to the bowl, of the rim of the bowl: to determine the number and the periods of the small oscillations of the system, supposing the motion to be such that all the molecules of the system move parallel to one vertical plane.

There will be two coexistent oscillations, the periods of which are equal to the two values of $\frac{\pi}{\sqrt{\rho}}$; ρ being given by the equation

$$\rho^2 - \frac{23}{4}\cdot\frac{g}{r}\cdot\rho + \frac{3}{2}\cdot\frac{g^2}{r^2} = 0,$$

r denoting the radius of the bowl.

(7) One of the scales of a common balance having been slightly displaced from its position of rest, in a vertical plane passing through the beam; to investigate the nature of the oscillatory motions of the two scales and of the beam, to which the displacement will give rise.

Let O (fig. 236) be the point of suspension of the whole balance, G its centre of gravity, AB the beam, P and Q the scales, which are here supposed to be material points. Draw aOb horizontal, $aA\alpha$, $bB\beta$, vertical. Let $AC = a = BC$, $OC = b$, $OG = c$, $AP = l = BP$, $Mk^2 =$ the moment of inertia of the beam about O, $m =$ the mass of P and of Q supposed to be equal.

Let ϕ be the angle which, at any time t, the beam makes with the horizon; let $\angle PA\alpha = \eta$, $\angle QB\beta = \theta$. Also put

$$\frac{g}{l} = n^2, \quad \frac{b}{l} = h, \quad \frac{Mc + 2mb}{Mk^2} g = p^2, \quad -\frac{mbg}{Mk^2} = q,$$

and let $-\mu_1^2$, $-\mu_2^2$, represent the two roots of the quadratic

$$z^2 + (n^2 + p^2 - 2hq)\,z + n^2p^2 = 0.$$

Then, bearing in mind that, initially,

$$\phi = 0, \qquad \eta = \epsilon, \qquad \theta = 0,$$
$$\frac{d\phi}{dt} = 0, \qquad \frac{d\eta}{dt} = 0, \qquad \frac{d\theta}{dt} = 0,$$

where ϵ is a known constant, we shall obtain for the complete expression of the motions

$$\phi = \frac{2\epsilon h p^2 q^2}{\mu_2^2 - \mu_1^2} (\cos \mu_1 t - \cos \mu_2 t),$$

$$2\eta = \frac{2\epsilon h p^2 q}{\mu_2^2 - \mu_1^2} \left(\frac{\cos \mu_1 t}{\mu_1^2 - p^2} - \frac{\cos \mu_2 t}{\mu_2^2 - p^2}\right) + \epsilon \cos nt,$$

$$2\theta = \frac{2\epsilon h p^2 q}{\mu_2^2 - \mu_1^2} \left(\frac{\cos \mu_1 t}{\mu_1^2 - p^2} - \frac{\cos \mu_2 t}{\mu_2^2 - p^2}\right) - \epsilon \cos nt.$$

(8) One of the scales of a common balance having been slightly displaced from its position of rest, in a vertical plane at right angles to the beam; to investigate the nature of the oscillatory motions of the two scales and of the beam.

Let AB (fig. 237) be the original position of the beam, PQ its position at any time t; p, q, the projections, on the directions AP, BQ, respectively, of the positions of the scales, considered as material points, at the same time. Let $AC = a = BC$, $AP = z = BQ$, $Pp = x$, $Qq = y$, $Mk^2 =$ the moment of inertia of the beam round C, $m =$ the mass of each scale, $l =$ the length of the string by which each scale is suspended. If we put, for simplicity,

$$\frac{g}{l} = n^2, \quad \frac{g}{l}\left(1 + 2\frac{ma^2}{Mk^2}\right) = n'^2,$$

we shall have, for the complete expression of the motions, the initial value of x being c, while those of y, $\frac{dx}{dt}$, $\frac{dy}{dt}$, are all zero,

$$x = c \cos\left(\frac{n' - n}{2} t\right) \cos\left(\frac{n' + n}{2} t\right),$$

$$y = -c \sin\left(\frac{n' - n}{2} t\right) \sin\left(\frac{n' + n}{2} t\right),$$

$$z = \frac{2g}{l} \frac{ma^2}{Mk^2} \frac{c}{n'^2} \sin^2 \frac{n't}{2}.$$

Investigations of the last two problems are given in a paper on the Sympathy of Pendulums, in the *Cambridge Mathematical Journal*, Vol. II. p. 120, by D. F. Gregory and A. Smith.

CHAPTER XII.

IMPULSIVE FORCES.

If two rigid bodies impinge against each other, their motions both of translation and of rotation will generally experience modification, the determination of the nature of which, in the case of bodies of which the positions and motions are assigned at the instant before impact, constitutes the general problem of collision. The process of collision may be divided into two stages of indefinitely small duration: in the former stage, by the force of compression, which we will denote by R, the two points at which the bodies touch each other are constrained to assume equal resolved velocities in the direction of the common normal to their surfaces; in the latter stage, by the force of restitution, if the bodies be not inelastic, an additional reaction eR takes place between them, where e denotes their common elasticity. Let ω_1, ω_2, ω_3, denote the angular velocities of one of the bodies about its principal axes and v_1, v_2, v_3, the components of the velocity of its centre of gravity, at the conclusion of the former stage of the collision; let ω_1', ω_2', ω_3', v_1', v_2', v_3', denote the analogous quantities in relation to the other body. Then, for the expression of the motion of the former body, as modified by the force of compression, we shall have six equations involving, together with known quantities, the symbols ω_1, ω_2, ω_3, v_1, v_2, v_3, R; and in like manner for the latter body we shall have six equations involving ω_1', ω_2', ω_3', v_1', v_2', v_3', R. Thus we shall have in all twelve equations involving thirteen variables. Another equation is supplied by the condition that the points of the two bodies at which their contact takes place shall have an equal resolved velocity in the direction of the common normal. Thus we shall be able to determine completely the modification of the motions of the two bodies due to the force of compression as well as the magnitude

of this force. An additional modification must be applied, in the case of elastic bodies, in consequence of the force of restitution eR, which, from the investigation for the former stage of the collision, has become a known force. If one of the bodies be immoveable, the simplification of the method of investigation which we have described is obvious, the thirteen equations of which we made mention being reduced in this case to seven, and the common normal velocity of the two points of contact being zero. For ample information on this subject the student is referred to Poisson's *Traité de Mécanique*, Tom. II. p. 254, seconde édition.

SECT. 1. *Single Body. Smooth Surfaces. Axes of Rotation, before and after impulsive action, parallel to each other.*

(1) A beam of imperfect elasticity, moving anyhow in a vertical plane, impinges upon a smooth horizontal plane: to determine the initial motion of the beam after impact.

We will commence with supposing the beam to be inelastic; in this case the extremity of the beam which strikes the horizontal plane will continue after impact to slide along it without detaching itself. Let PQ (fig. 238) represent the beam at any time after impact; KL being the section of the horizontal plane made by the vertical plane through PQ; G the centre of gravity of PQ; draw GH at right angles to KL. Let $GH = y$, $QG = a$, $\angle\, GQH = \theta$, $k =$ the radius of gyration about G, $m =$ the mass of the beam; ω, ω', the angular velocities of the beam about G estimated in the direction of the arrows in the figure, just before and just after impact; u, v, the vertical velocities of G estimated downwards just before and just after impact; B the blow of impact.

Then, $\omega' - \omega$ being the angular velocity communicated by the blow, we shall have, if β be the value of θ at the instant of impact,

$$\omega' - \omega = \frac{Ba \cos \beta}{mk^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

and, $u - v$ being the velocity of G which is destroyed by the blow,

$$u - v = \frac{B}{m} \text{ .... (2).}$$

Again, by the geometry, we have

$$y = a \sin \theta ;$$

and therefore, t denoting the interval between the instant of impact and the arrival of the beam at the position represented in the figure,

$$\frac{dy}{dt} = a \cos \theta \frac{d\theta}{dt} :$$

hence, $-v$, $-\omega'$, being the values of $\frac{dy}{dt}$, $\frac{d\theta}{dt}$, at the instant after impact, we have

$$v = a \cos \beta \,.\, \omega' \text{ .... (3).}$$

From (1), (2), (3), we get

$$u - \frac{B}{m} = a \cos \beta \left(\omega + \frac{Ba \cos \beta}{mk^2}\right),$$

$$\frac{B}{m}\left(1 + \frac{a^2}{k^2} \cos^2 \beta\right) = u - a\omega \cos \beta,$$

$$B = mk^2 \frac{u - a\omega \cos \beta}{a^2 \cos^2 \beta + k^2} \text{ .... (4).}$$

Hence, from (1),

$$\omega' = \omega + a \cos \beta \frac{u - a\omega \cos \beta}{a^2 \cos^2 \beta + k^2} = \frac{au \cos \beta + k^2 \omega}{a^2 \cos^2 \beta + k^2};$$

and, from (2),

$$v = u - k^2 \frac{u - a\omega \cos \beta}{a^2 \cos^2 \beta + k^2} = a \cos \beta \frac{au \cos \beta + k^2 \omega}{a^2 \cos^2 \beta + k^2}.$$

Next let us suppose the beam to be imperfectly elastic, its elasticity being denoted by e; in this case the value of B given in (4) must be increased in the ratio of $1 + e$ to 1; and therefore, instead of the equation (4), we have

$$B = (1 + e)\, mk^2 \frac{u - a\omega \cos \beta}{a^2 \cos^2 \beta + k^2},$$

which determines the magnitude of the blow of impact: substituting this value of B in (1), we get

$$\omega' = \omega + (1+e)\,a\cos\beta\,\frac{u - a\omega\cos\beta}{a^2\cos^2\beta + k^2}$$

$$= \frac{(k^2 - a^2 e\cos^2\beta)\,\omega + (1+e)\,au\cos\beta}{a^2\cos^2\beta + k^2};$$

and, substituting in (2),

$$v = u - (1+e)\,k^2\,\frac{u - a\omega\cos\beta}{a^2\cos^2\beta + k^2}$$

$$= \frac{a^2 u\cos^2\beta - ek^2 u + (1+e)\,k^2 a\omega\cos\beta}{a^2\cos^2\beta + k^2}.$$

The velocity of G parallel to the plane KL will be the same before and after impact. The end B of the beam will evidently after impact detach itself from the horizontal plane, since v is less and ω' greater when e has a finite value than when it is equal to zero.

(2) The edge BC, (fig. 239), of a vertical lamina is placed on a line Oy of greatest slope on an inclined plane: after sliding a given distance along the plane, it impinges against a small obstacle at C: to determine the impulsive reaction of the obstacle and the motion of the lamina immediately after impact.

Let G be the centre of gravity of the lamina; draw GH at right angles to Oy; Ox parallel to HG. Let $GH = a$, $CH = b$, m = the mass of the lamina, k = the radius of gyration about G; c = the velocity of G immediately before impact. We will commence with supposing the lamina to be perfectly inelastic; in this case the point C of the lamina will remain during impact in contact with the obstacle, the lamina rotating about this point. Let R, S, denote the impulsive reactions of the obstacle parallel to Ox, yO; and let u, v, denote the velocities of G parallel to Ox, Oy, on the completion of the impact; also let ω represent the angular velocity of rotation about G at the same instant.

Then we have, for the motion of translation,

$$mu = R \quad\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$mv = mc - S \quad\ldots\ldots\ldots\ldots\ldots(2);$$

and, for the motion of rotation,

$$mk^2\omega = Sa - Rb \text{.....................}(3).$$

Again, the velocity of the point C of the lamina, estimated parallel to Ox, will be

$$u - \omega CG \cos \angle GCH \text{ or } u - b\omega,$$

the former term of this expression arising from the motion of G, and the latter from the rotation of the lamina about G.

Also the velocity of the point C, parallel to Oy, will be

$$v - \omega \,.\, CG \sin \angle GCH \text{ or } v - a\omega,$$

the former term being due to the motion of G, and the latter to the rotation about G. But the point C of the lamina, which is perfectly inelastic, remains at rest during the impact: hence, evidently,

$$u - b\omega = 0 \text{......}(4); \qquad v - a\omega = 0 \text{..............}(5).$$

From (1) and (4) we have

$$R = mb\omega \text{...........................}(6),$$

and from (2), (5),

$$S = m(c - a\omega) \text{.......................}(7):$$

substituting these values of R and S in (3), we obtain

$$k^2\omega = ac - a^2\omega - b^2\omega,$$

and therefore,

$$\omega = \frac{ac}{a^2 + b^2 + k^2}, \quad u = \frac{abc}{a^2 + b^2 + k^2}, \quad v = \frac{a^2c}{a^2 + b^2 + k^2}:$$

hence also, from (6),

$$R = \frac{mabc}{a^2 + b^2 + k^2};$$

and, from (7),

$$S = m\left(c - \frac{a^2c}{a^2 + b^2 + k^2}\right) = \frac{mc(b^2 + k^2)}{a^2 + b^2 + k^2}.$$

If the lamina be supposed to be elastic, we must increase these values of R and S in the ratio of $1 + e$ to 1, e denoting the elasticity. Hence

$$R = \frac{m(1 + e)abc}{a^2 + b^2 + k^2}, \quad S = \frac{mc(1 + e)(b^2 + k^2)}{a^2 + b^2 + k^2};$$

and therefore, from (1), (2), (3),

$$u = \frac{(1+e)\,abc}{a^2+b^2+k^2}, \quad v = c\,\frac{a^2 - e\,(b^2+k^2)}{a^2+b^2+k^2}, \quad \omega = \frac{(1+e)\,ac}{a^2+b^2+k^2}.$$

(3) A beam AB (fig. 240) is originally in a vertical position, hanging from the point O along the line Oy: supposing the extremity A of the beam to be projected from O with a given velocity along a smooth horizontal groove Ox, to determine the motion of the beam.

Let AB be the position of the beam after a time t from the projection of A, G its centre of gravity; draw GH at right angles to Ox: let $OH = x$, $GH = y$, $\angle OAG = \theta$, $AG = a$; $m =$ the mass of the beam, $k =$ its radius of gyration about G.

Then, for the motion of the beam at any time after the projection, we have, by the Principle of the Conservation of Vis Viva,

$$m\left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + k^2\frac{d\theta^2}{dt^2}\right) = C + 2mgy \;\ldots\ldots\ldots\ldots (1);$$

and, by the Principle of the Conservation of the Motion of the Centre of Gravity,

$$\frac{dx}{dt} = C' \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From (1) and (2), we have

$$m\left(\frac{dy^2}{dt^2} + k^2\frac{d\theta^2}{dt^2}\right) = C'' + 2mgy:$$

but, from the geometry, we see that $y = a\sin\theta$: hence

$$m\,(a^2\cos^2\theta + k^2)\,\frac{d\theta^2}{dt^2} = C'' + 2mga\sin\theta \ldots\ldots\ldots (3).$$

Let B denote the blow of projection which is impressed upon the end A of the beam; u the velocity of A's projection, and ω the angular velocity of the beam about G immediately after the blow. Also let v be the velocity communicated to G by the blow.

Then we shall have

$$mv = B, \qquad mk^2\omega = Ba \;\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Again, the velocity of A along Ox will be equal to

$$v + a\omega,$$

the former term being due to the motion of G, and the latter to the rotation about G; but the velocity of A is also u by the hypothesis: hence

$$v + a\omega = u:$$

but, from the equations (4), we have $k^2\omega = av$: we obtain, therefore,

$$u = v + \frac{a^2}{k^2}v, \quad v = \frac{k^2 u}{a^2 + k^2}, \quad \omega = \frac{au}{a^2 + k^2}.$$

Now, $\theta = \frac{1}{2}\pi$, $\frac{d\theta}{dt} = \omega$, simultaneously: hence, from (3),

$$mk^2\omega^2 = C'' + 2mga;$$

and therefore

$$(a^2 \cos^2\theta + k^2)\frac{d\theta^2}{dt^2} = k^2\omega^2 - 2ga(1 - \sin\theta)$$

$$= \frac{k^2 a^2 u^2}{(a^2 + k^2)^2} - 2ga(1 - \sin\theta) \;\ldots\ldots\; (5).$$

Also, the value of $\frac{dx}{dt}$ being constant, as is shewn by the equation (2),

$$\frac{dx}{dt} = v = \frac{k^2 u}{a^2 + k^2}, \quad x = \frac{k^2 ut}{a^2 + k^2};$$

which gives the velocity of G parallel to Ox, and the value of x at any time of the motion: the angular velocity of the beam for every position is given by (5).

(4) An inelastic beam AB, (fig. 241), capable of moving in a vertical plane about a fixed horizontal axis through A, falls from a given position, and impinges against an immoveable obstacle at C: to determine the shock on the axis.

Let G be the centre of gravity of the beam; AM a horizontal line through A; let m = the mass of the beam; $\angle GAM = \theta$ at any time t of the descent; α = the initial value of θ; k = the radius of gyration about G; $AG = a$.

Then, for the motion of the beam in its fall,

$$m(a^2+k^2)\frac{d^2\theta}{dt^2}=mga\cos\theta;$$

multiplying by $2\frac{d\theta}{dt}$ and integrating,

$$m(a^2+k^2)\frac{d\theta^2}{dt^2}=2mag\sin\theta+C;$$

but $\theta=\alpha$ when $\frac{d\theta}{dt}=0$; hence

$$0=2mag\sin\alpha+C,$$

and therefore $(a^2+k^2)\frac{d\theta^2}{dt^2}=2ag(\sin\theta-\sin\alpha)$.

Let $\angle CAM=\beta$ and, at the instant before impact, let $\frac{d\theta}{dt}=\omega$: then

$$(a^2+k^2)\,\omega^2=2ag(\sin\beta-\sin\alpha)\ldots\ldots\ldots\ldots\ldots(1).$$

Let R, R', denote the impulsive reactions of the obstacle C and the axis A, at the instant of impact; both of which will evidently be at right angles to the length of the beam. Now the effect of the reaction R is to destroy the whole of the angular velocity of the beam about A, by impressing upon it an equal and opposite angular velocity: hence, putting $CA=c$,

$$m\omega(a^2+k^2)=Rc\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Again, the difference of the moments of R and R' about the centre of gravity of the beam being

$$R(c-a)-R'a,$$

we must have

$$R(c-a)-R'a=mk^2\omega\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

From (2) and (3) we obtain

$$m\omega(a^2+k^2)(c-a)-R'ac=mk^2c\omega,$$

$$R'ac=m\omega\{(c-a)(a^2+k^2)-ck^2\},$$

$$R'=m\omega\left\{a-\frac{a^2+k^2}{c}\right\};$$

and therefore, from (1),

$$R' = m\,(2ag)^{\frac{1}{2}} \left(\frac{\sin\beta - \sin\alpha}{a^2 + k^2}\right)^{\frac{1}{2}} \left(a - \frac{a^2 + k^2}{c}\right).$$

If $R' = 0$, we have

$$a - \frac{a^2 + k^2}{c} = 0, \quad c = \frac{a^2 + k^2}{a};$$

and therefore C must be the centre of oscillation of the beam at the moment of impact.

If the beam be elastic, we must increase the value of R given by (2) in the ratio of $1 + e$ to 1, e denoting the elasticity; we shall then have, from (3),

$$R' = \frac{m\omega}{ac}\{(1 + e)(c - a)(a^2 + k^2) - ck^2\}.$$

(5) An inelastic beam, which is moving without rotation along a smooth horizontal plane, impinges upon a fixed rod at right angles to the plane: to determine the impulsive reaction of the rod and the motion of the beam subsequent to the impact.

Let AB (fig. 242) be the position of the beam at the instant of impact; O the place of the obstacle; G the centre of gravity of the beam; $G'G$ the line of G's motion before impact. Produce OB indefinitely to x, and draw the indefinite line yOy' at right angles to Ox and meeting GG' at G'. Let $R =$ the impulsive reaction of O, which will be exerted along the line Oy'; $u =$ the velocity of G before impact; $\angle OG'G = \alpha$; $OG = c$; $k =$ the radius of gyration of AB about G; $m =$ the mass of the beam; let v_x, v_y, be the velocities of G parallel to Ox, Oy, just after impact, and ω the angular velocity about G.

Then, by the equations of impulsive motion,

$$mv_x = mu\sin\alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$mv_y = mu\cos\alpha - R \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$mk^2\omega = Rc \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Again, the velocity of the point O of the beam in the direction Oy, the instant after impact, must be $v_y - c\omega$, v_y being its

velocity due to the velocity of G, and $-c\omega$ its velocity due to the rotation of the beam about G; but, the beam being inelastic, the effect of the impact is to destroy the resolved part of O's velocity at right angles to AB; hence v_y must be equal to $c\omega$.

We have, then, from (2),

$$mc\omega = mu\cos\alpha - R,$$

and therefore, by the aid of (3),

$$mc^2\omega = mcu\cos\alpha - mk^2\omega,$$

or

$$\omega = \frac{cu\cos\alpha}{c^2+k^2}.$$

Hence, from (3),

$$R = \frac{mk^2u\cos\alpha}{c^2+k^2};$$

and consequently, from (2),

$$v_y = u\cos\alpha - \frac{k^2u\cos\alpha}{c^2+k^2} = \frac{c^2u\cos\alpha}{c^2+k^2}.$$

Also, from (1), $\qquad v_x = u\sin\alpha.$

Thus we have determined completely the instantaneous motions of the beam after the impact, and the impulsive reaction of the rod at O.

It may be ascertained that, if the original motion be precisely such as our particular figure represents it, on the consummation of the impact, the beam will detach itself from the obstacle and will then move along freely with the velocities v_x, v_y, ω, which we have obtained above. In fact we should find, if we were to assume the beam always to touch the obstacle, that the obstacle would have to exert a continuous attraction instead of a reaction.

(6) A uniform horizontal stick, falling to the ground by the action of gravity, strikes at one end against a stone: to compare the blow it receives with what it would have received had both ends struck simultaneously against two stones, the blows being supposed to take place at right angles to the stick.

The blow it actually receives is half the blow it would have received, on the latter hypothesis, at each stone.

(7) One end of a straight brittle rod is held still in the hand, while the other is tapped against a table till the rod snaps: to determine the point of fracture.

The fracture will take place at a distance from the fixed end equal to $\frac{a}{\sqrt{3}}$, where a is the length of the rod.

(8) A thin uniform brittle rod, capable of turning about one fixed extremity, is struck by a given impulse at a given point: to find the point at which there will be the greatest tendency to snap in two.

Let l be the length of the rod, a the distance of the point of impact from the free end.

If a be not greater than $\frac{1}{3}l$, the point where the rod is most likely to snap is at a distance from the fixed end equal to

$$l\left(\frac{\frac{1}{3}l-a}{l-a}\right)^{\frac{1}{2}}.$$

If a be greater than $\frac{1}{3}l$, the required point coincides with the point of impact.

(9) A perfectly inelastic rod slides in the direction of its length down an inclined plane, and eventually strikes a horizontal plane: to find the impulses experienced by the two ends of the rod at the instant of impact.

If V be the velocity of the rod the instant before impact, m its mass, and α the inclination of the plane, the blows experienced by the upper and lower ends are respectively equal to

$$\frac{\frac{1}{4}mV\sin 2\alpha}{2+\cos 2\alpha}, \quad \frac{mV\sin\alpha}{2+\cos 2\alpha}.$$

(10) A hollow circular cylinder, open at both ends, is moveable about a diameter of one end: to find the distance of the line of the centres of percussion from the fixed diameter.

Let a be the radius of the cylinder, and b its length: then the required distance is equal to

$$\frac{a^2}{b} + \frac{2}{3} b.$$

SECT. 2. *Single Body. Smooth Surfaces. Determination of Instantaneous Axes of Rotation, &c.*

(1) A rigid system at rest is struck by any system of simultaneous blows: to determine the position and velocity of the Spontaneous Axis of Rotation, that is, of a straight line, rigidly connected with the system, which, on the application of the blows, has no motion but in the direction of its length.

Let the centre of gravity of the system be taken as the origin of co-ordinates: the system of blows may be reduced to three impulsive pressures X, Y, Z, at the origin, along the axes of x, y, z, respectively, and three impulsive couples the moments of which are L, M, N, in the planes yz, zx, xy, respectively.

Let V_x, V_y, V_z be the components of the absolute velocity of a particle δm, (the co-ordinates of which are x, y, z), just after the impacts, parallel to the axes of co-ordinates; V_x', V_y', V_z', the components of the velocity of the same particle relatively to the centre of gravity; $\overline{V}_x$, $\overline{V}_y$, $\overline{V}_z$ the components of the velocity of the centre of gravity; ω_1, ω_2, ω_3, the angular velocities impressed upon the system about the three axes.

Then we have

$$V_x = \overline{V}_x + V_x', \qquad V_y = \overline{V}_y + V_y', \qquad V_z = \overline{V}_z + V_z';$$

and

$$\left.\begin{aligned} V_x' &= z\omega_2 - y\omega_3 \\ V_y' &= x\omega_3 - z\omega_1 \\ V_z' &= y\omega_1 - x\omega_2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now we have, for the motion about the centre of gravity, the equations

$$\left.\begin{aligned}\Sigma\delta m\,(yV_z' - zV_y') &= L\\ \Sigma\delta m\,(zV_x' - xV_z') &= M\\ \Sigma\delta m\,(xV_y' - yV_x') &= N\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(2);$$

which, by substituting for V_x', V_y', V_z', the values given above, become

$$\left.\begin{aligned}\omega_1\Sigma\,(y^2+z^2)\,\delta m - \omega_2\Sigma xy\delta m - \omega_3\Sigma xz\delta m &= L\\ \omega_2\Sigma\,(z^2+x^2)\,\delta m - \omega_3\Sigma yz\delta m - \omega_1\Sigma yx\delta m &= M\\ \omega_3\Sigma\,(x^2+y^2)\,\delta m - \omega_1\Sigma zxdm - \omega_2\Sigma zy\delta m &= N\end{aligned}\right\}\ldots\ldots\ldots(3).$$

To simplify these equations, suppose the axes of co-ordinates to coincide with the principal axes through the centre of gravity, and let A, B, C, represent the principal moments of inertia of the system: then we have

$$\left.\begin{aligned}A\omega_1 &= L\\ B\omega_2 &= M\\ C\omega_3 &= N\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Again, for the motion of the centre of gravity we have, if m be the whole mass of the system,

$$\left.\begin{aligned}m\overline{V}_x &= X\\ m\overline{V}_y &= Y\\ m\overline{V}_z &= Z\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5);$$

and therefore, for the components of the absolute velocity of any particle δm, we shall have

$$\left.\begin{aligned}V_x = \overline{V}_x + V_x' &= \frac{X}{m} + z\frac{M}{B} - y\frac{N}{C}\\ V_y = \overline{V}_y + V_y' &= \frac{Y}{m} + x\frac{N}{C} - z\frac{L}{A}\\ V_z = \overline{V}_z + V_z' &= \frac{Z}{m} + y\frac{L}{A} - x\frac{M}{B}\end{aligned}\right\}\ldots\ldots\ldots\ldots(6).$$

Considering V_x, V_y, V_z as constant, any two of the equations (6) will represent a straight line: multiplying them in order by $\frac{L}{A}$, $\frac{M}{B}$, $\frac{N}{C}$, we get, as a condition to which V_x, V_y, V_z are subject,

$$\frac{L \,.\, V_x}{A} + \frac{M \,.\, V_y}{B} + \frac{N \,.\, V_z}{C} = \frac{L \,.\, X}{mA} + \frac{M \,.\, Y}{mB} + \frac{N \,.\, Z}{mC} \quad \ldots\ldots\ldots (7).$$

The direction-cosines of the line are, as appears from its equations, proportional to

$$\frac{L}{A}, \quad \frac{M}{B}, \quad \frac{N}{C}:$$

but, if the line be the spontaneous axis, these cosines, as is evident from the definition, must also be proportional to V_x, V_y, V_z:

hence, putting

$$\left.\begin{aligned} V_x &= \frac{kL}{A} \\ V_y &= \frac{kM}{B} \\ V_z &= \frac{kN}{C} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8);$$

we see, from (7), that

$$k = \frac{\dfrac{LX}{mA} + \dfrac{MY}{mB} + \dfrac{NZ}{mC}}{\dfrac{L^2}{A^2} + \dfrac{M^2}{B^2} + \dfrac{N^2}{C^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (9).$$

From (8) and (9), V denoting the velocity of the spontaneous axis, we see that

$$\begin{aligned} V &= (V_x^2 + V_y^2 + V_z^2)^{\frac{1}{2}} \\ &= k\left(\frac{L^2}{A^2} + \frac{M^2}{B^2} + \frac{N^2}{C^2}\right)^{\frac{1}{2}} \\ &= \frac{\dfrac{LX}{mA} + \dfrac{MY}{mB} + \dfrac{NZ}{mC}}{\left(\dfrac{L^2}{A^2} + \dfrac{M^2}{B^2} + \dfrac{N^2}{C^2}\right)^{\frac{1}{2}}}. \end{aligned}$$

The equations (6) become, by (8),

$$\frac{kL}{A} = \frac{X}{m} + z\frac{M}{B} - y\frac{N}{C},$$

$$\frac{kM}{B} = \frac{Y}{m} + x\frac{N}{C} - z\frac{L}{A},$$

$$\frac{kN}{C} = \frac{Z}{m} + y\frac{L}{A} - x\frac{M}{B};$$

which are the equations to the spontaneous axis, k being supposed to have the value given by the equation (9).

For further information on this problem the student is referred to two papers in the *Cambridge Mathematical Journal*, Vol. IV. November 1844; the former paper having been contributed by Mr Goodwin, now Bishop of Carlisle.

(2) The extremity of the minor axis of an elliptic board is fixed: it is struck in a direction perpendicular to its plane through one focus: to determine the eccentricity of the ellipse in order that the axis of initial rotation may pass through the other focus.

Let A, B, be the moments of inertia of the ellipse about the tangent and normal to the curve at the fixed point: let P be the magnitude of the blow, and ω_1, ω_2, the instantaneous angular velocities about the tangent and normal.

Since the tangent and normal are principal axes, we have

$$A\omega_1 = Pb, \quad B\omega_2 = -P\,.\,ae.$$

But $A = \frac{5}{4}\pi ab^3$, $B = \frac{1}{4}\pi a^3 b$: hence $\frac{\omega_1}{\omega_2} = -\frac{a}{5be}$. But, since the initial axis of rotation is to pass through the other focus, $\frac{\omega_1}{\omega_2} = -\frac{ae}{b}$: hence $e^2 = \frac{1}{5}$.

(3) A rectangular parallelepiped is rotating with a given angular velocity about a diagonal, when one of its edges, which does not meet the diagonal, suddenly becomes fixed: to determine the angular velocity about this edge.

Let ω be the given angular velocity: let the fixed edge OC be taken as the axis of z, the edges OA, OB, being the axes of x, y, respectively.

Let Mk'^2 denote the moment of inertia of the parallelepiped about OC, and ω' the required angular velocity about it. Since all the impulses on the body pass through the axis of z, the moment of the momentum of the body about this axis will not be affected by the impulsive action : hence

$$Mk'^2\omega' = \Sigma m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right).$$

Let $2a$, $2b$, $2c$, be the lengths of the edges of the parallelepiped, a, b, c, being therefore the co-ordinates of its centre of gravity.

Let x', y', z', be the co-ordinates of any particle referred to axes through the centre of gravity, parallel to OA, OB, OC. Then

$$Mk'^2\omega' = \Sigma m\left\{(a+x')\frac{dy'}{dt} - (b+y')\frac{dx'}{dt}\right\}.$$

Let ω_1, ω_2, ω_3, be the angular velocities, before the impulse, about the axes of x', y', z' : then

$$\frac{dx'}{dt} = \omega_2 z' - \omega_3 y', \qquad \frac{dy'}{dt} = \omega_3 x' - \omega_1 z' :$$

hence, observing that the new origin is the centre of gravity and that the new axes are principal axes, we have

$$Mk'^2\omega' = \Sigma\{m\omega_3(x'^2 + y'^2)\},$$

and therefore, Mk^2 being the moment of inertia about the axis of z',

$$k'^2\omega' = k^2\omega_3.$$

It may easily be ascertained that $\dfrac{k^2}{k'^2} = \dfrac{1}{4}$: hence

$$\omega' = \frac{1}{4}\omega_3$$

$$= \frac{1}{4}\omega\cos\gamma,$$

where γ denotes the inclination of the fixed edge to the diagonal about which the body was originally revolving.

(4) A lamina in the form of a quadrant of a circle, fixed at one extremity of its arc, is struck by a blow at right angles to its plane at the other extremity: to find the position of the instantaneous axis.

Let the fixed point be taken as the origin of co-ordinates, the axis of x passing through the centre of the circle, that of y being a tangent to the circle, that of z perpendicular to the plane of the circle. Let ω_x, ω_y, ω_z, be the instantaneous angular velocities about the three axes. Let Z be the blow. The general formulæ are*

$$A\omega_x = \omega_y \Sigma (mxy) + \omega_z \Sigma (mxz) + L,$$
$$B\omega_y = \omega_z \Sigma (myz) + \omega_x \Sigma (myx) + M,$$
$$C\omega_z = \omega_x \Sigma (mzx) + \omega_y \Sigma (mzy) + N.$$

In the present case, a being the radius of the circle, these formulæ reduce themselves to

$$A\omega_x = \omega_y \Sigma (mxy) + Za,$$
$$B\omega_y = \omega_x \Sigma (mxy) - Za,$$
$$C\omega_z = 0.$$

Let μ denote the mass of a unit of area of the lamina: then

$$\Sigma (mxy) = \mu \int_0^{\frac{\pi}{2}} \int_0^a r d\theta dr \,.\, (a - r\cos\theta) \,.\, r\sin\theta$$

$$= \frac{1}{12} \mu a^4 \int_0^{\frac{\pi}{2}} (4 - 3\cos\theta) \sin\theta d\theta$$

$$= \frac{5}{24} \mu a^4.$$

Again $$A = \frac{1}{4} a^2 \,.\, \frac{1}{4} \mu\pi a^2 = \frac{\pi\mu a^4}{16}.$$

* Routh: *Rigid Dynamics*, 2nd Edn. p. 208.

Also $$B = \int_0^{\frac{\pi}{2}}\int_0^a \mu r d\theta dr \,.\, x^2$$

$$= \mu\int_0^{\frac{\pi}{2}}\int_0^a r d\theta dr \,.\, (a - r\cos\theta)^2$$

$$= \frac{1}{24}\mu a^4\int_0^{\frac{\pi}{2}}(15 - 16\cos\theta + 3\cos 2\theta)\, d\theta$$

$$= \frac{\mu a^4}{48}(15\pi - 32).$$

Hence $$\frac{\pi\mu a^4}{16} \,.\, \omega_x = \frac{5\mu a^4}{24}\omega_y + Za,$$

$$\frac{\mu a^4}{48}(15\pi - 32) \,.\, \omega_y = \frac{5\mu a^4}{24}\omega_x - Za,$$

whence $$(15\pi - 42)\,\omega_y = (10 - 3\pi)\,\omega_x,$$

and therefore, ϕ being the inclination of the instantaneous axis to the axis of x,

$$\tan\phi = \frac{10 - 3\pi}{15\pi - 42}.$$

(5) A free rigid body is moving about its centre of gravity, which is at rest: a given point of the body suddenly becomes fixed: to determine the position of the instantaneous axis at the subsequent instant, the motion of the body the instant before being supposed to be known.

Let h, k, l, be the co-ordinates of the given point, referred to Gx', Gy', Gz', the principal axes at G, the centre of gravity.

Let ω_x, ω_y, ω_z be the angular velocities of the body about the axes Gx', Gy', Gz', the instant before the point became fixed, Ω_x, Ω_y, Ω_z those about parallel axes through the point the instant after it became fixed. Let A', B', C', be the moments of inertia about Gx', Gy', Gz', and A, B, C, those about the parallel axes. Then*

* Routh: *Rigid Dynamics*, 2nd Edn. p. 204.

$$A'\omega_x = A\Omega_x - \Sigma(mxy).\Omega_y - \Sigma(mxz).\Omega_z,$$
$$B'\omega_y = B\Omega_y - \Sigma(myz).\Omega_z - \Sigma(myx).\Omega_x,$$
$$C'\omega_z = C\Omega_z - \Sigma(mzx).\Omega_x - \Sigma(mzy).\Omega_y.$$

Let a, b, c, be the principal radii of gyration at the centre of gravity: then, μ denoting the mass of the body,

$$\begin{aligned} A = \Sigma\{m(y^2+z^2)\} &= \Sigma m\{(y'-k)^2+(z'-l)^2\} \\ &= \Sigma m\{y'^2+z'^2-2ky'-2lz'+k^2+l^2\} \\ &= \mu(a^2+k^2+l^2). \end{aligned}$$

Also
$$\begin{aligned} \Sigma(mxy) &= \Sigma\{m(x'-h)(y'-k)\} \\ &= \Sigma\{m(x'y'-hy'-kx'+hk)\} \\ &= \mu hk. \end{aligned}$$

Hence
$$a^2\omega_x = (a^2+k^2+l^2)\,\Omega_x - hk\Omega_y - hl\Omega_z.$$

Similarly
$$b^2\omega_y = (b^2+l^2+h^2)\,\Omega_y - kl\Omega_z - kh\Omega_x,$$
$$c^2\omega_z = (c^2+h^2+k^2)\,\Omega_z - lh\Omega_x - lk\Omega_y.$$

By cross multiplication and obvious transformations we shall find that, r^2 denoting $h^2+k^2+l^2$,

$$\begin{aligned} \Omega_x\{(r^2+a^2)(r^2+b^2)(r^2+c^2) - h^2(r^2+b^2)(r^2+c^2) - k^2(r^2+c^2)(r^2+a^2) \\ - l^2(r^2+a^2)(r^2+b^2)\} \\ = hr^2(ha^2\omega_x + kb^2\omega_y + lc^2\omega_z) \\ + a^2\omega_x\{h^2(b^2+c^2) + b^2k^2 + c^2l^2 + b^2c^2\} \\ + b^2c^2h(k\omega_y + l\omega_z), \end{aligned}$$

the expressions for Ω_y, Ω_z, being thence obvious from symmetry.

COR. From the above results we may readily infer the following relation, viz.:

$$h\Omega_x + k\Omega_y + l\Omega_z = \frac{\dfrac{a^2h\omega_x}{r^2+a^2} + \dfrac{b^2k\omega_y}{r^2+b^2} + \dfrac{c^2l\omega_z}{r^2+c^2}}{1 - \dfrac{h^2}{r^2+a^2} - \dfrac{k^2}{r^2+b^2} - \dfrac{l^2}{r^2+c^2}}.$$

(6) One point of a rotating rigid body is fixed: the body receives a blow of given magnitude passing through a given

point of the body: to determine the condition that the initial axis of rotation after the blow may be perpendicular to the axis of rotation before the blow.

Let the axes of co-ordinates at the instant under consideration coincide with the principal axes at the fixed point. Let (h, k, l) be the given point of the body, (X, Y, Z) the components of the blow. Let A, B, C, be the principal moments of inertia at the fixed point, and $(\omega'_x, \omega'_y, \omega'_z)$ the component angular velocities generated by the blow. Then

$$A\omega'_x = Yl - Zk, \quad B\omega'_y = Zh - Xl, \quad C\omega'_z = Xk - Yh.$$

Let $(\omega_1, \omega_2, \omega_3)$ be the component angular velocities the instant before the blow: then, $(\omega_x, \omega_y, \omega_z)$ being the component angular velocities the instant after the blow,

$$\omega_x = \omega_1 + \omega'_x, \quad \omega_y = \omega_2 + \omega'_y, \quad \omega_z = \omega_3 + \omega'_z;$$

and therefore

$$A\omega_x = A\omega_1 + Yl - Zk,$$
$$B\omega_y = B\omega_2 + Zh - Xl,$$
$$C\omega_z = C\omega_3 + Xk - Yh.$$

Hence the equations to the initial axis of rotation after the blow are

$$\frac{Ax}{A\omega_1 + Yl - Zk} = \frac{By}{B\omega_2 + Zh - Xl} = \frac{Cz}{C\omega_3 + Xk - Yh}.$$

In order that this axis of rotation may be at right angles to the instantaneous axis before the blow, we must have

$$\frac{\omega_1}{A}(A\omega_1 + Yl - Zk) + \frac{\omega_2}{B}(B\omega_2 + Zh - Xl) + \frac{\omega_3}{C}(C\omega_3 + Xk - Yh) = 0.$$

Let x, y, z, be the co-ordinates of any point in the line of action of the blow, referred now to co-ordinate axes passing through the point (h, k, l) and parallel to the principal axes of the body at the fixed point: then, R denoting the blow,

$$\frac{X}{x} = \frac{Y}{y} = \frac{Z}{z} = \frac{R}{(x^2 + y^2 + z^2)^{\frac{1}{2}}},$$

and therefore, putting $\omega_1^2 + \omega_2^2 + \omega_3^2 = \omega^2$,

$$\frac{\omega^4}{R^2}(x^2+y^2+z^2) = \left\{\frac{\omega_1}{A}(ly-kz) + \frac{\omega_2}{B}(hz-lx) + \frac{\omega_3}{C}(kx-hy)\right\}^2,$$

an equation which shews that the locus of the blow's direction is a right circular cone, the equations to its axis of figure being

$$\frac{1}{x}\left(\frac{l\omega_2}{B} - \frac{k\omega_3}{C}\right) = \frac{1}{y}\left(\frac{h\omega_3}{C} - \frac{l\omega_1}{A}\right) = \frac{1}{z}\left(\frac{k\omega_1}{A} - \frac{h\omega_2}{B}\right).$$

Cor. Since

$$h\left(\frac{l\omega_2}{B} - \frac{k\omega_3}{C}\right) + k\left(\frac{h\omega_3}{C} - \frac{l\omega_1}{A}\right) + l\left(\frac{k\omega_1}{A} - \frac{h\omega_2}{B}\right) = 0,$$

it follows that the axis of the cone is perpendicular to the line joining the given point of the body to the fixed point.

(7) A given point of a rigid quiescent body is fixed: to determine the position of an axis, fixed in the body, through this point, such that the body, being struck by a given blow, may acquire the greatest possible vis viva.

Let L, M, N, be the components, in relation to the principal axes at the given point, of the moment of the blow about the given point. Let l, m, n, be the direction-cosines of the fixed axis. The principal moments of inertia being A, B, C, the angular velocity impressed on the body by the blow will be equal to

$$\frac{Ll + Mm + Nn}{Al^2 + Bm^2 + Cn^2},$$

and therefore the vis viva acquired will be equal to

$$\frac{(Ll + Mm + Nn)^2}{Al^2 + Bm^2 + Cn^2}.$$

Let $l = \frac{L}{A} + u$, $m = \frac{M}{B} + v$, $n = \frac{N}{C} + w$: then the numerator of the expression for the vis viva becomes

$$\left(\frac{L^2}{A}+\frac{M^2}{B}+\frac{N^2}{C}\right).\left(\frac{L^2}{A}+\frac{M^2}{B}+\frac{N^2}{C}\right.$$
$$\left.+2Lu+2Mv+2Nw+Au^2+Bv^2+Cw^2\right)$$
$$-\left(\frac{L^2}{A}+\frac{M^2}{B}+\frac{N^2}{C}\right).(Au^2+Bv^2+Cw^2)+(Lu+Mv+Nw)^2;$$

and the denominator becomes

$$A\left(\frac{L}{A}+u\right)^2+B\left(\frac{M}{B}+v\right)^2+C\left(\frac{N}{C}+w\right)^2$$
$$=\frac{L^2}{A}+\frac{M^2}{B}+\frac{N^2}{C}+2Lu+2Mv+2Nw+Au^2+Bv^2+Cw^2.$$

Hence the vis viva is equal to

$$\frac{L^2}{A}+\frac{M^2}{B}+\frac{N^2}{C}$$
$$-\frac{\left\{N\left(\frac{B}{C}\right)^{\frac{1}{2}}v-M\left(\frac{C}{B}\right)^{\frac{1}{2}}w\right\}^2+\left\{L\left(\frac{C}{A}\right)^{\frac{1}{2}}w-N\left(\frac{A}{C}\right)^{\frac{1}{2}}u\right\}^2+\left\{M\left(\frac{A}{B}\right)^{\frac{1}{2}}u-L\left(\frac{B}{A}\right)^{\frac{1}{2}}v\right\}^2}{A\left(\frac{L}{A}+u\right)^2+B\left(\frac{M}{B}+v\right)^2+C\left(\frac{N}{C}+w\right)^2}.$$

This result shews that the vis viva is greatest when u, v, w, and therefore l, m, n, are proportional to $\frac{L}{A}, \frac{M}{B}, \frac{N}{C}$: hence the fixed axis coincides with what would have been the instantaneous axis had the body been free to move without any constraint but that due to the given fixed point. (Euler's Theorem*.)

(8) To determine the nature of the impulses which must be impressed upon a free quiescent cube in order that, *ipso motus initio*, a diagonal of the cube may remain at rest.

Let X, Y, Z, be the components of the resultant force through O, the centre of gravity of the cube, along rectangular axes Ox, Oy, Oz, parallel to the edges; let L, M, N, be the components of the resultant couples about these axes: and let $x=y=z$, be the equations to the quiescent diagonal; then

$$X=0, \quad Y=0, \quad Z=0,$$

and $$L=M=N;$$

* Lagrange: *Mécanique Analytique;* Tom. I. p. 294. Thomson and Tait: *Natural Philosophy*, Vol. I. p. 216.

results which shew that the resultant force must be zero, and that the plane of the resultant couple must be at right angles to the quiescent diagonal.

(9) To determine the nature of the impulses in the preceding problem, in order that an edge of the cube may remain for an instant quiescent.

Let the equations to the quiescent edge be

$$y = a, \; z = a,$$

$2a$ being the length of an edge: then

$$X = 0, \; M = 0, \; N = 0,$$

$$Y = \frac{3L}{2a}, \; Z = -\frac{3L}{2a}:$$

results which shew that the resultant force is at right angles to the diagonal plane through the quiescent edge, and that the plane of the resultant couple is perpendicular to this edge.

(10) A lamina, in the form of a semi-ellipse bounded by the axis minor, is moveable about the centre as a fixed point, and falls from the position in which its plane is horizontal:

(1) to determine the impulse which must be applied at the centre of gravity, when the lamina is vertical, in order to reduce it to rest; (2) if this force be applied perpendicularly to the lamina at the extremity of an ordinate through the centre of gravity, instead of being applied at the centre of gravity itself, to ascertain the position of the axis of revolution the instant afterwards.

The required impulse is equal to $M\left(\frac{3}{8}\pi g a\right)^{\frac{1}{2}}$, M being the mass of the ellipse, and the required axis is the major axis.

Mackenzie and Walton: *Solutions of the Cambridge Problems* for 1854.

(11) The angular point A of a triangular lamina ABC is fixed: if a blow be impressed upon the lamina, at B or C, at

right angles to its plane, to determine the position of the instantaneous axis.

The instantaneous axis will pass through one of the points of trisection of BC.

(12) A square lamina is moving freely about a diagonal with a given angular velocity: if one of the ends of the other diagonal become fixed, to determine the impulsive pressure on the fixed point and also the instantaneous angular velocity.

Let ω be the given angular velocity, m the mass of the lamina, and c the length of a semidiagonal: then the impulsive pressure and the instantaneous angular velocity are respectively equal to $\frac{1}{7} mc\omega$ and $\frac{1}{7}\omega$.

(13) A square lamina, one angular point of which is fixed, is struck by two equal blows, one along a side not terminating at the fixed point, the other perpendicular to the plane of the lamina through an angular point, not the fixed one nor in the line of the former blow: to determine the initial instantaneous axis and the position of the invariable plane.

Let the side, terminating at the fixed point, which is perpendicular to the former blow, be the axis of x, the other side, which terminates at the fixed point, being that of y; the axis of z being perpendicular to the lamina. Then the equations to the instantaneous axis are

$$x : y : z :: 32 : 24 : 7,$$

and the equation to the invariable plane is

$$16x + 12y + 7z = 0.$$

(14) A rectangular lamina, the centre of which is fixed, is struck perpendicularly at a point in a given line through its centre: to find the position of the axis of instantaneous rotation.

Let 2α be the angle between the diagonals of the rectangle and β the inclination of the given line to one of its sides:

let θ be the inclination of the required axis to the same side: then

$$\tan\theta = \frac{\tan^2\alpha}{\tan\beta}.$$

Griffin: *Solutions of the Examples on the Motion of a Rigid Body*, p. 57.

(15) A free rectangular lamina is struck perpendicularly at a given point: to find the position of the axis of instantaneous rotation.

Let a, b, be the lengths of the sides of the lamina, and axes of co-ordinates, Ox, Oy, be taken in the plane of the lamina through its centre O, parallel to these sides: let h, k, be the co-ordinates of the given point: then the equation to the initial axis will be

$$\frac{hx}{a^2} + \frac{ky}{b^2} + \frac{1}{12} = 0.$$

Griffin: *Ib.* p. 57.

(16) A circular lamina, of radius a, revolving about a diameter with an angular velocity ω, is struck perpendicularly to its plane at the extremity of the diameter which is at right angles to the former, and afterwards has an angular velocity ω': to find the distance between the original and latter axis of rotation.

The required distance is equal to

$$\frac{a}{4} \cdot \frac{\omega' - \omega}{\omega'}.$$

Griffin: *Ib.* p. 58.

(17) A cube is struck by three blows along three of its edges which neither meet nor are parallel: to ascertain whether it has an initial instantaneous axis or not.

There is no instantaneous axis at a finite distance from the body. Griffin: *Ib.* p. 60.

(18) A rigid body is struck by a couple: to find the maximum value of the angle between the axis of the couple and the axis about which the body will begin to rotate.

The cosine of the maximum value of the angle is equal to

$$\frac{2}{\left(\frac{B}{C}\right)^{\frac{1}{2}}+\left(\frac{C}{B}\right)^{\frac{1}{2}}},\quad \frac{2}{\left(\frac{C}{A}\right)^{\frac{1}{2}}+\left(\frac{A}{C}\right)^{\frac{1}{2}}},\quad \frac{2}{\left(\frac{A}{B}\right)^{\frac{1}{2}}+\left(\frac{B}{A}\right)^{\frac{1}{2}}};$$

where A, B, C, are the principal moments of inertia with respect to the centre of gravity.

(19) A quiescent ellipsoid is struck by a system of blows, the resultants of which are a force, through its centre of gravity, the direction-cosines of which are l, m, n, and a couple, the direction-cosines of the axes of which are λ, μ, ν: to determine the ratio of the velocity of the spontaneous axis of rotation to the velocity of the centre of gravity of the ellipsoid.

The required ratio is equal to

$$\frac{\dfrac{l\lambda}{b^2+c^2}+\dfrac{m\mu}{c^2+a^2}+\dfrac{n\nu}{a^2+b^2}}{\left\{\dfrac{\lambda^2}{(b^2+c^2)^2}+\dfrac{\mu^2}{(c^2+a^2)^2}+\dfrac{\nu^2}{(a^2+b^2)^2}\right\}^{\frac{1}{2}}}.$$

SECT. 3. *Several Bodies. Smooth Surfaces.*

(1) A heavy sphere P (fig. 243) falls from a given altitude upon a body at rest, the upper surface of which is a smooth inclined plane; the body is capable of sliding along a smooth horizontal plane, its lower surface being flat: the vertical plane through the centres of gravity of the sphere and the body intersects the inclined plane in the direction of its greatest slope: to determine the initial motions of the sphere and of the body, both of which are supposed to be perfectly inelastic.

Let ABH denote the section of the body made by the vertical plane passing through its centre of gravity and that of the sphere, AH being a line in the horizontal plane.

Let V be the velocity of the sphere just before impact; u, v, the resolved parts of its velocity after impact, perpendicular and parallel to the hypotenuse BA of the triangle BAH; u' the

velocity of the body parallel to AH after the impact; m, m', the respective masses of the sphere and body, and B the blow of collision; let $\angle BAH = \alpha$.

Then, for the motion of the sphere, the blow being at right angles to BA,

$$mu = mV\cos\alpha - B \qquad\qquad (1),$$

$$v = V\sin\alpha \qquad\qquad (2);$$

and, for the motion of the body,

$$m'u' = B\sin\alpha \qquad\qquad (3).$$

These three equations involve four unknown quantities, u, v, u', B: for the solution of the problem, then, another equation will be necessary. This will be obtained by the consideration that, the sphere and the body being both perfectly inelastic, the effect of their collision is merely to prevent the penetration of the one into the interior of the other, without causing any recoil, which could result only from the existence of elasticity: hence the velocity of the ball, after collision, at right angles to the line BA, must be equal to the velocity of any assigned point in this line estimated in the same direction.

Now the velocity of any assigned point in BA at right angles to this line is evidently $u'\sin\alpha$; and therefore we have

$$u'\sin\alpha = u \qquad\qquad (4).$$

From the equations (1), (3), (4), we obtain

$$mB\sin^2\alpha = mm'V\cos\alpha - m'B,$$

and therefore $$B = \frac{mm'V\cos\alpha}{m\sin^2\alpha + m'} \qquad\qquad (5),$$

which gives the magnitude of the blow.

From (3) and (5) we get

$$u' = \frac{mV\sin\alpha\cos\alpha}{m\sin^2\alpha + m'},$$

which determines the motion of the body; and therefore, by (4),

$$u = \frac{mV\sin^2\alpha\cos\alpha}{m\sin^2\alpha + m'}.$$

D'Arcy; *Mémoires de l'Académie des Sciences de Paris*, 1747, p. 344.

(2) Two billiard balls B and C (fig. 244) are lying in contact on the table: to find the direction in which the ball B must be struck by a third ball A so as to go off in a given direction BD; the balls being of equal volume and weight, and perfectly smooth.

Let the direction AB, which joins the centres of A and B at the instant of their collision, make an angle θ with the straight line CBE, and let $\angle CBD = \alpha$. We will first suppose the balls to be inelastic. Let a, a', denote the resolved parts of the velocity of A in the direction AB, before and after collision respectively; and b, c, the velocities of B, C. Let m represent the mass of each of the balls, R the blow between A, B, and S that between B, C.

Then, for the motion of B, resolving forces at right angles to EC,

$$mb \sin \alpha = R \sin \theta \quad \text{.......................(1)},$$

and, resolving parallel to EC,

$$mb \cos \alpha = R \cos \theta - S \quad \text{.....................(2)}.$$

Also, for the motion of C, we have

$$mc = S \quad \text{..............................(3)}.$$

Again, since after collision the velocities of B and C in the direction EC must be equal, there is

$$b \cos \alpha = c \quad \text{...........................(4)}.$$

From (3) and (4) we get $mb \cos \alpha = S$,
and therefore, from (2),

$$mb \cos \alpha = R \cos \theta - mb \cos \alpha,$$

$$R \cos \theta = 2mb \cos \alpha \quad \text{......................(5)}.$$

From (1) and (5) there is

$$mb \sin \alpha \cos \theta = 2mb \cos \alpha \sin \theta,$$

and therefore $$\tan \theta = \tfrac{1}{2} \tan \alpha,$$

which determines the point at which A must come into collision with B.

If we introduce the consideration of elasticity, the magnitudes of R and S will each have to be increased in the ratio of $1 + e$ to 1. Now the direction of B's motion will evidently not be affected by any alteration in the absolute magnitudes of R and S, provided that the ratio between their intensities be not changed. Thus we see that the consideration of elasticity will not modify the solution of the problem.

(3) A billiard ball impinges simultaneously upon two other billiard balls which are resting in contact: to determine the motions of the three balls after collision.

Let A (fig. 245) denote the centre of the impinging ball at the moment of impact; A', A'', those of the other two balls. Draw the lines AA', AA'', and produce them indefinitely to points a', a''; draw Aa a common tangent to the two balls A', A''. Then evidently after collision A's motion will be confined to the straight line Aa; while A', A'', will proceed to move along $A'a'$, $A''a''$. Let u, v, be the velocities of A before and after impact; v' the velocity after impact of each of the balls A', A''. Since $AA'A''$ is evidently an equilateral triangle, $\angle aAa' = \frac{1}{6}\pi = \angle aAa''$; let B be the blow of collision between A, A', and A, A''; and m the mass of each of the three balls. Then

$$mv = mu - 2B\cos\tfrac{1}{6}\pi \;\ldots\ldots\ldots\ldots\ldots\ldots\; (1),$$

$$mv' = B \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (2).$$

Let us first suppose the three balls to be perfectly inelastic; then the instant after impact the balls A, A', will move in contact, as well as the balls A, A''; hence

$$v' = v\cos\tfrac{1}{6}\pi:$$

we have, therefore, from (2),

$$B = mv\cos\frac{\pi}{6};$$

and consequently, from (1),

$$mv = mu - 2mv\cos^2\frac{\pi}{6}, \qquad v = \frac{u}{1 + 2\cos^2\frac{\pi}{6}} = \frac{2u}{5};$$

and therefore $$v' = v \cos \frac{\pi}{6} = \frac{3^{\frac{1}{2}}}{5} u,$$

and $$B = \frac{3^{\frac{1}{2}} mu}{5} \text{..........................} (3).$$

If the balls be elastic, and e denote their elasticity, we must increase the value of B in (3) in the ratio of $1 + e$ to 1; hence we have

$$B = (1 + e) \frac{3^{\frac{1}{2}} mu}{5} \text{.....................} (4),$$

and therefore, from (1),

$$mv = mu - \tfrac{3}{5} (1 + e)\, mu,$$

$$v = \tfrac{1}{5} (2 - 3e)\, u;$$

and, from (2), (4), $$mv' = (1 + e) \frac{3^{\frac{1}{2}} mu}{5},$$

$$v' = \frac{3^{\frac{1}{2}}}{5} (1 \div e)\, u.$$

Maclaurin; *Treatise of Fluxions.* D'Alembert; *Traité de Dynamique*, p. 227.

(4) A ball C (fig. 246) impinges upon an inelastic beam AB with a given velocity, at right angles to its length: to determine the magnitude of the blow and the initial circumstances of the motion of the beam and ball.

Let G be the centre of gravity of the beam; m' its mass, k the radius of gyration about G; let $EG = a$, $u =$ the velocity of the ball before impact, $v =$ its velocity immediately afterwards; $R =$ the magnitude of the mutual impulse: let v' be the velocity of G and ω the angular velocity of the beam about G just after collision.

Then, for the initial motion of the ball after collision, m denoting its mass,

$$mv = mu - R \text{......................} (1);$$

and, for the initial motion of the beam,

$$m'v' = R \text{..........................} (2),$$

$$m'k^2\omega = Ra \text{.........................} (3).$$

Again, the velocity of the point E of the beam will be equal to

$$v' + a\omega,$$

the former term of the expression being due to the motion of G, and the latter to the rotation about G; but, the beam and ball being inelastic, the velocity of the point E of the beam after collision must be equal to that of the point E of the ball, and therefore of the point C: hence we have

$$v = v' + a\omega \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

From (1), (2), (3), (4), we obtain

$$u - \frac{R}{m} = \frac{R}{m'} + \frac{Ra^2}{m'k^2},$$

and therefore

$$R = \frac{u}{\dfrac{1}{m} + \dfrac{1}{m'} + \dfrac{a^2}{m'k^2}} = \frac{mm'k^2u}{(m+m')\,k^2 + ma^2}.$$

Hence, from (1), we get

$$v = u - \frac{m'k^2u}{(m+m')\,k^2 + ma^2} = \frac{m\,(k^2 + a^2)\,u}{(m+m')\,k^2 + ma^2},$$

from (2), $$v' = \frac{mk^2u}{(m+m')\,k^2 + ma^2},$$

and, from (3), $$\omega = \frac{mau}{(m+m')\,k^2 + ma^2}.$$

(5) A cylinder is revolving with a given angular velocity round its axis, which is horizontal, when it suddenly begins to draw up a weight, consisting of inelastic materials, by means of an inextensible string wound round the cylinder: to determine the time the system will continue in motion, and the original distance of the weight from the cylinder, in order that, at the instant the motion ceases, the weight may just touch the cylinder.

Let a = the radius of the cylinder, m = its mass, k = the radius of gyration about its axis; m' = the mass of the weight; let

ω, ω', denote the angular velocities of the cylinder just before and just after beginning to draw up the weight; let u = the velocity of the weight at the commencement of its motion, B = the impulsive force exerted initially by the string on the weight.

Then we shall have

$$\omega' = \omega - \frac{Ba}{mk^2}, \qquad u = \frac{B}{m'}:$$

but $u = a\omega'$: hence

$$\frac{B}{m'} = a\omega - \frac{Ba^2}{mk^2},$$

and therefore

$$B = \frac{mm'a\omega k^2}{m'a^2 + mk^2}, \qquad u = \frac{ma\omega k^2}{m'a^2 + mk^2} \,\dots\dots\dots\, (1).$$

Let θ denote the angle through which the cylinder has revolved about its axis at the end of the time t from the commencement of the raising of the weight; and let x be the corresponding distance of the weight below the horizontal plane through the axis of the cylinder. Then, by the Principle of the Conservation of Vis Viva,

$$m'\frac{dx^2}{dt^2} + mk^2\frac{d\theta^2}{dt^2} = 2m'gx + C \,\dots\dots\dots\dots\dots\, (2):$$

but, if b denote the value of x at the commencement of the raising of the weight, it is clear from the geometry that

$$x + a\theta = b, \quad \text{and therefore } a\frac{d\theta}{dt} = -\frac{dx}{dt}:$$

hence, from (2),

$$\frac{m'a^2 + mk^2}{a^2}\frac{dx^2}{dt^2} = 2m'gx + C:$$

differentiating with respect to t, and dividing by $2\dfrac{dx}{dt}$,

$$(m'a^2 + mk^2)\frac{d^2x}{dt^2} = m'a^2g:$$

integrating with respect to t, we have

$$(m'a^2 + mk^2)\frac{dx}{dt} = C + m'a^2gt:$$

but $\frac{dx}{dt} = -u$ when $t = 0$: hence $C = -(m'a^2 + mk^2)\,u$, or, by (1), $C = -ma\omega k^2$, and therefore

$$(m'a^2 + mk^2)\frac{dx}{dt} = m'a^2gt - ma\omega k^2 \ldots\ldots\ldots (3):$$

integrating again with respect to t, we get

$$(m'a^2 + mk^2)\,x = C' + \tfrac{1}{2}m'a^2gt^2 - ma\omega k^2t:$$

but $x = b$ when $t = 0$: hence $C' = (m'a^2 + mk^2)\,b$, and therefore

$$(m'a^2 + mk^2)\,x = (m'a^2 + mk^2)\,b + \tfrac{1}{2}m'a^2gt^2 - ma\omega k^2t \ldots\ldots (4).$$

Let t' denote the time when the motion ceases for an instant: then, from (3), since $\frac{dx}{dt} = 0$ when $t = t'$,

$$0 = m'a^2gt' - ma\omega k^2, \quad t' = \frac{m\omega k^2}{m'ag}.$$

Hence also, from (4), since $x = 0$ when $t = t'$,

$$(m'a^2 + mk^2)\,b = ma\omega k^2t' - \tfrac{1}{2}m'a^2gt'^2 = \frac{m^2\omega^2k^4}{2m'g},$$

which gives the required value of b.

(6) Two inelastic spheres, of which A and a (fig. 247) are the centres, are attached to rigid rods CA and ca, which are capable of motion in a single plane about axes through C and c at right angles to the plane: supposing the spheres to impinge against each other with given velocities, it is required to determine their initial velocities after impact.

Join Aa, and produce it indefinitely both ways to points α, β; from C, c, draw CG, cg, at right angles to $\alpha\beta$ at the moment of collision. Let C, c, represent the lines CG, cg; Ω, ω, the angular velocities of CA, ca, about C, c, respectively, immediately before, and Ω', ω', immediately after collision, the angular motions being estimated in the directions indicated by the arrows in the figure; B the blow of collision; I the moment of inertia of the sphere A with its rod AC about the axis through C, i the moment of inertia of the other sphere and rod about c.

Then, $\Omega' - \Omega$ being the angular velocity which CA gains, and $\omega - \omega'$ that which ca loses by the shock, we shall have

$$I(\Omega' - \Omega) = B \,.\, C, \quad i(\omega - \omega') = B \,.\, c \text{..............(1)}.$$

But, the spheres being inelastic, the point P of the sphere a will, the instant after collision, have the same velocity along $\alpha\beta$ as the point P of the sphere A: hence

$$\Omega' \,.\, CP \,.\, \sin \angle CPG = \omega' \,.\, cP \,.\, \sin \angle cPg,$$

or

$$\Omega' C = \omega' c \text{..............................(2)}.$$

Now, from the equation (1),

$$cI(\Omega' - \Omega) + Ci(\omega' - \omega) = 0,$$

and therefore, by (2), we get

$$cI(c\omega' - C\Omega) + C^2 i(\omega' - \omega) = 0,$$

$$(c^2 I + C^2 i)\,\omega' = C(cI\Omega + Ci\omega);$$

and therefore $\quad (c^2 I + C^2 i)\,\Omega' = c(cI\Omega + Ci\omega);$

which two equations give the values of Ω', ω'.

The solution of a particular case of this problem was unsuccessfully attempted by John Bernoulli, son of the celebrated John Bernoulli, in the *Mémoires de St. Petersbourg*, Tom. VII.: the correct solution of the problem, in its most general form, was given by D'Alembert, *Traité de Dynamique*, p. 221; second edition.

(7) An impulsive tension in the direction of the tangent is applied at one extremity of a uniform perfectly flexible heavy string, lying on a smooth table: to investigate the form of the string in order that all the particles may start with equal velocities.

Let t be the impulsive tension at a point of the string the co-ordinates of which are x, y; and let the initial velocities of that point, parallel to the axes, be v_x, v_y: then, μ being the mass of a unit of length of the string, we have the following equations, viz.

$$\left.\begin{aligned}\frac{d}{ds}\left(t\frac{dx}{ds}\right)&=\mu v_x\\ \frac{d}{ds}\left(t\frac{dy}{ds}\right)&=\mu v_y\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)^*.$$

From these equations we have

$$\mu^2(v_x^2+v_y^2)=\left(t\frac{d^2x}{ds^2}+\frac{dx}{ds}\frac{dt}{ds}\right)^2+\left(t\frac{d^2y}{ds^2}+\frac{dy}{ds}\frac{dt}{ds}\right)^2$$

$$=\left(\frac{dx^2}{ds^2}+\frac{dy^2}{ds^2}\right)\frac{dt^2}{ds^2}+2t\frac{dt}{ds}\left(\frac{dx}{ds}\frac{d^2x}{ds^2}+\frac{dy}{ds}\frac{d^2y}{ds^2}\right)$$

$$+t^2\left\{\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2\right\}$$

$$=\frac{dt^2}{ds^2}+\frac{t^2}{\rho^2},$$

where ρ denotes the radius of curvature of the string at the point x, y.

Since the initial velocities of all the points of the string are supposed constant throughout the string, we have, c denoting a constant quantity,

$$\frac{dt^2}{ds^2}+\frac{t^2}{\rho^2}=c\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Again, from the equations (1) we may shew that

$$\frac{d^2t}{ds^2}=\frac{t}{\rho^2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)\dagger.$$

From (2) and (3) we see that

$$\frac{dt^2}{ds^2}+t\frac{d^2t}{ds^2}=c,$$

$$t\frac{dt}{ds}=cs+c',$$

$$t^2=cs^2+2c's+c'',$$

where c' and c'' are constants.

* Tait and Steele: *Dynamics of a Particle*, 2nd *Edition*, p. 294.
† *Ibid.* p. 295.

Hence $$\rho^2 = \frac{t}{\frac{d^2t}{ds^2}} = \frac{t^2}{t\frac{d^2t}{ds^2}} = \frac{t^2}{c - \frac{dt^2}{ds^2}}$$

$$= \frac{t^4}{ct^2 - (cs + c')^2}$$

$$= \frac{\left(s^2 + \frac{2c'}{c}s + \frac{c''}{c}\right)^2}{\frac{cc'' - c'^2}{c^2}},$$

whence, putting $s + \frac{c'}{c} = \sigma$, we have

$$\rho = \frac{\sigma^2 + a^2}{a} \quad\text{.........................}(4),$$

where $$a = \left(\frac{cc'' - c'^2}{c^2}\right)^{\frac{1}{2}}.$$

The equation (4) shews that the initial form of the string is that of a catenary, unless $a = 0$, or $cc'' = c'^2$, in which case $\rho = \infty$, and its form is rectilinear.

(8) A given inelastic mass is let fall from a given height on one scale of a balance, and two inelastic masses are let fall from different heights on the other scale, so that the three impacts take place simultaneously: to find the relations between the masses and heights in order that the balance may remain permanently at rest.

If m be the given mass, m', m'', the other two masses, and h, h', h'', respectively, the three heights, then

$$m : m' : m'' :: h'^{\frac{1}{2}} - h''^{\frac{1}{2}} : h^{\frac{1}{2}} - h''^{\frac{1}{2}} : h'^{\frac{1}{2}} - h^{\frac{1}{2}}.$$

(9) An inelastic sphere (A) slides down an inclined plane and comes into contact with an equal inelastic sphere (B) lying on a horizontal plane and touching the inclined plane: to determine the velocities of the two spheres just after collision, and the angular velocity of the line joining the centres of A and B the instant before its becoming horizontal.

Let $l =$ the length of the portion of the inclined plane down which the sphere A has slid from rest before collision with B, $\alpha =$ the inclination of the plane, $r =$ the radius of either sphere; let v, v', be the velocities of A, B, respectively, just after collision, and ω the required angular velocity of the distance between their centres: then

$$v = \frac{\cos^2\alpha}{1+\cos^2\alpha} \cdot (2gl\sin\alpha)^{\frac{1}{2}}, \quad v' = \frac{\cos\alpha}{1+\cos^2\alpha} \cdot (2gl\sin\alpha)^{\frac{1}{2}},$$

$$\omega^2 = \frac{g\sin^2\alpha}{2r^2} \cdot \frac{(l+2r)\cos^2\alpha + 2r}{(1+\cos^2\alpha)^2}.$$

(10) A uniform bar, moveable in a vertical plane about one of its ends, falls from a horizontal position and strikes a perfectly elastic ball: to determine the greatest velocity which it can communicate to the ball, and to find the position in which the ball must be struck to receive this velocity.

Let a denote the length of the bar, m, m', the masses of the bar and ball respectively; then the greatest velocity which can be communicated to the ball is equal to $\left(ga \cdot \frac{m}{m'}\right)^{\frac{1}{2}}$; and, in order to acquire this velocity, the ball must be placed at a distance, vertically below the point of suspension, equal to $a\left(\frac{m}{3m'}\right)^{\frac{1}{2}}$.

(11) Two equal inelastic balls, connected by a rigid rod, without weight, in the line of their centres, slide along a smooth vertical and a smooth horizontal plane, to the intersection of which the rod is perpendicular: if the lower ball impinge directly upon an equal ball at rest, to determine the angular velocity of the rod just after the collision, its angular velocity and position just before collision being known.

If α denote the inclination, and ω the angular velocity of the rod just before the collision, its angular velocity just afterwards will be equal to

$$\frac{\omega}{1+\sin^2\alpha}.$$

(12) A rectangular door, which is open, is struck perpendicularly at the outer edge by a body, the mass of which is one third of that of the door, and which is moving with a given velocity: to determine the velocity communicated to the outer edge of the door, the hinges being supposed smooth and the colliding bodies inelastic.

The outer edge of the door will acquire a velocity equal to half that of the impinging body.

(13) A slender homogeneous rod lies on a smooth horizontal plane: it is divided into two portions by a joint at its middle point: it is set in motion by a blow at one extremity perpendicular to its length: to compare the initial velocity of the middle point of the rod with that which it would have had supposing the rod had not been jointed.

The velocity of the middle point of the rod, when jointed, is twice as great as if it had been left in one piece, and in the opposite direction.

(14) One extremity of a beam, placed upon a smooth horizontal plane, is fixed: a ball A is placed on the plane in contact with the beam at a given distance from the fixed extremity: to determine at what point of the beam, on the other side of it, another ball A' must impinge directly, so that the greatest possible velocity may be communicated to A by the impact, the beam and balls being inelastic.

Let a, a', be the distances, at the instant of impact, of A, A', respectively, from the fixed end of the beam; m, m', the respective masses of A, A'; and μk^2 the moment of inertia of the beam about its fixed end. Then

$$a' = \left\{\frac{1}{m'}(ma^2 + \mu k^2)\right\}^{\frac{1}{2}}.$$

O'Brien and Ellis: *Solutions of the Senate-House Problems for* 1844.

(15) A rectangular door of weight W, initially open and at rest, is closed by means of a weight W', suspended at one end

of a cord, which passes over a pully at the edge of the door when shut: the cord winds on and off the arc of a horizontal circle the radius of which is equal to the breadth of the door: to determine the angular velocity of the door the instant before and the instant after its being shut.

Let α = the inclination of the initial position of the door to its position when shut, b = the breadth of the door; and let ω, ω', be respectively the required angular velocities: then

$$\omega^2 = \frac{6W'g\alpha}{b(W+3W')}, \quad \omega' = \frac{W-3W'}{W+3W'} \cdot \omega.$$

If $W = 3W'$, it appears that the door will remain at rest when closed.

(16) Two straight rods ACB, CD, of equal thickness and density, lie on a smooth horizontal plane at right angles to each other, the end C of the latter being in contact with the former: to determine the point at which ACB may be struck by a blow without consequent rotation.

Let $AC = a$, $BC = b$, $CD = c$. Of the two parts of the rod ACB let AC be the greater: then the required point of impact lies in CA at a distance from C equal to

$$\frac{a^2 - b^2}{2(a+b+c)}.$$

(17) Two equal smooth uniform rods, one end of the one being freely jointed to one end of the other, are laid upon a smooth horizontal plane: to find the point at which either must be struck, in order that the system may begin to move as if it were rigid.

The blow must be impressed at a distance from the joint equal to $a \sin^2 \alpha$, $2a$ being the length of either rod, and 2α the angle between them.

(18) Four equal rods are at rest, freely jointed together in the form of a square: a blow is struck at one corner in the direction of one of the sides: to compare the initial velocities of the centres of the four rods.

The required initial velocities are in the proportion of 2, 5, 2, -1.

(19) A rectangle, formed of four uniform rods, which are connected by hinges at their ends, is revolving about its centre on a smooth horizontal plane with a given angular velocity, when a point in one of the sides suddenly becomes fixed: to find the angular velocity of either of the sides adjacent to the side with the fixed point, immediately after it becomes fixed.

If $2a$ be the length of the side of which a point becomes fixed, $2b$ the length of an adjacent side, and ω the given angular velocity of the rectangle, the required angular velocity is equal to

$$\frac{3a+b}{6a+4b}\cdot\omega.$$

(20) A perfectly inelastic and smooth ellipsoid the semi-axes of which are a, b, c, revolving with an angular velocity ω round one axis c, impinges with a velocity v upon a quiescent sphere of equal magnitude: the instant before collision, the semi-axis a lies in the direction of the motion of the centre of gravity of the ellipsoid: at the instant of impact, the sphere touches the ellipsoid at the extremity of the latus rectum of its principal section containing a and b: supposing the eccentricity of that principal section to be equal to $\sqrt{\frac{2}{3}}$, to determine the relation between v and ω in order that there may be no rotatory motion in the ellipsoid after collision.

The required relation is

$$\frac{v}{\omega}=2a.$$

SECT. 4. *Rough Surfaces.*

(1) An inelastic cylinder O (fig. 248) having rolled down a perfectly rough plane CA, impinges upon a perfectly rough plane $C'A$, the axis of the cylinder being parallel to the

intersection of the two planes: to find the velocity with which the cylinder will commence its ascent up the second plane, and the limiting angle of inclination of the two planes for which the ascent is possible.

Let $\angle CAC' = \alpha$, $k =$ the radius of gyration of the cylinder about its axis, $a =$ the radius of the cylinder, $m =$ its mass; $u =$ the velocity of the centre O of a circular section of the cylinder just before impact, and $v =$ the velocity after impact up the plane AC'; $R =$ the impulsive force of friction exerted by the plane AC' upon the cylinder at the moment of impact to secure perfect rolling.

Then, for the motion of the centre of gravity of the cylinder parallel to AC', we have

$$mv = R - mu \cos \alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

and, for the value of the decrement of the angular velocity of the cylinder about its axis owing to the impulse R, we have the expression

$$\frac{Ra}{mk^2},$$

which, by (1), is equal to

$$\frac{a(v + u \cos \alpha)}{k^2}:$$

but, the planes being both perfectly rough, it is evident that $\frac{u}{a}$ is the angular velocity before, and $\frac{v}{a}$ after the impact: hence

$$\frac{u}{a} - \frac{v}{a} = \frac{a(v + u \cos \alpha)}{k^2};$$

$$u - v = \frac{a^2}{k^2}(v + u \cos \alpha):$$

but $a^2 = 2k^2$: hence

$$u - v = 2v + 2u \cos \alpha;$$

and therefore, $\qquad v = \frac{1}{3}(1 - 2 \cos \alpha)\, u,$

which gives the velocity with which the cylinder begins to ascend the plane AC'.

Since, from the nature of the case, v cannot have a negative value, it is clear that the ascent is impossible unless α be greater than $\frac{1}{8}\pi$.

(2) An inelastic cylinder rolls without sliding along a plane, and impinges upon a perfectly rough fixed point, the circular section of the cylinder through the rough point being supposed to bisect the axis of the cylinder: to determine the least distance of the point from the plane in order that the cylinder may be reduced to rest by the impact.

Let ω be the angular velocity of the cylinder just before and ω' just after impact; the cylinder being supposed to turn over the fixed point C, (fig. 249). Then, a being the radius of the cylinder, the centre O of a transverse section will have a horizontal velocity $a\omega$ before impact, and a velocity $a\omega'$, at right angles to the radius CO, just after impact. Let c denote the distance of C from the plane on which the cylinder is rolling, R the normal and S the tangential reaction at C.

Then, for the motion of translation at right angles to CO, we have

$$ma\omega' = ma\omega . \frac{a-c}{a} + S \text{..................}(1),$$

and, for rotation about O, there is

$$\frac{1}{2} ma^2\omega' = \frac{1}{2} ma^2\omega - Sa \text{..................}(2).$$

From (1) and (2) we have

$$\omega' = \omega . \frac{3a-2c}{3a}.$$

This result shews that the cylinder will roll over the fixed point if c be less than $\frac{3}{2}a$.

The following is a different solution of the same problem.

The motion the instant before impact is made up of two

motions, the one of translation, the velocity being $a\omega$, and the other of rotation, the angular velocity being ω.

Let P be any point in the area of the circular section through C; let $OP=r$, and let $\theta=$ the inclination of OP to CO, and ϕ its inclination to the horizontal line AB. Then the moment of the momentum of the circular section about C, due to the rotation, is equal to

$$\int_0^a\int_0^{2\pi} \rho r\,d\theta\,dr\,.\,r\omega\,.\,(r+a\cos\theta)$$

$$=\rho\omega\int_0^a\int_0^{2\pi} r^2 d\theta\,dr\,(r+a\cos\theta)$$

$$=\rho\omega a^4\int_0^{2\pi} d\theta\left(\frac{1}{4}+\frac{1}{3}\cos\theta\right)=\frac{1}{2}\pi\rho\omega a^4.$$

The moment of the momentum, due to translation, is equal to

$$\int_0^a\int_0^{2\pi} a\omega\rho r\,d\phi\,dr\,(r\sin\phi+a-c)$$

$$=a^3\omega\rho\int_0^{2\pi} d\phi\left\{\frac{1}{3}a\sin\phi+\frac{1}{2}(a-c)\right\}$$

$$=\pi\rho\omega a^3(a-c).$$

Hence the whole moment is equal to

$$\frac{1}{2}\pi\rho\omega a^3(3a-2c),$$

which will not be positive unless c be less than $\frac{3}{2}a$, that is, if the cylinder be reduced to rest, c will be not less than $\frac{3}{2}a$.

(3) A ball, sliding without rotation along a smooth horizontal plane, impinges obliquely against a perfectly rough vertical plane: to determine the subsequent motion of the ball.

Let Ox, Oy, Oz, (fig. 250), be three rectangular axes, the plane xOy being horizontal and passing through the centre C of the ball, and the plane xOz being the rough vertical plane against which the ball impinges. Let E be the point at which the ball strikes against the vertical plane; CF the direction of

the motion of C before impact. Let u be the velocity of C before impact, α the inclination of CF to Ox; v_x, v_y, the resolved parts of the velocity of C parallel to Ox, Oy, after the impact; ω the angular velocity of the ball about C after impact; X, Y, the impulsive reactions of the rough plane along xO, and parallel to Oy, during impact; m the mass of the ball; a the radius of the ball, k the radius of gyration about a diameter.

First we will suppose the ball to be inelastic. For the motion of the ball after impact, we have

$$mv_x = mu\cos\alpha - X \qquad (1),$$

$$mv_y = Y - mu\sin\alpha \qquad (2),$$

$$mk^2\omega = Xa \qquad (3).$$

Now, the ball being perfectly inelastic, the velocity of C at right angles to the vertical plane will be destroyed by the impact, or $v_y = 0$; hence, from (2),

$$Y = mu\sin\alpha \qquad (4).$$

Also, the vertical plane being perfectly rough, the ball will roll without sliding after impact: hence $a\omega = v_x$, and therefore, from (1), (3), we get

$$k^2(mu\cos\alpha - X) = Xa^2,$$

$$X = \frac{mk^2u\cos\alpha}{a^2+k^2} \qquad (5).$$

Next, let us suppose the ball to be elastic, e denoting the elasticity; then, v_x', v_y', ω', denoting on the new supposition what v_x, v_y, ω, were taken to denote on the old one, we shall have

$$mv_x' = mu\cos\alpha - (1+e)X \qquad (6),$$

$$mv_y' = mu\sin\alpha - (1+e)Y \qquad (7),$$

$$mk^2\omega' = (1+e)Xa \qquad (8).$$

From the equations (5) and (6), we obtain

$$v_x' = u\cos\alpha - \frac{(1+e)k^2u\cos\alpha}{a^2+k^2} = \frac{a^2-ek^2}{a^2+k^2}u\cos\alpha;$$

from (4) and (7),

$$v_y' = u\sin\alpha - (1+e)u\sin\alpha = -eu\sin\alpha;$$

and, from (5), (8),

$$\omega' = \frac{(1+e)\,au\cos\alpha}{a^2+k^2};$$

which values of v_x', v'_y, ω', completely determine the subsequent motion of the ball.

(4) A perfectly rough sphere is placed upon a perfectly rough horizontal plane which is made to rotate with a uniform angular velocity about a vertical axis: to determine the path described by the sphere in space.

Let Oz (fig. 251) be the vertical axis about which the plane revolves, O being a point in the plane; let Ox, Oy, be any two horizontal lines fixed in space and at right angles to each other; P the point of contact of the sphere with the revolving plane at any time t; draw PM parallel to yO. Let $OM = x$, $PM = y$; a = the radius of the sphere, m = its mass, mk^2 = its moment of inertia about a diameter; ω = the angular velocity of the revolving plane about Oz, the motion being supposed to take place in the direction indicated by the arrow in the plane xOy in the figure; let X, Y, denote the resolved parts of the friction exerted by the plane on the sphere, estimated parallel to Ox, Oy, respectively; ω_x, ω_y, the angular velocities of the sphere about diameters parallel to Ox, Oy, the directions of these velocities being estimated in the manner indicated by the arrows in the planes yOz, zOx.

For the motion of the centre of gravity of the sphere we have

$$m\frac{d^2x}{dt^2} = X \quad\text{..........................(1)},$$

$$m\frac{d^2y}{dt^2} = Y \quad\text{..........................(2)};$$

and, for the rotation of the sphere,

$$mk^2\frac{d\omega_x}{dt} = Ya \quad\text{..........................(3)},$$

$$mk^2\frac{d\omega_y}{dt} = -Xa \quad\text{..........................(4)}.$$

From (1) and (4) we have, eliminating X,

$$a\frac{d^2x}{dt^2} = -k^2\frac{d\omega_y}{dt} \quad\text{....................(5)}$$

and, from (2), (3), eliminating Y,

$$a\frac{d^2y}{dt^2}=k^2\frac{d\omega_x}{dt}\text{ .... (6).}$$

Integrating the equations (5), (6), and adding arbitrary constants, we have

$$a\frac{dx}{dt}=C-k^2\omega_y\text{ .... (7),}$$

$$a\frac{dy}{dt}=C'+k^2\omega_x\text{ .... (8).}$$

Now the linear velocity of the centre of gravity of the sphere relatively to the rough plane, in consequence of the rolling of the sphere, will be

$$a\omega_y \text{ parallel to } Ox, \quad -a\omega_x \text{ parallel to } Oy;$$

and the linear velocity due to the rotation of the rough plane will be

$$-\omega y \text{ parallel to } Ox, \quad \omega x \text{ parallel to } Oy:$$

hence, $\frac{dx}{dt}, \frac{dy}{dt}$, being the whole linear velocity of the centre of the sphere parallel to Ox, Oy, respectively, we have

$$\frac{dx}{dt}=-\omega y+a\omega_y, \quad \frac{dy}{dt}=\omega x-a\omega_x;$$

and therefore, by the aid of (7) and (8), eliminating ω_x and ω_y,

$$a\frac{dx}{dt}=C-\frac{k^2}{a}\left(\omega y+\frac{dx}{dt}\right), \quad \left(1+\frac{a^2}{k^2}\right)\frac{dx}{dt}=\frac{aC}{k^2}-\omega y,$$

$$a\frac{dy}{dt}=C'+\frac{k^2}{a}\left(\omega x-\frac{dy}{dt}\right), \quad \left(1+\frac{a^2}{k^2}\right)\frac{dy}{dt}=\frac{aC'}{k^2}+\omega x:$$

eliminating t between these two equations, we get

$$(aC-k^2\omega y)\,dy=(aC'+k^2\omega x)\,dx:$$

integrating and adding an arbitrary constant C'', we obtain

$$2aCy-k^2\omega y^2=2aC'x+k^2\omega x^2+C'',$$

or

$$x^2+y^2+\frac{2aC'}{k^2\omega}x-\frac{2aC}{k^2\omega}y+\frac{C''}{k^2\omega}=0\text{ .... (9).}$$

We proceed now to the determination of the arbitrary constants. Let the initial distance of the centre of the sphere from the axis of z be b, and let the axis of x be so chosen as to pass through the initial position of the point of contact of the sphere with the rough plane. Then, since the initial impulse of the friction of the revolving plane upon the sphere is at right angles to the axis of x, we shall have initially $\frac{dx}{dt}=0$, $\omega_y=0$: hence, from (7), we see that $C=0$. Again, F denoting the impulse of the friction when the sphere is just placed upon the revolving plane, and $\left(\frac{dy}{dt}\right)$, (ω_x), denoting the values of $\frac{dy}{dt}$, ω_x, just after the impulse, we shall have

$$m\left(\frac{dy}{dt}\right)=F \qquad\qquad (10),$$

$$mk^2(\omega_x)=Fa \qquad\qquad (11).$$

But, since there is no sliding between the sphere and the plane, it is clear that

$$\left(\frac{dy}{dt}\right)=b\omega-a(\omega_x) \qquad\qquad (12),$$

where $b\omega$ is the velocity of the centre of the sphere parallel to Ox due to the rotation of the plane, and $-a(\omega_x)$ the velocity estimated in the same direction due to the rolling of the sphere along the plane: hence from (10) we have

$$m\{b\omega-a(\omega_x)\}=F,$$

and therefore, by (11),

$$a\{b\omega-a(\omega_x)\}=k^2(\omega_x), \qquad (\omega_x)=\frac{ab\omega}{a^2+k^2};$$

and then, by (12),

$$\left(\frac{dy}{dt}\right)=b\omega-\frac{a^2b\omega}{a^2+k^2}=\frac{k^2b\omega}{a^2+k^2}.$$

But from (8) we have

$$a\left(\frac{dy}{dt}\right)=C'+k^2(\omega_x):$$

hence, putting for $\left(\frac{dy}{dt}\right)$ and (ω_x) their values,

$$\frac{abk^2\omega}{a^2+k^2} = C' + \frac{abk^2\omega}{a^2+k^2}, \quad C' = 0.$$

Since $C = 0$ and $C' = 0$, we have, from (9),

$$x^2 + y^2 + \frac{C''}{k^2\omega} = 0:$$

but $x = b$ when $y = 0$, and therefore

$$b^2 + \frac{C''}{k^2\omega} = 0,$$

and we get for the equation to the path of P,

$$x^2 + y^2 = b^2,$$

the equation to a circle having O for its centre.

The following elegant solution of this problem was communicated to me by the late Robert Leslie Ellis:

"A sphere, resting on a perfectly rough horizontal plane, receives a tangential impulse when the plane is made to move in its own plane. This impulse gives a velocity to the centre of the sphere and produces an angular velocity about a horizontal axis. The centre of the sphere moves parallel to the impulse, the axis of rotation is perpendicular to it; therefore the point of contact moves parallel to the impulse and therefore to the direction of motion of the centre. Therefore, as there is no sliding, the centre moves in the same direction as that of the motion of the plane supposed rectilineal. Moreover it is easily seen that the velocity of the centre is to that of the point of contact, or, which is the same thing, to that of the plane, as $1 : 1 + \frac{a^2}{k^2}$, a being the radius of the sphere, k its least radius of gyration. While the direction and velocity of the plane's motion remain unaltered, no farther action occurs; when a change takes place, a new tangential impulse is given to the sphere, producing a new velocity of the centre parallel

to its own direction, and a new velocity of rotation about an axis at right angles to it. The new velocity of rotation bearing to the old the same ratio as the new velocity of the centre to the old, the result is a compound velocity of the centre bearing the same ratio as before to the velocity of the point of contact, and as before parallel to it, and therefore still parallel to the direction of motion of the plane; and so on; whether the motion of the plane varies continuously or discontinuously, in direction, in velocity, or in both. In the case proposed the motion of the plane (by which throughout I mean the element thereof in contact with the sphere) is always normal to a line drawn to a fixed point. Therefore the motion of the centre is so too, therefore the centre describes a circle whose centre is perpendicularly over the said fixed point. Q.E.D."

(5) An inelastic homogeneous cylinder, the axis of which is horizontal, rolls down a perfectly rough inclined plane which terminates in a perfectly rough horizontal plane: to find the velocity of the cylinder along the horizontal plane, and the blow which it receives when it first impinges upon it.

Let $\alpha =$ the inclination of the inclined plane to the horizon, $m =$ the mass of the cylinder, $u =$ the velocity of its axis the instant before and u' the instant after impact, $F =$ the initial impulse of the friction of the horizontal plane, estimated in the direction of the sphere's motion along it, and $B =$ the normal impulse of the horizontal plane; then

$$u' = \tfrac{1}{3}(1 + 2\cos\alpha)\,u, \quad F = \tfrac{1}{3}mu(1 - \cos\alpha), \quad B = mu\sin\alpha.$$

(6) A homogeneous cylinder, the axis of which is horizontal, slides, without rolling, down an inclined plane which is for a certain space quite smooth, and, after acquiring a given velocity, is suddenly caused by the roughness of the surface to roll without sliding: to determine the velocity of the axis of the cylinder the instant rolling commences, and to find the initial impulse of friction.

If u be the velocity of the cylinder the instant before and u' the instant after the commencement of perfect rolling,

m the mass of the cylinder, F the initial impulse of friction; then

$$u' = \tfrac{2}{3} u, \quad F = \tfrac{1}{3} mu.$$

(7) Two wheels, revolving uniformly in the same plane, about axes, perpendicular to the plane, through their centres, are suddenly brought into contact, and their axes are kept fixed: to determine what alteration will take place in their angular velocities, the friction being sufficient to prevent all sliding.

Let $m =$ the mass of one wheel, $k =$ its radius of gyration about its axis of rotation, $a =$ its radius; let α be its angular velocity before and α' after collision. Let n, l, b, β, β', denote like quantities in relation to the other wheel. Then, the revolutions of the two wheels being supposed opposite in character before being brought into contact,

$$\alpha' - \alpha = nal^2 \cdot \frac{b\beta - a\alpha}{mb^2k^2 + na^2l^2},$$

and
$$\beta' - \beta = mbk^2 \cdot \frac{a\alpha - b\beta}{na^2l^2 + mb^2k^2}.$$

(8) A book $ABCD$ is placed, in a vertical plane, with one angle A on a table: to find the greatest ratio which the side BC can bear to the side AB in order that, after the impact, the book may not tilt over the angle B, the table being supposed to be perfectly rough and the book to be inelastic.

The ratio of BC to AB cannot possibly be greater than $\dfrac{1}{\sqrt{2}}$: how much less the ratio should be is indeterminate, being dependent upon the physical nature of the contact between the side AB of the book and the table.

(9) A spherical ball of given elasticity, moving with a given velocity, and revolving uniformly round a horizontal axis through its centre and perpendicular to the plane of the motion of its centre, impinges upon a horizontal plane of such a nature as to prevent all sliding: to determine whether the angle of reflection from the plane is increased or diminished by increasing the velocity of rotation before impact; and to

find how many revolutions the ball will make after impact before it again strikes the plane.

Let the angles of incidence and reflection be α, α', respectively, and conceive the rotation to be estimated in the direction indicated by the arrows in the diagram: fig. (252): let $r =$ the radius of the ball: let u, v, be the components of the velocity of incidence, parallel and perpendicular to the plane, and ω the angular velocity, the instant before impact.

If ω, being positive, be increased, α' will increase. If ω be negative, or the rotation of an opposite character to that indicated in the figure, then α' will decrease as ω increases: if

$$\omega = -\frac{5v}{2r}, \quad \alpha' = 0,$$

or the ball will rebound in the normal: if ω be a greater negative quantity than $-\frac{5v}{2r}$, α' will be negative or the angle of reflection will be on the same side of the normal as the angle of incidence.

The required number of revolutions will be equal to

$$\frac{4\pi eu}{g} \cdot \frac{7r}{2\omega r + 5v}.$$

(10) An inelastic rod rests in a horizontal position on two perfectly rough pegs equidistant from its centre of gravity: if it be turned about one of them, in the vertical plane in which they are situated, and then allowed to fall, to determine whether its motion will cease or not after impact.

Its motion will cease or not as the distance between the pegs is greater or less than $\frac{a}{\sqrt{3}}$, where a is the length of the rod.

Griffin: *Solutions of the Examples on the Motion of a Rigid Body*, p. 102.

(11) A perfectly rough plane, moving with a certain velocity parallel to four of the edges of a rigid inelastic cube placed upon it, is suddenly brought to rest: to determine the velocity in order that the cube may just turn over its edge.

If c = the length of each edge, and v = the required velocity,

$$v^2 = \tfrac{8}{3}(\sqrt{2} - 1)\, cg.$$

(12) A perfectly rough cube rests with one of its faces on a perfectly rough rectangular board, which rests on a smooth horizontal plane, the centre of the base of the cube coinciding with that of the board and the edges of the face being parallel to those of the board: a blow is applied to the board at the middle point of one of its edges in a direction perpendicular to the nearest vertical face of the cube: to find the impulsive stress between the board and the cube, and their motion just after the application of the blow.

Let m be the mass of the cube, m' of the board, B the blow, a the length of an edge of the cube. The horizontal and vertical components of the impulsive stress are equal respectively to

$$\frac{5mB}{5m + 8m'}, \quad \frac{3mB}{5m + 8m'};$$

the board's instantaneous velocity is equal to

$$\frac{8B}{5m + 8m'},$$

while the cube revolves for an instant, relatively to the board, about the lower edge nearest the point of impact with an angular velocity equal to

$$\frac{6B}{a\,(5m + 8m')}.$$

(13) A homogeneous sphere, rotating about a horizontal diameter, falls upon a perfectly rough inclined plane through such a height that its angular velocity is not affected by the first impact, and then proceeds to descend the plane directly by bounds: to find the velocity of the sphere along the plane just after the n^{th} impact, and to determine the range which the sphere describes upon the plane before it ceases to hop.

Let α be the inclination of the plane to the horizon, h the height through which the sphere falls, e its elasticity: then the required velocity and range are respectively equal to

$$(2gh)^{\frac{1}{2}} . \sin\alpha . \left(1 + \frac{10}{7} \cdot \frac{e - e^2}{1 - e}\right),$$

$$\frac{4eh \sin\alpha}{(1-e)^2} \cdot \left(1 - \frac{4}{7} \cdot \frac{e^2}{1+e}\right).$$

(14) An imperfectly elastic homogeneous rough sphere is projected obliquely, without rotation, against a fixed plane: to determine ρ, the ratio of the tangential forces of restitution and compression, in terms of α, α', the angles of incidence and reflection, and e, the coefficient of elasticity for direct impact.

The value of ρ is given by the equation

$$2\rho = 5 - 7e \tan\alpha' \cot\alpha.$$

Ferrers and Jackson: *Solutions of the Cambridge Problems*, 1848 to 1851.

(15) A series of perfectly rough semicylinders are fixed, side by side, upon their flat faces directly across a straight road of constant inclination: to determine the inclination of the road in order that a rough circular inelastic hoop, just started downwards from the summit of one of the cylindrical ridges, may travel directly along the road with a uniform mean velocity.

Let a = the radius of the hoop, a_1 = that of one of the cylinders, β = the inclination of the road: then, α being given by the formula

$$\sin\alpha = \frac{a_1}{a + a_1},$$

β is determined by the equation

$$\cos^2\alpha = \frac{\sin\dfrac{\alpha - \beta}{2}}{\sin\dfrac{\alpha + \beta}{2}}.$$

Mackenzie and Walton: *Solutions of the Cambridge Problems for* 1854.

CHAPTER XIII.

LIVE THINGS.

(1) A FLEA is resting on a needle AB at a given point E: the needle lies on a smooth table: the flea then skips to a given point F of the needle: to determine the least initial velocity of the flea.

Let V be the velocity with which the flea skips, α the inclination of V to the horizon, u the velocity of the needle during the flight of the flea, t the time of flight, m, m', the masses of the needle and flea respectively, and let $EF = c$.

Then, since the centre of gravity of the flea and needle will not be affected by the skip,

$$-mu + m'V\cos\alpha = 0 \quad\text{.....................}(1).$$

Also
$$V\sin\alpha = \tfrac{1}{2}gt \quad\text{..................................}(2).$$

Now the whole range of the flea is equal to the distance EF diminished by the space through which F has slid backwards during the time of flight; and therefore

$$\frac{V^2\sin 2\alpha}{g} = c - ut = c - \frac{m'}{m}V\cos\alpha\,.\,t, \text{ by (1)},$$

$$= c - \frac{m'V^2\sin 2\alpha}{mg}, \text{ by (2)}:$$

hence
$$V^2 = \frac{mcg}{m+m'}\,.\,\text{cosec}\,2\alpha.$$

The least possible value of V is therefore equal to

$$\left(\frac{mcg}{m+m'}\right)^{\frac{1}{2}}.$$

(2) A beetle, placed upon a moveable inclined plane, which rests upon a smooth horizontal plane, sets off to crawl up it at a given uniform velocity relatively to the inclined plane: to determine the velocity of the plane and the pressure exerted upon it by the beetle.

Let P (fig. 253) be the position of the beetle at any time on the inclined plane AB; Ox the smooth horizontal plane; let $\angle BAx = \alpha$, $u =$ the uniform relative velocity of the beetle; $OA = x$, $V = \frac{dx}{dt}$; let N, T, be the impulsive and N', T', the finite actions between the plane and beetle, m, m', being their respective masses.

Since the centre of gravity of the beetle and plane can have no horizontal motion, we have

$$mV + m'(V + u\cos\alpha) = 0,$$

and therefore

$$V = -\frac{m'u\cos\alpha}{m+m'},$$

which shews that the plane travels in the direction xO with a velocity equal to

$$\frac{m'u\cos\alpha}{m+m'}.$$

Again, for the impulsive actions, resolving parallel and perpendicularly to the plane,

$$T = m'(u + V\cos\alpha)$$

$$= m'u \cdot \frac{m + m'\sin^2\alpha}{m+m'},$$

and

$$N = -m'V\sin\alpha$$

$$= \frac{m'^2 u\sin\alpha\cos\alpha}{m+m'}.$$

For the finite actions, since the beetle has no acceleration,

$$T' = m'g\sin\alpha, \qquad N' = m'g\cos\alpha.$$

(3) A monkey is put at the top of a pole, the lower end of which is placed on a smooth horizontal plane, its higher resting against a smooth vertical wall: the monkey contrives to clamber down the pole in such a manner that the pole remains quiescent: to determine the velocity of the monkey when he gets to the lower end of the pole.

Let P (fig. 254) be the position of the monkey on the pole AB at any time t after the commencement of the motion: let m, m', be the masses of the pole and monkey respectively; let R, S, be the pressures exerted by the two planes on the ends of the pole; let N, T, be the normal and longitudinal actions between the pole and the monkey; let $\alpha =$ the inclination of AB to the vertical, $AB = 2a$, $AP = x$.

Then, for the motion of the monkey along the pole,

$$m' \frac{d^2x}{dt^2} = T + m'g \cos \alpha \dots\dots\dots\dots\dots\dots (1),$$

and, for the preservation of its contact with the pole,

$$N = m'g \sin \alpha \dots\dots\dots\dots\dots\dots\dots\dots (2).$$

For the equilibrium of the pole, resolving vertically,

$$S + T \cos \alpha = N \sin \alpha + mg \dots\dots\dots\dots\dots\dots (3),$$

and, taking moments about A,

$$Nx + mga \sin \alpha = 2aS \sin \alpha \dots\dots\dots\dots\dots\dots (4).$$

From (2), (3), (4), we may readily ascertain that

$$T = m'g \frac{\sin^2 \alpha}{\cos \alpha} - \frac{m'gx}{2a \cos \alpha} + \frac{mg}{2 \cos \alpha}.$$

Substituting this value of T in (1), we shall get

$$\frac{d^2x}{dt^2} + \frac{gx}{2a \cos \alpha} = \frac{g(m + 2m')}{2m' \cos \alpha} \dots\dots\dots\dots\dots\dots (5):$$

multiplying by $2\frac{dx}{dt}$ and integrating, we have, observing that $\frac{dx}{dt} = 0$ when $x = 0$,

$$\frac{dx^2}{dt^2} = \frac{gx}{2\cos\alpha}\left\{\frac{2}{m'}(m+2m') - \frac{x}{a}\right\} \text{.....................(6)}.$$

When therefore $x = 2a$, or the monkey arrives at B, his velocity is equal to

$$\left\{\frac{2ga(m+m')}{m'\cos\alpha}\right\}^{\frac{1}{2}}.$$

(4) A needle is suspended from one extremity; at the point of suspension there is a spider, the mass of which is equal to that of the needle: supposing the needle to be placed horizontally, and then to be projected downwards with a given angular velocity, to determine how long the spider will take to run to the lower end of the needle, the motion of the spider being such that the angular velocity of the needle may remain unchanged.

Let m = the mass of the needle or spider; let P = the place of the spider at any time on the needle CA (fig. 255), C being the point of suspension; let N, T, be the normal and longitudinal actions between the needle and spider; draw Cx vertically, and let $CP = x$, $CA = 2a$, $\angle ACx = \theta$; let ω = the angular velocity of the needle.

Then, since the needle's angular velocity is to be constant, we have

$$mga\sin\theta + Nx = 0 \text{............................ (1)}.$$

Again, for the motion of the spider,

$$m\frac{d^2}{dt^2}(x\cos\theta) = mg - N\sin\theta - T\cos\theta \text{ ...... (2)},$$

and $$m\frac{d^2}{dt^2}(x\sin\theta) = N\cos\theta - T\sin\theta \text{............ (3)}.$$

From (2) and (3) we see that

$$m\left\{x\sin\theta\frac{d^2}{dt^2}(x\cos\theta) - x\cos\theta\frac{d^2}{dt^2}(x\sin\theta)\right\} = mgx\sin\theta - Nx:$$

but $$x\sin\theta\frac{d^2}{dt^2}(x\cos\theta) - x\cos\theta\frac{d^2}{dt^2}(x\sin\theta)$$

$$= \frac{d}{dt}\left\{x \sin\theta \frac{d}{dt}(x\cos\theta) - x\cos\theta\frac{d}{dt}(x\sin\theta)\right\}$$

$$= -\frac{d}{dt}\left(x^2\frac{d\theta}{dt}\right):$$

hence
$$m\frac{d}{dt}\left(x^2\frac{d\theta}{dt}\right) = Nx - mgx\sin\theta,$$

and therefore, by (1), observing that $\frac{d^2\theta}{dt^2} = 0$ by the hypothesis,

$$2x\frac{dx}{dt}\frac{d\theta}{dt} = -g(x+a)\sin\theta:$$

but $\theta = \frac{1}{2}\pi - \omega t$, by the hypothesis: hence

$$2\omega x\frac{dx}{dt} = g(x+a)\cos\omega t,$$

$$\cos\omega t\,.\,dt = \frac{2\omega}{g}\cdot\frac{xdx}{x+a}:$$

integrating and bearing in mind that $x = 0$ when $t = 0$,

$$\frac{1}{\omega}\sin\omega t = \frac{2\omega}{g}\left\{x - a\log\frac{x+a}{a}\right\},$$

and therefore, for the required time,

$$\sin\omega t = \frac{2\omega^2}{g}(2a - a\log 3),$$

$$t = \frac{1}{\omega}\sin^{-1}\left\{\frac{2\omega^2 a}{g}(2 - \log 3)\right\}.$$

This problem may be solved also more briefly thus:

By D'Alembert's Principle we have, taking moments about the point of suspension for the system composed of the spider and needle,

$$\frac{d}{dt}\left(mx^2\frac{d\theta}{dt} + \frac{4}{3}ma^2\frac{d\theta}{dt}\right) = -mg\sin\theta(x+a),$$

and therefore, replacing $\frac{d\theta}{dt}$ by its constant value $-\omega$,

$$2\omega x\frac{dx}{dt} = g(x+a)\cos\omega t,$$

the differential equation obtained by the former method.

(5) A fly is standing upon a needle, which rests upon a smooth horizontal plane: supposing the fly to walk along the needle to its other end, how far would the needle be displaced?

If c be the length of the needle, and m, m', the masses of the needle and fly respectively, the displacement of the needle will be equal to

$$\frac{m'c}{m+m'}.$$

(6) Two beetles are standing upon the ends of a horizontal rod, which is supported on a smooth table: supposing one beetle to crawl to the middle of the rod, how far must the other crawl along it in order that the final position of the rod may be the same as its original one?

Let β be the mass of the former and β' of the latter beetle, and a the length of the rod. Then, if x be the distance the latter beetle should crawl,

$$x = \frac{a\beta}{2\beta'}.$$

(7) A circular plate is laid upon a smooth table; a snail is placed upon the plate at a given distance from the centre of the plate: supposing the snail to crawl along the plate in a circular path relatively to the plate's centre, to find the motion of the centre of the plate.

If m, m', denote the masses of the plate and snail respectively, and a the radius of the relative circle described by the snail, the centre of the plate will also describe a circle the radius of which is equal to

$$\frac{m'a}{m+m'}.$$

(8) A circular plate, moveable about a vertical axis through its centre, rests on a smooth table: an insect starts from the centre and crawls, relatively to the plate, in a circle, the diameter of which is a radius of the plate: to determine the angle through which the plate will have turned when the insect has returned to the centre.

Let m be the mass of the plate, μ of the insect: then the required angle is equal to

$$\pi\left\{1-\frac{m^{\frac{1}{2}}}{(m+2\mu)^{\frac{1}{2}}}\right\}.$$

(9) A circular disc, moveable about a vertical axis through its centre, rests on a smooth table: an equiangular spiral is traced on the disc, the centre being the pole of the spiral: an insect crawls along the curve, starting from the point at which it cuts the circumference: to determine the angle through which the disc will have revolved when the insect reaches the centre.

Let α be the angle of the spiral, m the mass of the disc, μ that of the insect: then the required angle is equal to

$$\frac{1}{2}\tan\alpha\log\left(1+\frac{2\mu}{m}\right).$$

(10) A uniform circular wire, moveable about a fixed point in its circumference, lies on a smooth horizontal plane: an insect, the mass of which is equal to that of the wire, crawls along it, starting from the point most distant from the fixed point, with a uniform velocity relatively to the wire: to find the angle through which the wire will have turned at the end of any time.

If u be the velocity of the insect relatively to the wire, the angle through which the wire will have turned at the end of any time t is equal to

$$\frac{ut}{2a}-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{ut}{2a}\right).$$

(11) A rigid needle of insensible mass, moveable about one end, is held horizontally, a fat fly resting upon it at a given distance from the fixed end: supposing the needle to be let go, and the fly to run towards the fixed end, so that the needle descends with a uniform angular acceleration, to determine the motion of the fly along the needle.

If r be the distance of the fly from the fixed end of the needle at any time t from the commencement of the motion, a being the initial value of r, then

$$2t\frac{dr}{dt}+r=a\cos\left(\frac{gt^2}{2a}\right).$$

(12) A uniform rod of mass m and length $2a$, which is inclined to the horizon at an angle α, is at rest, being supported by a fulcrum to which its middle point is fixed. A mouse of mass μ rests on the rod at the fulcrum. After a time the mouse begins to run along the rod with a velocity which is proportional to the tangent of the inclination, at any time, of the rod to the horizon, and, when it arrives at the end of the rod, the rod's inclination is β: to find the angular velocity of the rod when its angle of inclination is any angle θ between α and β.

The square of the required angular velocity is equal to

$$\frac{2g.\tan\beta.(\cos\alpha-\cos\theta)}{a\left(\frac{m}{3\mu}\tan^2\beta+\tan^2\theta\right)^2}.\left(\frac{m}{3\mu}\tan^2\beta-1+\sec\alpha\sec\theta\right).$$

(13) A monkey ascends a ladder with a uniform velocity, the lower end of the ladder being fixed, the higher resting against a vertical wall: to find the pressure of the ladder on the wall.

Let $2a$ denote the length of the ladder, α its inclination to the horizon, x the distance of the monkey, at any time, from the foot of the ladder, and R the corresponding pressure of the ladder on the wall; then, m, m', denoting the respective masses of the ladder and the monkey,

$$R=\frac{g\cot\alpha}{2a}(ma+m'x).$$

(14) Two little monkeys, the masses of which are m, m', cling to the two portions of a fine inextensible string which passes over a smooth fixed pully: the monkeys keep descending with the respective accelerations f, f': to find the tension of the string.

The required tension is equal to

$$\frac{mm'ff'}{m-m'}.$$

(15) A plank, upon which at the upper end a dog is standing, is placed directly along a smooth inclined plane: to determine how long it will take the dog to run down the plank, so that the plank may not stir till he is off it.

If l = the length of the plank, α = the inclination of the plane; then, m, m', being the masses of the plank and dog respectively, the required time is equal to

$$\left\{\frac{2m'l}{g\sin\alpha\,(m+m')}\right\}^{\frac{1}{2}}.$$

(16) A rope of inconsiderable weight is suspended over a smooth pully: two monkeys commence together clambering up, without jerking, the two portions of the rope, in such a manner that the rope does not slide over the pully, and that they both reach the pully at the same moment: to find the time of their ascent to the pully, their initial positions and their masses being given.

Let m, m', denote the masses of the monkeys, and a, a', their initial distances below the axis of the pully: then the required time is equal to

$$\left(\frac{2}{g}\right)^{\frac{1}{2}}.\left(\frac{ma-m'a'}{m'-m}\right)^{\frac{1}{2}}.$$

(17) A circular ring is placed vertically upon a perfectly rough table; and an earwig is put gently upon the ring: to investigate the angular velocity of the earwig in any position about the centre of the ring in order that the ring may not stir.

Let a be the radius of the ring, θ the inclination, to the horizon, of the earwig's distance from the ring's centre at any time, and let α be the initial value of θ. Then, ω being the angular velocity of the earwig in this position,

$$\omega^2=\frac{g}{a}.\frac{(\sin\alpha-\sin\theta).(2+\sin\alpha+\sin\theta)}{(1+\sin\theta)^2}.$$

(18) A perfectly rough circular hoop rolls with a uniform velocity directly up a perfectly rough inclined plane by the action of a mouse running along its circumference: to determine

the angular velocity of the mouse, in any position, about the centre of the hoop, the angular velocity of the mouse in a given position being known.

Let C (fig. 256) be the centre of the hoop at any time, H the point of contact of the hoop and the inclined plane AB, P the position of the mouse. Let $\angle PCH = \theta$, ω = the angular velocity of the mouse about C, m = the mass of the hoop, m' = the mass of the mouse, α = the inclination of AB to the horizon, a = the radius of the hoop: then

$$m'a\,(1-\cos\theta)^2\omega^2 = C + 2g\,(m+m')\sin\alpha\,(\theta - \sin\theta)$$
$$+ 2m'g\,\{\cos(\theta+\alpha) - \cos(2\theta+\alpha) + 2\theta\sin\alpha\},$$

C being a constant, which may be expressed in terms of given simultaneous values of ω and θ.

(19) A given hollow tube in the form of a common helix is fixed round a given quiescent cylinder, moveable about its axis, which is fixed and vertical: a given wasp flies to the lower end of the tube, where it gently alights: it then crawls through the tube and, on arriving at the upper end, gently flies away: prove that the subsequent angular velocity of the cylinder varies as the velocity of the wasp, relatively to the tube, at the instant before its final flight.

(20) A little squirrel clings to a thin rough hoop, the plane of which is vertical, and which is rolling along a perfectly rough horizontal plane: the squirrel makes a point of keeping at a constant altitude above the horizontal plane, and selects his place on the hoop so as to travel from a position of instantaneous rest the greatest possible distance in a given time: to find the inclination of the squirrel's distance from the centre of the hoop to the vertical.

If m be the mass of the squirrel and m' that of the hoop, the required inclination is equal to

$$\cos^{-1}\left(\frac{m}{m+2m'}\right).$$

(21) Two buckets of given weights are suspended by a fine inelastic string placed over a fixed pully: at the centre of the

base of one of the buckets a frog of given weight is sitting; at an instant of instantaneous rest of the buckets, the frog leaps vertically upwards so as just to arrive at the level of the rim of its bucket: to find the ratio of the absolute length of the frog's vertical ascent in space to the length of its bucket, and to ascertain the time which elapses before the frog again arrives at the base of its bucket.

If m denote the mass of the frog's bucket, m' that of the other bucket, μ that of the frog; h the length of the frog's bucket, h' that of its absolute ascent in space, and t the required time;

$$\frac{h'}{h} = \frac{2m'(m+m')}{(m+m'+\mu)^2}, \quad t^2 = \frac{4h}{m'g} \cdot (m+m').$$

(22) A small beetle, placed at an end of the horizontal diameter of a thin heavy motionless ring, which is moveable about its centre in a vertical plane, starts off to crawl round the ring, so as to describe in space equal angles in equal times about its centre: to determine its velocity, relatively to the ring, in any position, and its pressure on the ring.

Let P (fig. 257) be the position of the beetle at any time t after starting, O the centre of the ring, Ox a horizontal line. Let a = the radius of the ring, m = its mass, m' = the mass of the beetle, $\angle POx = \theta$, ω = the constant value of $\frac{d\theta}{dt}$. Let N, T, denote the normal and tangential pressures of the beetle on the ring.

Then the required relative velocity of the beetle is equal to

$$\frac{m'g}{m\omega} \cdot \sin \omega t + a\,\frac{m+m'}{m}\,\omega;$$

also $$N = m'(g \sin \omega t - a\omega^2), \qquad T = m'g \cos \omega t.$$

Mackenzie and Walton; *Solutions of the Cambridge Problems for* 1854.

(23) A given little animal clings at a given point to a uniform quiescent rod hanging by a smooth hinge at its upper end from the ceiling of a room: supposing the animal to make

a sudden spring to a bracket on the wall, in the same horizontal plane with the initial position of the animal and at a given distance from the rod, and that the angle through which the rod afterwards oscillates from the vertical is observed, to find what must have been the impulsive stress on the hinge at the instant when the animal started.

Let m, μ, be the respective masses of the rod and the animal; $2a$ the length of the rod, b the vertical height of the higher end of the rod above the bracket, c the initial distance of the bracket from the rod, and α the observed angle of oscillation. Then, X, Y, representing the horizontal and vertical impulses on the hinge,

$$X = \frac{m}{b}(4a - 3b)\left(\frac{ag}{3}\right)^{\frac{1}{2}} \sin\frac{\alpha}{2},$$

$$Y = \frac{bc\mu^2}{8ma \sin\frac{\alpha}{2}} \cdot \left(\frac{3g}{a}\right)^{\frac{1}{2}}.$$

(24) A rod AB (fig. 258), attached to a fixed horizontal rod Ox and a fixed vertical rod Oy by rings at its ends, is kept at rest by a man's hand, while a monkey is sitting upon a small platform C vertically above O: the monkey then springs horizontally from C and alights on the middle point G of the rod, to which it clings tightly: to determine the motion of the rod the instant after the monkey's arrival at G, supposing the man to have loosed his hold the instant before.

Let $AG = a = BG$, $\angle ABO = \theta$, $m =$ the mass of the moveable rod, $m' =$ the mass of the monkey, $u =$ the velocity with which the monkey springs from C, $h =$ the vertical altitude of C above G: then the angular velocity of AB, the instant after the monkey's arrival at G, is equal to

$$\frac{3m'}{a} \cdot \frac{(2gh)^{\frac{1}{2}} \cos\theta + u \sin\theta}{4m + 3m'}.$$

(25) A little animal is resting on the middle point of a thin uniform quiescent bar, the ends of which are attached by

small rings to two smooth fixed rods at right angles to each other in a horizontal plane: supposing the animal to start off along the bar with a given velocity, relatively to the bar, to find the angular velocity initially impressed upon the bar.

Let m be the mass of the animal, m' of the bar, $2a$ the length of the bar, θ the inclination of the bar to either rod, and V the given velocity of the animal: then the required angular velocity of the bar is equal to

$$\frac{3m}{3m+4m'} \cdot \frac{V \sin 2\theta}{a}.$$

CHAPTER XIV.

MISCELLANEOUS PROBLEMS.

(1) A PARTICLE, placed at a centre of attraction varying as the distance, is urged from rest by a constant force, which acts for one sixth of the time of a complete oscillation about the centre, ceases for the same period, and then acts as before: shew that the particle will then be retained at rest, and that the spaces moved through in the two periods are equal.

(2) A particle moves in a straight line under the action of a force directed towards a point in that line and varying inversely as the square of the distance from that point: if v, v', v'', be the velocities of the particle when at distances x, x', x'', from the centre of force, shew that

$$\frac{v'^2 - v''^2}{x} + \frac{v''^2 - v^2}{x'} + \frac{v^2 - v'^2}{x''} = 0.$$

(3) Two ships are sailing uniformly with velocities u, v, along lines inclined at an angle θ: shew that, if a, b, be their distances at one time from the point of intersection of their courses, the least distance of the ships is equal to

$$\frac{(av - bu)\sin\theta}{(u^2 + v^2 - 2uv\cos\theta)^{\frac{1}{2}}}.$$

(4) A particle is projected vertically upwards with a velocity u, in a medium the resistance of which varies as k times the square of the velocity: if t be the time which elapses before the particle returns to the point of projection, prove that $k^{\frac{1}{2}}.\left(\frac{2u}{g} - t\right)$ is positive and is the greater the greater k is.

(5) Shew that a boat must be rowed with a velocity, through the water, one half greater than that of the stream, so that it may be taken a given distance up a river with the least possible expenditure of work. (The resistance to the boat is supposed to be proportional to the square of its velocity through the water.)

(6) If OO' be the horizontal range of the path OPO' of a projectile P, prove that the angular velocities of OP, $O'P$, are to each other as the squares of the cosines of the angles POO', $PO'O$.

(7) If the product of the velocities at two points P, Q, of the parabolic path of a body, acted on by gravity, be constant, shew that the locus of the pole of PQ is a circle, the centre of which is at the focus of the parabola.

(8) A particle is projected horizontally with a given velocity: if the squares of the times from the instant of projection to the instants at which the particle arrives at a certain series of points in its path be in arithmetical progression, prove that the angular velocities of the tangents at these points are in harmonical progression.

(9) A body, acted on by gravity, is projected from a given point; and, when it has reached its greatest height, another body is projected from the same point in such a manner that it shall strike the first body: shew that, u, v, being the horizontal and vertical components of the velocity of projection of the former, and u', v', those of the latter body,

$$\frac{u'}{u} - 2\frac{v'}{v} = 1.$$

(10) A perfectly elastic ball is thrown into a smooth cylindrical well from a point in the circumference of the circular mouth: shew that, if the ball be reflected any number of times from the surface of the cylinder, the intervals between the reflections will be equal: shew also that, if the ball be projected horizontally in a direction making an angle $\frac{\pi}{n}$ with

the tangent to the mouth at the point of projection, it will reach the surface of the water at the instant of the n^{th} reflection, if the space due to the velocity of projection be equal to

$$\frac{(\text{radius})^2}{\text{depth}} \times \left(n \sin \frac{\pi}{n}\right)^2.$$

(11) A perfectly elastic ball impinges, with a velocity v and at an angle α to the horizon, on an inclined plane: the direction of impact is in a vertical plane parallel to the plane's intersection with the horizon: after rebounding it falls on this line of intersection: shew that

$$2v \sin \alpha \sin \lambda = (gh)^{\frac{1}{2}},$$

λ being the plane's inclination to the horizon, and h the distance of the first point of impact from the horizontal plane.

(12) A ball, thrown from any point in one of the walls of a rectangular room, returns, after striking the three others, to the point of projection, before it falls to the ground: shew that the space due to the velocity of projection is greater than the diagonal of the floor.

(13) A particle moves in one plane under the action of two forces, at right angles to each other, one of which tends towards a fixed point in the plane: supposing the centric force to vary as the time from a given instant, and the angular velocity of its direction to be constant, prove that, θ being the angle described by the particle about the fixed point, the other force is equal to $\alpha e^{\theta} + \beta e^{-\theta} + \gamma$, where α, β, γ, are constants.

(14) In a curve described by a particle about a centre of force, the angle between the radius vector and the tangent varies as the time: if a, b, c, be the radii vectores at any three points of the path, and α, β, γ, their inclinations to their corresponding tangents, prove that

$$(\sin \alpha)^{b^2-c^2}.(\sin \beta)^{c^2-a^2}.(\sin \gamma)^{a^2-b^2} = 1.$$

(15) In an ellipse of small eccentricity, described by a particle about a centre of force at the focus, the equation of the centre varies nearly as the velocity parallel to the axis major.

(16) If a polygon of a given number of sides be inscribed in the orbit of a planet, such that all its sides subtend equal angles at the Sun, prove that the sum of the angular velocities of the planet about the Sun, at the angular points of the polygon, is independent of the position of the polygon.

(17) If P be the perimeter of a closed curve described about a centre of force, τ the time of revolution, h twice the area described in a unit of time, and ρ the radius of curvature at the time t, prove that

$$P = h\int_0^\tau \frac{dt}{\rho}.$$

(18) A particle, attracted towards a fixed centre, the force varying as the distance, is projected with a given velocity from a given point: supposing the locus of the direction of projection to be a plane, the locus of the orbit is an ellipsoid, of which the particle's original position is an umbilical point.

(19) A particle describes an elliptic orbit about a centre of force at one of the foci: prove that the action is proportional to the area described about the other focus.

T.: *Quarterly Journal of Mathematics*, Vol. VII. p. 45.

(20) When a particle is attracted towards a fixed centre of force and moves in the brachistochrone, prove that the area described round the centre of force varies as the action.

(21) Shew that three bodies, attracting each other according to the law of gravitation, may move in a line in such wise that the mutual distances are in a constant ratio, the value of which depends on the masses.

Cayley: *Messenger of Mathematics*, Vol. IV. p. 211.

(22) Two points begin to move at the same instant and also stop simultaneously, and the product of their accelerations at any time varies inversely as the product of their velocities: if

a, b, be their initial and a', b', their final velocities, and t be the greatest time the motion can last, prove that

$$\frac{a'^2 - a^2}{2a\alpha} = t = \frac{b'^2 - b^2}{2b\beta},$$

where α, β, are respectively their initial accelerations.

(23) Two spheres, the molecules of which attract according to the law of the inverse square, were originally in contact: if W, W', W'', be the labouring forces which have been expended in pushing them asunder in the line of their centres, when the distances between their centres are respectively a, a', a''; prove that

$$W\left(\frac{1}{a'} - \frac{1}{a''}\right) + W'\left(\frac{1}{a''} - \frac{1}{a}\right) + W''\left(\frac{1}{a} - \frac{1}{a'}\right) = 0.$$

(24) A particle, acted on by a central force P, is moving in a medium the resistance of which varies as the velocity: prove that

$$\frac{d^2r}{dt^2} + P - \frac{h^2}{r^3}\cdot e^{-2ct} + c\frac{dr}{dt} = 0,$$

where c and h are constants.

(25) If two particles be projected, with equal velocities, from the higher extremity of the vertical diameter of a vertical circle, the one down the diameter, the other along any chord terminating at that extremity, prove that the chord will be described in a less time than the diameter.

(26) $ABCDE$ being a regular pentagon, of which the plane is vertical and the side CD horizontal and lowest, the times in which a particle would descend along BC, AC, EC, respectively, are in geometrical progression.

(27) A particle is placed on the surface of an ellipsoid, the centre of which is a centre of attractive force: to determine the direction in which the particle will begin to move.

If α, β, γ, be the co-ordinates of the place of the particle on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

the particle will begin to move in the line of which the equations are

$$\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} + \frac{\gamma z}{c^2} = 1,$$

and $$\frac{x}{\alpha}\left(\frac{1}{b^2} - \frac{1}{c^2}\right) + \frac{y}{\beta}\left(\frac{1}{c^2} - \frac{1}{a^2}\right) + \frac{z}{\gamma}\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = 0.$$

(28) The transverse axis of an equilateral hyperbola is vertical: prove that the sum of the squares of the times in which a molecule would fall down two straight lines drawn from a point in the circle, of which the transverse axis is a diameter, to the extremities of a given double ordinate of the lower branch of the hyperbola, is the same for every point in the circle.

(29) A particle is constrained to move in a circle and is acted on by a force tending to a fixed point and varying inversely as the distance: prove that the sum of the squares of the velocities of the particle at the extremities of any chord, drawn through the centre of force, is constant.

(30) If a particle be projected from one extremity of the axis major of an ellipse, the minor axis of which is vertical, so as to exactly shoot down a thin straight tube extending from the upper end of the minor axis to the other end of the axis major; shew that, α being the angle of projection, and $\cos\beta$ the eccentricity of the ellipse,

$$\tan\alpha = 3\sin\beta.$$

(31) If the circle which generates a cycloid roll along the directrix, supposed horizontal, with a certain uniform velocity, shew that the velocity of the describing point is in each position the same as if it had slid, acted on by gravity and without friction, along the cycloid from rest at a cusp. (The vertex of the cycloid lies below the directrix.)

(32) A heavy particle is attached to a fixed point by means of an elastic string, of which the natural length is $3a$ and modulus of elasticity six times the weight of the particle:

the particle is projected horizontally, with a velocity equal to $3\left(\frac{ag}{2}\right)^{\frac{1}{2}}$, from a point vertically above the fixed point and at the distance $3a$ from it: prove that the angular velocity of the string will be constant, and that the particle will describe the curve $r = a(4 - \cos\theta)$.

(33) A molecule descends from rest through a certain altitude down a curve in a vertical plane to a point of the curve where the tangent is horizontal: supposing the whole pressure exerted on the curve, viz. $\int R dt$, during this descent, to be a minimum, prove that the radius of curvature at the lowest point is equal to four times the vertical descent.

(34) A particle descends from rest from any point along an inverted cycloid to the vertex: to find the equation to the hodograph of the path of the particle.

The axis of the cycloid being the axis of x, and the origin of co-ordinates in the base, the equation to the hodograph is

$$(x^2 + y^2 + cg)^2 = c^2g^2 + 4agx^2,$$

where a is the radius of the generating circle and c the initial depth of the particle below the base of the cycloid.

(35) A particle is placed very near the centre of a smooth circular horizontal table: fine strings are attached to the particle and pass over smooth pullies which are placed at equal intervals round the margin of the table: to the other end of each of these strings is attached a particle of the same mass as the central particle: to find the time of an oscillation of the system.

If n be the number of the strings and a the radius of the table, the time of the oscillation is equal to

$$\pi\left(\frac{n+2}{n}\cdot\frac{a}{g}\right)^{\frac{1}{2}}.$$

(36) Two equal corpuscules, which attract each other according to any law, are moving in two narrow rectilinear tubes,

fixed in horizontal positions, and intersecting each other at right angles: prove that the centre of gravity of the corpuscules describes equal areas in equal times about the common point of the tubes.

(37) A body, hanging vertically, drew another body of half its weight, by means of a fine string, up an inclined plane: when the bodies had described a space c, the string broke, and it was found that the smaller body continued its motion in the same direction through an additional space c before it began to descend: to find the inclination of the plane.

$$\text{The required inclination} = \frac{\pi}{6}.$$

(38) A body, descending vertically, draws an equal body 25 feet in $2\frac{1}{2}$ seconds up a plane inclined at 30° to the horizon, by means of a fine string passing over a pully at the top of the plane: to determine the force of gravity.

$$\text{Gravity} = 32 \text{ feet.}$$

(39) A weight P, descending vertically, draws another weight Q through a space of $16\frac{1}{10}$ feet in 4 seconds up a plane inclined to the horizon at an angle of 30°, by means of a fine string passing over a pully at the top of the plane: shew that

$$P : Q :: 3 : 5.$$

(40) A railway train of given mass is travelling due south at a uniform rate along a line which runs due north and south: prove that, the earth being supposed perfectly spherical, the train will exert on the metals a strain towards the west, the magnitude of which varies as the product of the velocity of the train and the sine of the latitude of its position, and a strain towards the south, the magnitude of which varies simply as the sine of twice the latitude.

(41) An engine, the power of which is sufficient to generate in a second a velocity of 150 feet a second in a mass M, which is its own mass, is attached to a carriage, of mass $\frac{1}{2}M$, by means

of an inelastic weightless chain 3 feet long: this carriage again in exactly the same way to another, of mass $\frac{M}{2^2}$; this to a third, of mass $\frac{M}{2^3}$. The engine and carriages are in contact when the train starts: to determine the velocity with which the last carriage will begin to move.

The last carriage will begin to move with a velocity of $\sqrt{(1088)}$ feet per second.

(42) A particle slides down a smooth inclined plane, which moves with uniformly accelerated velocity directly forwards on a horizontal plane: the particle and plane begin to move at the same instant: shew that, if v be the velocity of the particle relatively to the plane, $v^2 = 2g's$, where s is the space described along the plane, and

$$g' = g \sin \alpha - f \cos \alpha,$$

α being the inclination of the plane and f its acceleration.

(43) The extremities of a fine thread, carrying a heavy bead, are fastened to two points in a vertical line: the bead and thread revolve uniformly about the vertical line, the bead's position on the thread being constant: to find the position of the bead.

The depth of the bead below the horizontal plane, which is equidistant from the two points, is equal to

$$\frac{b^2 g}{\omega^2 (b^2 - a^2)},$$

$2a$ being the distance betweeen the two points, $2b$ the length of the thread, and ω the angular velocity of revolution.

(44) The inclination of a smooth inclined plane is such that the pressure on it of a body, supported by a string parallel to the plane, is increased by one half when the string is cut and the body supported by moving the plane horizontally: shew that, if the plane be stopped suddenly at any time after the commencement of the motion, the body will strike it again after an equal interval.

(45) In a narrow horizontal tube, revolving uniformly about a vertical axis, a particle is placed at a given distance from the axis: the particle escapes from the end of the tube when the tube comes into a given position and falls on a horizontal plane: shew that, if the length of the tube be unknown, the angular velocity being given, the locus of the point of impact on the plane will be an equilateral hyperbola, but that, when the angular velocity is unknown, the length of the tube being assigned, the locus will be a straight line.

(46) A narrow rectilinear tube is made to revolve upwards with a constant angular velocity ω in a vertical plane, about a pivot through one extremity: a particle is placed in the tube, when it is in a horizontal position, at a distance a from the pivot: shew that, if ω be very small, the particle will reach the pivot in a time approximately equal to $\left(\dfrac{6a}{g\omega}\right)^{\frac{1}{3}}$.

(47) A bead is enclosed in a smooth circular tube, the centre of which moves uniformly in a straight line in the plane of the tube: shew that the velocity of the bead, relatively to the tube, is uniform.

(48) A bead is placed on a smooth circular wire, on which is impressed a given angular velocity in its own plane about a point of its circumference: prove that the initial velocity of the bead is half that of a point of the wire the distance of which from the bead is equal to that of the bead from the fixed point.

(49) A circular hoop of radius a revolves with a uniform angular velocity ω round a vertical diameter, and a smooth ring slides upon it: shew that, when $\omega < \left(\dfrac{g}{a}\right)^{\frac{1}{2}}$, the ring may perform small oscillations of the period

$$\frac{\pi}{\left(\dfrac{g}{a} - \omega^2\right)^{\frac{1}{2}}}$$

about the lowest point.

(50) A tube of small bore, in the form of a logarithmic spiral, revolves with a uniform angular velocity about an axis passing through its pole and perpendicular to its plane, which is horizontal, and contains a particle which moves freely in it: supposing the initial velocity of the particle, relatively to the tube, to be equal to the velocity of the point of the tube in contact with the particle, shew that the path of the particle is another logarithmic spiral.

(51) Two heavy particles are connected by a fine inelastic string, which is just stretched, and one of them is struck in a direction perpendicular to the string: shew that they will never approach each other.

(52) A particle, acted on by no forces, moves in a rough groove in one plane: shew that the difference between the logarithms of the velocities at any two points varies as the angle between the tangents to the groove at those points.

(53) If the arc of a circle, containing an angle $\cot^{-1}(-\mu)$, be placed so that its bounding chord is vertical, then the times of descent down all chords considered rough, which are drawn from the highest point, will be equal, μ being the coefficient of friction.

(54) Prove that the locus of an axis, passing through a fixed point of a solid body, such that the moment of inertia of the body round it is constant, is a cone of the second order, and that the circular sections of the cones corresponding to different values of the constant moment have the same directions.

Ferrers and Jackson: *Solutions of the Cambridge Problems,* 1848 to 1851, p. 319.

(55) Shew that the difference of the moments of inertia of a body round two axes in a given plane, which are equally

inclined to a fixed line in the plane, is proportional to the sine of the angle between those axes.

Ferrers and Jackson: *Solutions of the Cambridge Problems*, 1848 to 1851, p. 322.

(56) To find the relation between the co-ordinates of a point in a straight line where it is a principal axis of a body.

Let the principal axes at the centre of gravity of the body be the axes of co-ordinates, and let l, m, n, be the direction-cosines of the line: then, α, β, γ, being the radii of gyration of the body about the said principal axes, the co-ordinates of the point are connected by the equation

$$\frac{\beta^2-\gamma^2}{\frac{y}{m}-\frac{z}{n}}=\frac{\gamma^2-\alpha^2}{\frac{z}{n}-\frac{x}{l}}=\frac{\alpha^2-\beta^2}{\frac{x}{l}-\frac{y}{m}}=lx+my+nz.$$

(57) The form of a homogeneous solid of revolution, of given superficial area, described upon an axis of given length, is such that its moment of inertia about the axis is a maximum: prove that the normal at any point of the generating curve is three times as long as the radius of curvature.

Mackenzie and Walton: *Solutions of the Cambridge Problems for* 1854.

(58) Shew that if one of a free system of particles, which attract one another with forces varying as the mass and distance, have at any instant the same position and the same velocity, in magnitude and direction, as the centre of gravity of the system, it will coincide with it throughout the motion.

(59) When a circular lamina, the plane of which is vertical, rolls along a horizontal plane, shew, by calculating the value of each, that the diminution of pressure arising from the centrifugal force is compensated by the increase of pressure arising from the effective force on each particle in the direction of its motion.

(60) Of the two parts m, m', of a compound vibrating body the centres of gravity H, H', the centres of oscillation O, O', and their point of connection P, are in a straight line passing through S the axis of suspension: $SH=h$, $SH'=h'$, $SO=l$, $SO'=l'$, $SP=e$, and r, r', are the respective expansions of linear units of the masses for a given rise of temperature. Shew, first, that the change of l' for the given change of temperature is

$$r'l' + e(r-r')\left(2-\frac{l'}{h'}\right);$$

and then that the length L of the compound pendulum is unaltered by temperature if

$$\tfrac{1}{2}L = \frac{mrhl + m'r'h'l' + m'h'e(r-r')}{mrh + m'r'h' + m'e(r-r')}.$$

(61) Two solid globes, each of radius a and density Δ, attracting by the force of gravity two minute spheres at the extremities of the arms of a torsion balance in directions perpendicular to the arms and at a distance b from their centres, were observed to retain the spheres at a distance s from their position of rest; and the time of a free oscillation of the balance was found to be t seconds: shew that, if L be the length of a seconds pendulum, and R the Earth's radius, the density of the Earth is equal to

$$\frac{La^3t^2\Delta}{Rb^2s}, \text{ nearly.}$$

(62) A slender rod, suspended horizontally by two equal strings attached to two points equidistant from its ends, performs small oscillations round a vertical line through its middle point: prove that, if in the position of equilibrium the strings are inclined at equal angles to the vertical, the time of oscillation is the same as it would be if the strings were parallel, of a length equal to the projection of either of them on a vertical line, and at a distance equal to the mean pro-

portional between their distances at the points of suspension and attachment respectively.

(63) A uniform rod, the centre of gravity of which is initially at rest, moves in a plane under the action of a constant force in the direction of its length: prove that the square of the radius of curvature of the path of the rod's centre of gravity varies as the versed sine of the angle through which the rod has revolved at the end of any time from the beginning of the motion.

(64) A rigid hoop, completely cracked at one point, rolls, acted on by no force but gravity, in a vertical plane upon a perfectly rough horizontal plane: prove that the wrench-couple at the point of the hoop most remote from the crack will be a maximum whenever, the crack being lower than the centre, the inclination of the diameter through the crack to the horizon is equal to $\tan^{-1}\left(\frac{2}{\pi}\right)$.

(65) A fixed rod is inclined at any angle to the horizon: a moveable rod, to the lower end of which is attached a small ring moveable along the fixed rod, is placed at right angles to the fixed rod and in the same vertical plane with it: prove that the moveable rod will slide down without any motion of rotation.

(66) The end A of a fine string ABC is fastened to a fixed point, and two particles, the masses of which are m and m', are fastened to the points B and C of the string: the end C is then held so that BC is horizontal and AB inclined at an angle α to the vertical: if C be let go, to find the initial tensions of AB and BC.

If T, T', be the initial tensions of AB, BC, respectively,

$$T = \frac{mg\,(m+m')\cos\alpha}{m+m'\cos^2\alpha}, \quad T' = \frac{mm'g\sin\alpha\cos\alpha}{m+m'\cos^2\alpha}.$$

Besant: *Messenger of Mathematics*, Vol. IV. p. 2.

(67) Three particles A, B, C, repelling each other according to any law, are tied together by strings so as to form a triangle, and the system is in equilibrium: the string BC being cut, to find the changes of the tensions of the other two strings.

Let P, Q, R, be the forces of repulsion between B and C, C and A, A and B, respectively; and let L, M, N, be the masses of the particles at A, B, C, respectively: then the changes of the tensions of AB and AC are respectively

$$\frac{L.P.\cos B + M.Q.\cos A}{L+M}, \quad \frac{L.P.\cos C + N.R.\cos A}{L+N}.$$

Besant: *Ib.* Vol. IV. p. 2.

(68) A heavy string is at rest within a smooth tube of small bore, the axis of which is a circle in a vertical plane: the upper end of the string is fastened to the highest point of the tube: if the fastening be removed, to find the initial tension at any point.

Let A be the highest point of the string: let m be the mass of a unit of length of the string, and T the tension at a point P: then, θ, α, representing the angles subtended at the centre of the circle by the arc AP and the whole string respectively, and a denoting the radius of the circle,

$$T = maga\left(\frac{1-\cos\alpha}{\alpha} - \frac{1-\cos\theta}{\theta}\right).$$

Besant: *Ib.* Vol. IV. p. 4.

(69) An elliptic lamina, the plane of which is vertical and transverse axis horizontal, is supported by two weightless pins passing through its foci: if one of the pins be released, to determine the eccentricity of the ellipse in order that the pressure on the other may be initially unaltered.

The required eccentricity is equal to $\left(\frac{2}{5}\right)^{\frac{1}{2}}$.

Besant: *Ib.* Vol. IV. p. 6.

(70) Three equal smooth balls are kept in contact with each other on a smooth horizontal plane by a string passing round them, and a fourth equal ball rests upon the three: if the string be cut, to determine the initial change of pressure between the upper ball and each of the lower ones.

If R be the original and R' the initial pressure,

$$R' = \frac{6}{7} R.$$

Besant: *Ib.* Vol. IV. p. 6.

(71) Two equal rods are jointed together at their higher ends and rest symmetrically over a smooth sphere: the junction of the rods being severed, to find the initial pressure of each rod on the sphere.

Let W be the weight of either rod; let k be its radius of gyration about its centre of gravity, and k' about its point of contact with the sphere: let α be the inclination of each rod to the horizon: then the required pressure is equal to

$$W\left(\frac{k}{k'}\right)^2 \cos\alpha.$$

Besant: *Messenger of Mathematics*, Vol. IV. p. 8.

(72) If a rigid body be moving in any manner in three dimensions, prove that we can take moments about the instantaneous axis as if it were an axis fixed in space and in the body, provided that the moment of inertia of the body about the instantaneous axis is constant throughout the motion.

Routh: *Rigid Dynamics*, Second Edition, p. 417.

(73) A right cone rolls on its slant side on a perfectly rough inclined plane under the action of gravity: to find the motion.

Let 2α be the angle of the cone, h the length of its axis, β the inclination of the plane to the horizon, I the radius of gyration of the cone about a generating line, and ϕ the inclination of the line of contact of the plane and cone to the direction

of the component of gravity resolved along the plane: then the motion of the cone is defined by the equation

$$\frac{d^2\phi}{dt^2} = -\frac{3}{4}gh\frac{\sin^2\alpha\sin\beta}{k^2\cos\alpha}\sin\phi.$$

Routh: *Ib.* p. 417.

(74) A lamina, one point of which is fixed, is in motion, no forces acting on it: prove that the sum of the squares of the angular velocities about the two principal axes through the fixed point in the plane of the lamina is constant.

(75) If, throughout the motion of a rigid body about a fixed point, under the action of a system of forces, the angular acceleration of the body about the instantaneous axis bear to the moment of inertia about this axis and to the forces the same relation as if the axis were fixed, prove that, if the three principal moments of inertia at the fixed point be not all equal, the locus of the axis relatively to the body is a cone of the second order.

(76) A rigid body, a given point of which is fixed, revolves with a uniform angular velocity about a permanent axis under the action of a couple: prove that there are six axes of maximum reluctance (viz. six axes of rotation corresponding to maximum values of the constraining couple), two in each principal plane, each two bisecting the angles between the principal axes in the plane in which they are.

Quarterly Journal of Mathematics, Vol. VII. p. 376.
Routh: *Dynamics of a System of Rigid Bodies*, Second Edition, p. 93.

(77) A rigid body is in motion, one of its points being fixed: if, at any instant during the motion, the axis of rotation passes through the centre of gravity of the body, prove that, at that instant, the pressure on the fixed point is at right angles to the axis of rotation.

(78) Two rigid bodies move, acted on by no force, about fixed points: the principal moments of inertia of the one about

its fixed point are in the duplicate ratio of the corresponding moments of the other: prove that the initial circumstances may be so adjusted that, if the angles which the instantaneous axis makes with the corresponding principal axes are at any instant equal each to each in the two bodies, their angular velocities are also equal.

(79) A sphere, revolving about a diameter and not acted on by any extraneous force, expands symmetrically: prove that its vis viva varies inversely as its moment of inertia about a diameter.

(80) A homogeneous globe is placed upon a perfectly rough table, very near to a given centre of force in the surface of the table, the law of attraction being that of the inverse square: prove that the square of the time of an oscillation varies as the volume of the globe.

(81) If two particles move in one plane so that their centre of gravity moves along a straight line, and the sum of the areas conserved about a point in this line vanishes, prove that the line joining the particles will move parallel to itself.

(82) In a straight line $OABC$, O is a fixed end of a fine straight inelastic string OAB, to which particles of equal masses are fixed at A and B: C is a centre of force attracting directly as the distance, and the points A, B, trisect the line OC. Supposing the particles to receive slight disturbances in a plane through OC without slackening the string, prove that, μr being the attractive force at a distance r, and $\frac{2\pi}{\rho_1}$, $\frac{2\pi}{\rho_2}$, the periods of the oscillations of the particles, the product of the quantities ρ_1, ρ_2, is equal to 3μ, and that their ratio is independent of μ.

(83) When a horizontal lamina, resting upon a horizontal plane, receives a blow, shew that, in the time which the system takes to make a complete revolution, the centre of gravity will

advance over a space equal to the circumference of the circle described about the spontaneous centre of rotation as centre and passing through the centre of gravity.

(84) A uniform beam is revolving in a vertical plane about a horizontal axis through its middle point; and, at the instant it is passing through its horizontal position, a perfectly elastic ball, the mass of which is one-third that of the beam, is projected horizontally from a point vertically above the axis, so as to hit the beam at one extremity, then to rebound to the other, and so on for ever, bounding and rebounding along the same path. If θ be the angle, on each side of its horizontal position, through which the beam revolves, prove that $\theta = \cot\theta$.

(85) A rigid body, moveable about a fixed point, is struck by a blow of given magnitude at a given point: if the angular velocity thus impressed upon the body be the greatest possible, prove that, a, b, c, being the co-ordinates of the given point in relation to the principal axes through the fixed point, and l, m, n, being the direction-cosines of the blow,

$$\frac{a}{l}\left(\frac{1}{B^2}-\frac{1}{C^2}\right)+\frac{b}{m}\left(\frac{1}{C^2}-\frac{1}{A^2}\right)+\frac{c}{n}\left(\frac{1}{A^2}-\frac{1}{B^2}\right)=0,$$

A, B, C, being the moments of inertia of the body about the principal axes at the fixed point.

(86) Two equal particles of mass m are fixed at the extremities of the axis of a prolate spheroid, the mass of which is M, the eccentricity of the generating ellipse being e. The spheroid is struck by a couple and then left to move freely. Shew that throughout the motion it will constantly have contact with a single plane if $m = \frac{1}{10} Me^2$.

(87) Two equal particles are fixed at the extremities of the axis of a prolate spheroid: the spheroid is struck by a couple and then left to move freely: prove that, throughout the

motion, it will constantly have contact with a single plane, if the ratio of the mass of one of the particles to the mass of the spheroid be equal to $\frac{e^2}{10}$, where e is the eccentricity of the generating ellipse.

Ferrers and Jackson: *Solutions of the Cambridge Problems*, 1848 to 1851, p. 311.

(88) A fox, pursued by a hound, is running with uniform velocity over a frail arch in the form of a cycloid: the hound stops at a weak point of the arch, then tumbles through, and reaches the level ground with a velocity equal to that of the fox: prove that the fox exerted no normal pressure on the arch at the point where the hound fell through.

(89) A man, of weight W, stands on smooth ice: prove that if, when he gradually parts his legs, kept straight, with his feet in contact with the ice, the pressure of his feet on the ice be constant, his head will descend with a uniform acceleration; and that, if f be the acceleration of his head, when his feet exert no pressure on the ice, their pressure on the ice, were f' the acceleration of his head, would be equal to

$$\frac{f-f'}{f}\cdot W.$$

(90) Dato Pendulo turbinante, composito ex ponderibus non in communi turbinationis plano, sed vel in alio, vel in aliis diversis planis inhærentibus; demissisque rectis perpendicularibus ad commune planum turbinationis ex ponderibus; si pondera singula ducantur in distantias suarum perpendicularium ab axe turbinationis et porro in altitudines superficierum conicarum, quas rectæ a ponderibus ad verticem turbinationis eductæ describunt; deinde summa productorum dividatur per id, quod fit ducendo ponderum summam in distantiam centri gravitatis communis omnium ab axe turbinationis: habebitur distantia Centri turbinationis, seu longitudo Penduli simplicis circuitus minimos iisdem cum composito temporibus facientis, sive altitudo super-

ficiei conicæ, quam Pendulum quodlibet simplex describens Pendulo dato composito erit isochronum[1].

John Bernoulli: *Acta Erud. Lips.* 1715. Jun. pag. 242. *Opera, Tom.* II. p. 187.

[1] Let G (fig. 259) be the centre of gravity of any rigid body, acted on by gravity, and revolving conically about a fixed point C to which it is fixed, the path of each of its particles being a horizontal circle. Let CO be a vertical line. The body is called a *turbinating pendulum*, C the *vertex of turbination*, CO the *axis of turbination*, the plane OCG the *plane of turbination*; the *centre of turbination* of the compound pendulum is a point in the axis of turbination at a distance from the vertex of turbination equal to the altitude of the conical superficies described by a simple pendulum turbinating isochronously with the compound pendulum. A simple pendulum is said to make *circuitus minimos* when the conical angle is indefinitely small.

APPENDIX.

THE COPY of Bernoulli's programme[1] which had been received by the celebrated David Gregory, was seen some years ago by the author of this work, in the possession of the lamented D. F. Gregory, late Fellow of Trinity College. The following reprint of the challenge will probably be acceptable to those who take an interest in the antiquities of science.

Acutissimis qui toto orbe florent Mathematicis.

S. P. D.

JOHANNES BERNOULLI, MATH. P.P.

"Cum compertum habeamus, vix quicquam esse quod magis excitet generosa ingenia ad moliendum quod conducit augendis scientiis, quam difficilium pariter et utilium quæstionum propositionem, quarum enodatione, tanquam singulari si qua alia via, ad nominis claritatem perveniant sibique apud posteritatem æterna extruant monumenta: Sic me nihil gratius Orbi Mathematico facturum speravi quam si imitando exemplum tantorum Virorum Mersenni, Pascalii, Fermatii, præsertim recentis illius Anonymi Ænigmatistæ Florentini, aliorumque, qui idem ante me fecerunt, præstantissimis hujus ævi Analystis proponerem aliquod problema, quo, quasi lapide Lydio, suas methodos examinare, vires intendere, et, si quid invenirent, nobiscum communicare possent, ut quisque suas exinde promeritas laudes a nobis, publice id profitentibus, consequeretur.

"Factum autem illud est ante semestre in Actis Lips. m. Jun. pag. 269, ubi tale problema proposui cujus utilitatem cum jucunditate conjunctam videbunt omnes qui cum successu ei se applicabunt. Sex mensium spatium a prima publicationis

[1] See page 338.

die Geometris concessum est, intra quod, si nulla solutio prodiret in lucem, me meam exhibiturum promisi. Sed ecce elapsus est terminus, et nihil solutionis comparuit; nisi quod Celeb. Leibnitius, de profundiore Geometria præclare meritus, me per literas certiorem fecerit, se jam feliciter dissolvisse nodum pulcherrimi hujus, uti vocabat, et inauditi antea problematis; insimulque humaniter rogavit, ut præstitutum limitem ad proximum Pascha extendi paterer; quo interea apud Gallos Italosque idem illud publicari posset, nullusque adeo superesset locus ulli de angustia termini querelæ. Quam honestam petitionem non solum indulsi, sed ipse hanc prorogationem promulgare decrevi, visurus num qui sint qui nobilem hanc et arduam quæstionem aggressuri, post longum temporis intervallum, tandem Enodationis compotes fierent. Illorum interim in gratiam, ad quorum manus Acta Lipsiensia non perveniunt, propositionem hic repeto.

PROBLEMA MECHANICO-GEOMETRICUM DE LINEA CELERRIMI DESCENSUS.

"*Determinare lineam curvam data duo puncta, in diversis ab horizonte distantiis, et non in eadem recta verticali posita, connectentem, super qua mobile, propria gravitate decurrens et a superiori puncto moveri incipiens, citissime descendat ad punctum inferius.*

"Sensus problematis hic est: ex infinitis lineis, quæ duo illa data puncta conjungunt, vel ab uno ad alterum duci possunt, eligatur illa, juxta quam, si incurvetur lamina tubi canalisve formam habens, ut ipsi impositus globulus et liberè dimissus iter suum ab uno puncto ad alterum emetiatur tempore brevissimo.

"Ut vero omnem ambiguitatis ansam præcaveamus, scire B.L. volumus, nos hîc admittere Galilæi hypothesin de cujus veritate, seposita resistentia, jam nemo est saniorum Geometrarum qui ambigat, *Velocitates scilicet acquisitas gravium cadentium esse in subduplicata ratione altitudinum emensarum*, quanquam alias nostra solvendi methodus universaliter ad quamvis aliam hypothesin sese extendat.

"Cum itaque nihil obscuritatis supersit, obnixe rogamus omnes et singulos hujus ævi Geometras, accingant se promte, tentent, discutiant quicquid in extremo suarum methodorum recessu absconditum tenent. Rapiat qui potest præmium quod Solutori paravimus; non quidem auri non argenti summam, quo abjecta tantum et mercenaria conducuntur ingenia, a quibus ut nihil laudabile, sic nihil, quod scientiis fructuosum, expectamus; sed cum virtus sibi ipsi sit merces pulcherrima, atque gloria immensum habeat calcar, offerimus præmium quale convenit ingenui sanguinis Viro, consertum ex honore, laude, et plausu; quibus magni nostri Apollinis perspicacitatem, publice et privatim, scriptis et dictis coronabimus, condecorabimus, et celebrabimus.

"Quod si vero festum Paschatis præterierit, nemine deprehenso qui quæsitum nostrum solverit, nos quæ ipsi invenimus publico non invidebimus: Incomparabilis enim Leibnitius solutiones tum suam, tum nostram, ipsi jam pridem commissam protinus ut spero in lucem emittet; quas si Geometræ, ex penitiori quodam fonte petitas perspexerint, nulli dubitamus quin angustos vulgaris Geometriæ limites agnoscant, nostraque prein inventa tanto pluris faciant, quanto pauciores eximiam nostram quæstionem soluturi extiterint, etiam inter illos ipsos, qui per singulares, quas tantopere commendant, methodos, interioris Geometriæ latibula non solum intime penetrasse, sed etiam ejus pomœria, Theorematis suis aureis, nemini ut putabant cognitis, ab aliis tamen jam longe prius editis, mirum in modum extendisse gloriantur.

PROBLEMA ALTERUM PURE GEOMETRICUM, QUOD PRIORI SUBNECTIMUS ET STRENÆ LOCO ERUDITIS PROPONIMUS.

"Ab Euclidis tempore vel Tyronibus notum est; Ductam utcunque a puncto dato rectam lineam, a circuli peripheria ita secari, ut rectangulum duorum segmentorum, inter punctum datum et utramque peripheriæ partem interceptorum, sit eidem constanti perpetuo æquale. Primus ego ostendi in eod. Actor. Jun. pag. 265, hanc proprietatem infinitis aliis curvis convenire, illamque adeo circulo non esse essentialem. Arrepta hinc occa-

sione, proposui Geometris determinandam curvam, vel curvas, in quibus non rectangulum, sed solidum sub uno et quadrato alterius segmentorum æquetur semper eidem: sed a nemine hactenus solvendi modus prodiit; exhibebimus eum quandocunque desiderabitur. Quoniam autem non nisi per curvas transcendentes quæsito satisfacimus, en aliud, cujus solutio per mere algebraicas, in nostra est potestate. *Quæritur Curva, ejus proprietatis, ut duo illa segmenta, ad quamcunque potentiam datam elevata et simul sumpta, faciant ubique unam eandemque summam.*

"Casum simplicissimum, existente sc. numero potentiæ 1, ibidem in Actis, pag. 266. jam solutum dedimus; generalem vero solutionem, quam etiamnum premimus, Analystis eruendam relinquimus."

Dabam Groningæ ipsis Cal. Jan. 1697.

Groningæ, Typis Catharinæ Zandt, Provincialis Academiæ Typographæ, 1697.

THE END.

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